On complex singularity analysis of holomorphic solutions of linear partial differential equations.

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We construct formal series solutions of linear PDE as linear combinations of powers of solutions of a first order nonlinear ODE : the *tanh method*.

Initiated by W. Malfliet : an effective algebraic method for exact solutions for nonlinear PDEs

$$H\left(u, \partial_t u, \partial_x u, \partial_x^2 u, \dots\right) = 0$$

using finite expansions

$$u(t,x) = \sum_{j \in J} u_j \left(\varphi(\kappa(x - wt))\right)^j$$

where φ is a solution of a Ricatti equation

$$\varphi' = a + b\varphi + c\varphi^2.$$

We will consider *special solutions* of a linear PDE

$$\partial_z^S X(t,z) = \sum_{\mathbf{k}=(k_0,k_1)\in\mathcal{J}} a_{\mathbf{k}}(t,z)\partial_t^{k_0}\partial_z^{k_1} X(t,z)$$

in the form

$$X(t,z) = \sum_{j\geq 0} X_j(t,z)(\phi(t))^j$$

where $\phi(t)$ is a solution of some nonlinear first order ODE

$$\phi'(t) = P\left(t,\phi(t)\right).$$

with

- $P(t,X) \in \mathbb{C}\left\{t\right\}\left[X\right]$
- $a_{\mathbf{k}}(t,z) \in \mathcal{O}\left[\mathcal{D}(t_0)\times\mathcal{D}(0)\right]$
- a finite set $\mathcal{J} \subset \left\{\mathbf{k} = (k_0,k_1) \in \mathbb{N}^2 \mid k_1 \leq S-1\right\}$.

Motivations

- *existence* of such a formal solutions
- *sufficient conditions* for which this formal solutions are holomorphic in some punctured polydiscs of $\mathbb{C}^2$
- *rate of growth* of this solutions near the singularities (example : $P(t,X) = X^2$, $\phi' = \phi^2$, $\phi(t) = -1\backslash(t+t_0))$

Schedule

- Formal solutions.
- Majorant series method.
  - An auxiliary linear Cauchy problem.
  - Banach space of entire functions with exponential growth.
  - A Cauchy-Kowalevskii theorem.
  - Classification of singularities for first order ODEs.
  - Main result.

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We look for *transseries* solutions $\hat{X}(t,z)$

$$\hat{X}(t,z)=\sum_{\beta\geq 0}\sum_{\ell\geq 0}X_{\ell,\beta}(t)\frac{\phi(t)^{\ell}}{\ell!}\frac{z^{\beta}}{\beta!}.$$

An induction relation for the coefficients $X_{\ell,\beta}(t)$

$$X_{\ell,\beta+S}=R\left((X_{\ell_1,\beta_1})_{\ell_1,\beta_1}\right)$$

is obtained by the Faa di Bruno formula

$$\partial_t^m\left(\frac{\phi(t)^{\ell}}{\ell!}\right)=\sum_{\substack{(p_1,\ldots,p_m)\in\mathbb{N}^m\\ p_1+2p_2+\ldots+mp_m=m}}\frac{m!}{p_1!\ldots p_m!}\frac{\phi(t)^{\ell-|\mathbf{p}|}}{(\ell-|\mathbf{p}|)!}\prod_{k=1}^{m}\left(\frac{\phi^{(k)}(t)}{k!}\right)^{p_k}$$

Proposition

Let $X_{\ell,\beta}(t)\in\mathcal{O}\left(\mathcal{D}(t_0)\right)$, $\ell\geq 0$, $0\leq\beta\leq S-1$.
Then, there exists a formal solution $\hat{X}(t,z)$ of the PDE for the given initial conditions

$$(\partial_z^j\hat{X})(t,0)=\sum_{\ell\geq 0}X_{\ell,j}(t)\frac{\phi(t)^{\ell}}{\ell!}\ ,\quad 0\leq j\leq S-1.$$

Let

$$v_{n_0,\ell,\beta} := \sup_{|t-t_0|\le r} |\partial_t^{n_0} X_{\ell,\beta}(t)|.$$

As $X_{\ell,\beta}(t)$ satisfies $X_{\ell,\beta+S} = R\left((X_{\ell_1,\beta_1})_{\ell_1,\beta_1}\right)$, $v_{n_0,\ell,\beta}$ satisfies $v_{n_0,\ell,\beta+S} \le \tilde{R}\left((v_{n_1,\ell_1,\beta_1})_{n_1,\ell_1,\beta_1}\right)$.

