Maillet Type Theorem and Gevrey Regularity in Time of Solutions to Nonlinear Partial Differential Equations

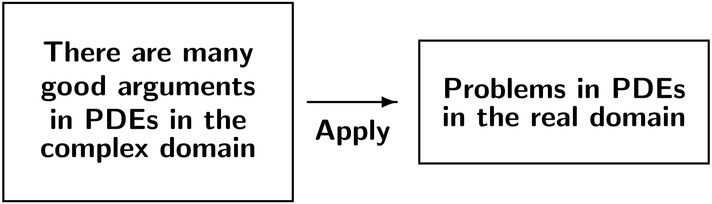
Hidetoshi TAHARA
(Sophia University, Tokyo, JAPAN)

August 11, 2011, Poland (Będlewo)

I am a researcher of partial differential equations in the complex domain. Recently, I am very much interested in applying complex method to problems in the real domain.

I am a researcher of partial differential equations in the complex domain. Recently, I am very much interested in applying complex method to problems in the real domain.

My intension is illustrated as follows.



In this talk, I will consider the equation

(E) $$t^\gamma \partial_t^m u = F\Big(t, x, \Big\{\partial_t^j \partial_x^\alpha u\Big\}_{j+|\alpha|\le L}\Big)$$

where $\gamma \ge 0$ **and** $L \ge m \ge 1$

In this talk, I will consider the equation

(E) $$t^\gamma \partial_t^m u = F\Big(t, x, \Big\{\partial_t^j \partial_x^\alpha u\Big\}_{j+|\alpha|\le L}\Big)$$

where $\gamma \ge 0$ and $L \ge m \ge 1$

and I will present two results:

Part I: Maillet type theorem
- this is a model in the complex domain -

Part II: Gevrey regularity in time of solutions of (E)
- this is a result in the real domain -

Part I
Maillet type theorem in the complex PDEs

0.1. Notations

t the time variable in $\mathbb{C}_t$,

$x = (x_1, \ldots, x_n)$ the space variables in $\mathbb{C}^n_x$,

$D_R = \{x \in \mathbb{C}^n\,;\, |x_i| \leq R\ \ (i = 1, \ldots, n)\,\}$.

0.1. Notations

$$t \quad \text{the time variable in} \quad \mathbb{C}_t,$$
$$x = (x_1, \ldots, x_n) \quad \text{the space variables in} \quad \mathbb{C}_x^n,$$
$$D_R = \{x \in \mathbb{C}^n\,;\, |x_i| \leq R \ (i = 1, \ldots, n)\}.$$

We will use the following notations:

$\mathcal{O}_R$: the set of all holomorphic functions in x on D_R,
$\mathcal{O}_R[[t]]$: the ring of formal power series in t with coefficients in $\mathcal{O}_R$,
$\mathcal{M}_R[[t]]$: the subset of all $f(t,x) \in \mathcal{O}_R[[t]]$ satisfying
$$f(0,x) \equiv 0.$$

0.2. Some definitions

Definition 0.1. For $s \geq 1$ we denote by $\mathcal{O}\{t\}_s$ (or $\mathcal{E}^{\{s\}}$) the set of all formal power series $\sum_{k\geq 0} a_k(x)t^k \in \mathcal{O}_R[[t]]$ satisfying the following: there are $C > 0$ and $h > 0$ such that

$$\max_{x\in D_R} |a_k(x)| \leq Ch^k k!^{s-1}, \quad \forall k \in \mathbb{N}.$$

0.2. Some definitions

Definition 0.1. For $s \geq 1$ we denote by $\mathcal{O}\{t\}_s$ (or $\mathcal{E}^{\{s\}}$) the set of all formal power series $\sum_{k \geq 0} a_k(x) t^k \in \mathcal{O}_R[[t]]$ satisfying the following: there are $C > 0$ and $h > 0$ such that

$$\max_{x \in D_R} |a_k(x)| \leq C h^k k!^{s-1}, \quad \forall k \in \mathbb{N}.$$

If $f(t,x) \in \mathcal{O}\{t\}_s$ (or $\in \mathcal{E}^{\{s\}}$), we say that $f(t,x)$ is a formal power series in the formal Gevrey class of order s.

0.2. Some definitions

Definition 0.1. For $s \geq 1$ we denote by $\mathcal{O}\{t\}_s$ (or $\mathcal{E}^{\{s\}}$) the set of all formal power series $\sum_{k\geq 0} a_k(x)t^k \in \mathcal{O}_R[[t]]$ satisfying the following: there are $C > 0$ and $h > 0$ such that

$$\max_{x\in D_R} |a_k(x)| \leq Ch^k k!^{s-1}, \quad \forall k \in \mathbb{N}.$$

If $f(t,x) \in \mathcal{O}\{t\}_s$ (or $\in \mathcal{E}^{\{s\}}$), we say that $f(t,x)$ is a formal power series in the formal Gevrey class of order s.

Definition 0.2. Let $f(t,x) = \sum_{k\geq 0} a_k(x)t^k \in \mathcal{O}_R[[t]]$. We define the valuation of $f(t,x)$ with respect to t by

$$val_t(f) = \min\{\, k \in \mathbb{N}\,;\, a_k(x) \not\equiv 0\}$$

(if $a_k(x) \equiv 0$ for all $k \in \mathbb{N}$, we set $val_t(f) = \infty$).

0.3. Equation and assumption

Let $\gamma \geq 0$ and $1 \leq m \leq L$ be integers, and let us consider

$$\text{(E)} \qquad t^{\gamma} \partial_t^m u = F\Big(t, x, \{\partial_t^j \partial_x^{\alpha} u\}_{j+|\alpha| \leq L}\Big)$$

0.3. Equation and assumption

Let $\gamma \geq 0$ and $1 \leq m \leq L$ be integers, and let us consider

$$\text{(E)} \qquad t^{\gamma}\partial_t^m u = F\Big(t, x, \{\partial_t^j \partial_x^{\alpha} u\}_{j+|\alpha| \leq L}\Big)$$

under the following assumptions:

c_1) $F(t,x,z)$ is a holomorphic function on Ω,
c_2) $\hat{u}(t,x) \in \mathcal{M}_R[[t]]$ is a formal solution of (E)

where Ω is a neighborhood of the origin.

0.3. Equation and assumption

Let $\gamma \geq 0$ and $1 \leq m \leq L$ be integers, and let us consider

$$\text{(E)} \qquad t^{\gamma}\partial_t^m u = F\Big(t, x, \{\partial_t^j \partial_x^{\alpha} u\}_{j+|\alpha|\leq L}\Big)$$

under the following assumptions:

c_1) $F(t,x,z)$ is a holomorphic function on Ω,
c_2) $\hat{u}(t,x) \in \mathcal{M}_R[[t]]$ is a formal solution of (E)

where Ω is a neighborhood of the origin. We set:

$$k_{j,\alpha} = val_t\left(\frac{\partial F}{\partial z_{j,\alpha}}(t,x,D\hat{u}(t,x))\right), \quad D\hat{u} = \{\partial_t^j \partial_x^{\alpha} \hat{u}\}_{j+|\alpha|\leq L},$$

and suppose

$$c_3) \qquad \begin{cases} k_{j,\alpha} \geq \gamma - m + j, & \text{if } |\alpha| = 0, \\ k_{j,\alpha} \geq \gamma - m + j + 1, & \text{if } |\alpha| > 0. \end{cases}$$

0.4. Maillet type theorem

Then, we have the following result:

Theorem 0.3 (Gérard-Tahara). **Suppose the conditions $c_1)$, $c_2)$ and $c_3)$: then, the formal solution $\hat{u}(t,x)$ in $c_2)$ satisfies**

$$\hat{u}(t,x) \in \mathcal{O}\{t\}_s \ \text{(or} \in \mathcal{E}^{\{s\}}\text{)} \quad \text{for any} \quad s \geq s_0$$

where

$$s_0 = 1 + \max\left[0, \max_{|\alpha|>0} \left(\frac{j+|\alpha|-m}{k_{j,\alpha}-\gamma+m-j} \right) \right].$$

(Essentially, the proof was given in the book of Gérard-Tahara.)