We define the formal series

$$U(t,z,T) = \sum_{\beta\ge 0}\sum_{\ell\ge 0}\sum_{n_0\ge 0} u_{n_0,\ell,\beta}\frac{t^{n_0}}{n_0!}\frac{T^\ell}{\ell!}\frac{z^\beta}{\beta!}$$

where $u_{n_0,\ell,\beta}$ is the *unique* solution of $u_{n_0,\ell,\beta+S} = \tilde{R}\left((u_{n_1,\ell_1,\beta_1})_{n_1,\ell_1,\beta_1}\right)$.

Proposition

The formal series $U(t,z,T)$ is the unique solution of the Cauchy problem

$$\partial_z^S U(t,z,T) = \sum_{\mathbf{q}=(q_0,q_1,q_2)\in\mathcal{Q}} B_{\mathbf{q}}(t,z,T)\partial_t^{q_0}\partial_T^{q_1}\partial_z^{q_2}U(t,z,T)$$

for the given initial conditions $(\partial_z^j U)(t,0,T) = \sum_{\ell\ge 0}\sum_{n_0\ge 0} v_{n_0,\ell,j}\frac{t^{n_0}}{n_0!}\frac{T^\ell}{\ell!}$, $0\le j\le S-1$.

- $\mathcal{G}_q(\delta_1, \delta_2; \sigma)$: subspace of the vector space of entire functions / to T and holomorphic / to (t, z) :

$$V(t,z,T) = \sum_{n,\beta \geq 0} v_{n,\beta}(T) \frac{t^n}{n!} \frac{z^\beta}{\beta!} \in \mathcal{G}_q(\delta_1, \delta_2; \sigma),$$

such that $\displaystyle\sum_{n,\beta \geq 0} ||v_{n,\beta}(T)||_{\beta;\sigma} \frac{\delta_1^n \delta_2^\beta}{(n+\beta)!} < +\infty$, where

$$||v_{n,\beta}(T)||_{\beta;\sigma} = \sup_{T \in \mathbb{C}} |v_{n,\beta}(T)| (1+|T|)^{-m} \exp(-\sigma r_b(\beta)|T|^q).$$

- $||.||_{\delta_1,\delta_2;\sigma}$:

$$||V(t,z,T)||_{\delta_1,\delta_2;\sigma} = \sum_{n,\beta \geq 0} ||v_{n,\beta}(T)||_{\beta;\sigma} \frac{\delta_1^n \delta_2^\beta}{(n+\beta)!}.$$

Theorem

If

$$\partial_z^S U(t,z,T) = \sum_{\mathbf{q}\in\mathcal{Q}} B_{\mathbf{q}}(t,z,T)\partial_t^{q_0}\partial_T^{q_1}\partial_z^{q_2} U(t,z,T)$$

satisfies some conditions and if for all $0 \le j \le S-1$,

$$\left(\partial_z^j U\right)(t,0,T) = \psi_j(t,T) \in \mathcal{G}_q\left(\delta_{1,0}, \delta_{2,0}; \sigma_0\right).$$

Then, there exists δ_1, δ_2, $\sigma > 0$, *such that the Cauchy problem has a unique solution* $U(t,z,T) \in \mathcal{G}_q\left(\delta_1, \delta_2; \sigma\right)$.