Part II
Gevrey regularity in time in the real domain

In this PART II, I will consider the equation

(E) $$t^{\gamma}\partial_t^m u = F\Big(t, x, \Big\{\partial_t^j \partial_x^{\alpha} u\Big\}_{j+|\alpha|\le L, j<m}\Big)$$

where $\gamma \ge 0$ **and** $L \ge m \ge 1$

in Gevrey classes,

In this PART II, I will consider the equation

$$\textbf{(E)} \qquad t^{\gamma}\partial_t^m u = F\Big(t, x, \Big\{\partial_t^j \partial_x^{\alpha} u\Big\}_{j+|\alpha|\le L, j<m}\Big)$$

where $\gamma \geq 0$ **and** $L \geq m \geq 1$

in Gevrey classes, and give an answer to the following problem on time regularity:

Problem. Let $u(t,x) \in C^{\infty}([0,T], \mathcal{E}^{\{\sigma\}}(V))$ **(with** $\sigma \geq 1$**) be a solution of (E); then can we have the property:**

$$u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$$

for some $s \geq 1$ **?**

The plan of part II is as follows:

- 1. Notations, Definitions of Gevrey classes, etc
- 2. Problem and examples
- 3. Main theorems
  - sufficient condition for time regularity -
- 4. Necessity of the condition

§1. Notations, definitions, etc

1.1. Notations

t the time variable in $\mathbb{R}_t$,

$x = (x_1, \ldots, x_n)$ the space variables in $\mathbb{R}^n_x$,

$\mathbb{N} = \{0, 1, 2, \ldots\}, \quad \mathbb{N}^* = \{1, 2, \ldots\}$

$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n,$

$\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ with $\partial_{x_i} = \partial/\partial x_i \ (i = 1, \ldots, n),$

$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}.$

1.2. Functions in Gevrey class (1)

Let $\sigma \geq 1$ and V be an open subset of $\mathbb{R}^n_x$.

(1) We denote by $\mathcal{E}^{\{\sigma\}}(V)$ the set of all functions $f(x) \in C^\infty(V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{x \in K} \left|\partial_x^\alpha f(x)\right| \leq C h^{|\alpha|} |\alpha|!^\sigma, \quad \forall \alpha \in \mathbb{N}^n.$$

Let $\sigma \geq 1$ and V be an open subset of $\mathbb{R}^n_x$.

(1) We denote by $\mathcal{E}^{\{\sigma\}}(V)$ the set of all functions $f(x) \in C^\infty(V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{x \in K} \left| \partial_x^\alpha f(x) \right| \leq C h^{|\alpha|} |\alpha|!^\sigma, \quad \forall \alpha \in \mathbb{N}^n.$$

A function in $\mathcal{E}^{\{\sigma\}}(V)$ is called a function of Gevrey class of order σ.

1.2. Functions in Gevrey class (1)

Let $\sigma \geq 1$ and V be an open subset of $\mathbb{R}^n_x$.
(1) We denote by $\mathcal{E}^{\{\sigma\}}(V)$ the set of all functions $f(x) \in C^\infty(V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{x \in K} \left| \partial_x^\alpha f(x) \right| \leq C h^{|\alpha|} |\alpha|!^\sigma, \quad \forall \alpha \in \mathbb{N}^n.$$

A function in $\mathcal{E}^{\{\sigma\}}(V)$ is called a function of Gevrey class of order σ. If $1 < s_1 < s_2 < \infty$ we have

$$\mathcal{A}(V) = \mathcal{E}^{\{1\}}(V) \subset \mathcal{E}^{\{s_1\}}(V) \subset \mathcal{E}^{\{s_2\}}(V) \subset C^\infty(V).$$

By this, we can say that functions in $\mathcal{E}^{\{s_1\}}(V)$ is more regular than those in $\mathcal{E}^{\{s_2\}}(V)$.