For the proof, we need

$$\|\left(T^s \partial_t^\nu \partial_T^\kappa \partial_z^{-S} V\right)(t,z,T)\|_{\delta_1,\delta_2;\sigma} \le C\delta_1^{-\nu}\delta_2^s \|V(t,z,T)\|_{\delta_1,\delta_2;\sigma},$$

we have to estimate $\|\left(T^s \partial_T^\kappa v_{n+\nu,\beta-S}\right)(T)\|_{\beta,\sigma}$, we use the Cauchy-integral formula

$$\left(T^s\partial_T^\kappa v_{n+\nu,\beta-S}\right)(T) = \frac{\kappa!}{2i\pi}\int_{|\xi-T|=a}\frac{T^s v_{n+\nu,\beta-S}}{(\xi-T)^{\kappa+1}}d\xi$$

to obtain our result, we use a good choice for the radius a (introduced by Y. Dubinskï)

$$a = \left(|T|^q+1\right)^{1/q} - |T|$$

and we yield the conditions

$$\frac{b(s+\kappa(q-1))}{q} + \nu < S.$$

For

$$\phi'(t) = P\left(t, \phi(t)\right),$$

by P. Painlevé, the only movable sigularities in $\mathbb{C}$ of $\phi(t)$ are poles and /or algebraic branch points.

We define $D_\theta(t_0, r) = D(t_0, r) \backslash \left[t_0, re^{i\theta}\right)$.
Let $\phi(t)$ solution on $D_\theta(t_0, r_0)$, there can be represented by a *Puiseux series*

$$\phi(t) = \sum_{n \geq -n_0} f_n (t - t_0)^{n/\mu}$$

where $\mu, n_0 \in \mathbb{N}^*$ et $f_{-n_0} \neq 0$.

Theorem

Under some conditions, the formal series

$$X(t,z)=\sum_{\beta\geq 0}\sum_{\ell\geq 0}X_{\ell,\beta}(t)\frac{\phi(t)^{\ell}}{\ell!}\frac{z^{\beta}}{\beta!}$$

defines a holomorphic function on $D_{\theta}(t_0,r_0)\times D\left(0,\delta\right)$.
Moreover, there exists C_1, C_2 *such that*

$$\sup_{|z|\leq\delta}|X(t,z)|\leq C_1|t-t_0|^{-n_0m/\mu}\exp\left(C_2|t-t_0|^{-n_0q/\mu}\right)$$

for all $t\in\mathcal{D}_{\theta}(t_0,r_0)$.

Sketch of proof :

- $U(t,z,T)$ solution of the auxiliary problem belongs to $\mathcal{G}_q\left(\delta_1,\delta_2;\sigma\right)$ (Cauchy-Kowalevskii)

- $$W(t,z,T)=\sum_{\beta\le 0}\sum_{\ell\le 0}X_{\ell,\beta}(t)\frac{T^{\ell}}{\ell!}\frac{z^{\beta}}{\beta!}$$

- $$\sup_{|z|<\delta}\sup_{t\in\mathcal{D}}|W(t,z,T)|\le|U(0,\delta,|T|)|\le C(1+|T|)^{m}\exp\left(\sigma\zeta(b)|T|^{q}\right)$$

$W(t,z,T)$ is holomorphic / t, z and at most of exponential growth / T. Therefore $X(t,z)=W(t,z,\phi(z))$ is holomorphic solution.

$$\begin{cases} \phi' = \phi^2 & \text{Ricatti equation} \\ \partial_z^2 X = \partial_t X & \text{heat equation} \end{cases}$$

Auxiliary equation : $\partial_z^2 U = \partial_t U + T^2 \partial_T U$.

The condition from Cauchy-Kowalevskii theorem for $T^2 \partial_T$:

$$\frac{b(2+1(q-1)}{q} < 2 \Leftrightarrow \frac{b}{q} + b < 2 \quad (b > 1)$$

holds for q large enough.

S. Malek, C. Stenger : *On complex singularity analysis of holomorphic solutions of linear partial differential equations*, to appear in Advances in Dynamical Systems and Applications.

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