1.3. Functions in Gevrey class (2)

(2) We denote by $C^\infty([0,T],\mathcal{E}^{\{\sigma\}}(V))$ the set of all infinitely differentiable functions in $t\in[0,T]$ with values in $\mathcal{E}^{\{\sigma\}}(V)$ equipped with the usual local convex topology.

(2) We denote by $C^\infty([0,T],\mathcal{E}^{\{\sigma\}}(V))$ the set of all infinitely differentiable functions in $t \in [0,T]$ with values in $\mathcal{E}^{\{\sigma\}}(V)$ equipped with the usual local convex topology.

It is easy to see that $C^\infty([0,T],\mathcal{E}^{\{\sigma\}}(V))$ is the set of all functions $u(t,x) \in C^\infty([0,T] \times V)$ satisfying the following: for any compact subset K of V and any $k \in \mathbb{N}$ there are $C_k > 0$ and $h_k > 0$ such that

$$\max_{[0,T]\times K} \left| \partial_t^k \partial_x^\alpha u(t,x) \right| \leq C_k h_k^{|\alpha|} |\alpha|!^\sigma, \quad \forall \alpha \in \mathbb{N}^n.$$

1.4. Functions in Gevrey class (3)

(3) Let $s \geq 1$: we denote by $\mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ the set of all functions $u(t,x) \in C^\infty([0,T] \times V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{[0,T]\times K} \left| \partial_t^k \partial_x^\alpha u(t,x) \right| \leq C h^{k+|\alpha|} k!^s |\alpha|!^\sigma, \quad \forall (k,\alpha) \in \mathbb{N} \times \mathbb{N}^n.$$

1.4. Functions in Gevrey class (3)

(3) Let $s \geq 1$: we denote by $\mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ the set of all functions $u(t,x) \in C^\infty([0,T] \times V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{[0,T]\times K} \left|\partial_t^k \partial_x^\alpha u(t,x)\right| \leq C h^{k+|\alpha|} k!^s |\alpha|!^\sigma, \quad \forall (k,\alpha) \in \mathbb{N} \times \mathbb{N}^n.$$

This means that $u(t,x)$ is a function of the Gevrey class of order s in t and of the Gevrey class of order σ in x. We often write $\mathcal{E}^{\{\sigma\}}([0,T] \times V) = \mathcal{E}^{\{\sigma,\sigma\}}([0,T] \times V)$.

(3) Let $s \geq 1$: we denote by $\mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ the set of all functions $u(t,x) \in C^\infty([0,T] \times V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{[0,T]\times K} \left| \partial_t^k \partial_x^\alpha u(t,x) \right| \leq C h^{k+|\alpha|} k!^s |\alpha|!^\sigma, \quad \forall (k,\alpha) \in \mathbb{N} \times \mathbb{N}^n.$$

This means that $u(t,x)$ is a function of the Gevrey class of order s in t and of the Gevrey class of order σ in x. We often write $\mathcal{E}^{\{\sigma\}}([0,T] \times V) = \mathcal{E}^{\{\sigma,\sigma\}}([0,T] \times V)$.

The following is clear:

$$\mathcal{E}^{\{s,\sigma\}}([0,T] \times V) \subset C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V)).$$

§2. Problem and examples

2.1. Equation and problem

From now, I will consider the following nonlinear partial differential equation

$$\text{(E)} \qquad t^{\gamma}\partial_t^m u = F\Big(t, x, \{\partial_t^j \partial_x^{\alpha} u\}_{j+|\alpha|\le L, j<m}\Big)$$

where $\gamma \geq 0$ and $L \geq m \geq 1$ are integers, and $F(t, x, \{z_{j,\alpha}\}_{j+|\alpha|\le L, j<m})$ is a suitable function in a Gevrey class.

2.1. Equation and problem

From now, I will consider the following nonlinear partial differential equation

$$\text{(E)} \qquad t^{\gamma}\partial_t^m u = F\Big(t, x, \{\partial_t^j \partial_x^\alpha u\}_{j+|\alpha|\le L, j<m}\Big)$$

where $\gamma \geq 0$ and $L \geq m \geq 1$ are integers, and $F(t, x, \{z_{j,\alpha}\}_{j+|\alpha|\le L, j<m})$ is a suitable function in a Gevrey class. And, we will consider the following problem on Gevrey regularity in time:

Problem 1.1. Let $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ be a solution of (E); can we have the result

$$u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$$

for some $s \geq 1$?

2.2. Example (1)

Let us give three examples.

Example 2.1. Let us consider the periodic KdV equation:

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = 0, \quad u(0,x) = \varphi(x) \text{ on } \mathbb{T} \tag{2.1}$$

where $\varphi(x)$ is an analytic function on the torus $\mathbb{T}$.

2.2. Example (1)

Let us give three examples.

Example 2.1. Let us consider the periodic KdV equation:

$$(2.1) \qquad \partial_t u + \partial_x^3 u + 6u\partial_x u = 0, \;\; u(0,x) = \varphi(x) \;\text{ on } \mathbb{T}$$

where $\varphi(x)$ is an analytic function on the torus $\mathbb{T}$.
The following results are known:
(1) This problem (2.1) is well-posed in $H^s(\mathbb{T})$ for $s \gg 1$.
(2) Gorsky-Himonas showed:

$$u(t,x) \in C^\infty((-\delta,\delta), \mathcal{E}^{\{1\}}(\mathbb{T})).$$

(3) Hannah-Himonas-Petronilho showed:

$$u(t,x) \in \mathcal{E}^{\{3,1\}}((-\delta,\delta) \times \mathbb{T}).$$

The proof of (3) just gives an answer to our problem in the KdV case.

2.3. Example (2)

Example 2.2. Let $a > 0$, $k \in \mathbb{N}^*$ and let us consider

$$(t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t, x). \tag{2.2}$$

2.3. Example (2)

Example 2.2. Let $a > 0$, $k \in \mathbb{N}^*$ and let us consider

$$(t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t,x). \tag{2.2}$$

The following results are known:
(1) (2.2) is uniquely solvable in $C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ for any $\sigma \geq 1$.
(2) In the case $f(t,x) \in \mathcal{E}^{\{\sigma\}}([0,T] \times \mathbb{R})$, we have the time regularity:

$$\begin{cases} u(t,x) \in \mathcal{E}^{\{\sigma,\sigma\}}([0,T] \times \mathbb{R}), & \text{if } k \geq 2, \\ u(t,x) \in \mathcal{E}^{\{2\sigma-1,\sigma\}}([0,T] \times \mathbb{R}), & \text{if } k = 1. \end{cases} \tag{2.3}$$

The result (2.3) gives an answer to our problem in the case (2.2).

Example 2.3. Recently, Kinoshita-Taglialatela (Arkiv för Matematik, 49 (2011), 109-127) discussed time regularity problem for the following hyperbolic Cauchy problem:

$$
(2.4) \qquad \begin{cases} \partial_t^2 u - a(t)\partial_x^2 u = b(t)\partial_t u + c(t)\partial_x u, \\ u(0,x) = u_0(x), \;\; \partial_t u(0,x = u_1(x). \end{cases}
$$

2.4. Example (3)

Example 2.3. Recently, Kinoshita-Taglialatela (Arkiv för Matematik, 49 (2011), 109-127) discussed time regularity problem for the following hyperbolic Cauchy problem:

$$(2.4) \qquad \begin{cases} \partial_t^2 u - a(t)\partial_x^2 u = b(t)\partial_t u + c(t)\partial_x u, \\ u(0,x) = u_0(x), \;\; \partial_t u(0, x = u_1(x). \end{cases}$$

And they showed under a suitable assumption that the problem (2.4) is well-posed in $\mathcal{E}^{\{s,\sigma\}}([0,T] \times \mathbb{R})$ for $0 \leq \sigma - 1 \leq (s-1)/s$.

2.4. Example (3)

Example 2.3. Recently, Kinoshita-Taglialatela (Arkiv för Matematik, 49 (2011), 109-127) discussed time regularity problem for the following hyperbolic Cauchy problem:

$$(2.4) \qquad \begin{cases} \partial_t^2 u - a(t)\partial_x^2 u = b(t)\partial_t u + c(t)\partial_x u, \\ u(0,x) = u_0(x), \ \ \partial_t u(0, x = u_1(x). \end{cases}$$

And they showed under a suitable assumption that the problem (2.4) is well-posed in $\mathcal{E}^{\{s,\sigma\}}([0,T] \times \mathbb{R})$ for $0 \le \sigma - 1 \le (s-1)/s$.

This result can be improved to $1 \le \sigma \le s$ by solving our problem on time regularity.

2.5. Motivation

By looking at these examples, I have come to think that the mechanism of Gevrey regularity in time is very close to that of Maillet type theorem in the book of Gérard-Tahara.

And so, by applying the argument in the proof of Maillet type theorem we will be able to solve time regularity problem.

§3. Main theorems
- sufficient condition for time regularity -

3.1. Equation and assumptions

We will consider

$$\textbf{(E)} \qquad t^{\gamma}\partial_t^m u = F\Big(t, x, Du\Big), \;\; Du = \{\partial_t^j \partial_x^{\alpha} u\}_{\substack{j+|\alpha|\leq L \\ j<m}}$$

3.1. Equation and assumptions

We will consider

$$\text{(E)} \qquad t^{\gamma}\partial_t^m u = F\Big(t, x, Du\Big), \quad Du = \{\partial_t^j \partial_x^{\alpha} u\}_{\substack{j+|\alpha|\le L \\ j<m}}$$

Let Ω be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$, and let $F(t, x, z)$ be a C^{∞} function on Ω. Let $s_1 \ge 1$, $\sigma \ge 1$ and $s_2 \ge 1$, let $T > 0$, and let V be an open subset of $\mathbb{R}^n$.

3.1. Equation and assumptions

We will consider

$$\text{(E)} \qquad t^{\gamma}\partial_t^m u = F\Big(t, x, Du\Big), \quad Du = \{\partial_t^j \partial_x^\alpha u\}_{\substack{j+|\alpha|\le L \\ j<m}}$$

Let Ω be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$, and let $F(t, x, z)$ be a C^∞ function on Ω. Let $s_1 \geq 1$, $\sigma \geq 1$ and $s_2 \geq 1$, let $T > 0$, and let V be an open subset of $\mathbb{R}^n$.
The main assumptions are as follows.

a$_1$) $\gamma \geq 0$ and $L \geq m \geq 1$ are integers.

a$_2$) $s_1 \geq 1$ and $\sigma \geq s_2 \geq 1$ are real numbers.

a$_3$) $F(t, x, z) \in \mathcal{E}^{\{s_1,\sigma,s_2\}}(\Omega)$.

a$_4$) $u(t, x) \in C^\infty([0, T], \mathcal{E}^{\{\sigma\}}(V))$ is a solution of (E).

3.2. Additional assumption

Definition 3.1. For $f(t,x) \in C^\infty([0,T] \times V)$ we define the order of the zero of $f(t,x)$ at $t=0$ by

$$ord_t(f,V) = \min\{\, k \in \mathbb{N}\,;\, (\partial_t^k f)(0,x) \not\equiv 0 \text{ on } V\}.$$

3.2. Additional assumption

Definition 3.1. For $f(t,x) \in C^\infty([0,T] \times V)$ we define the order of the zero of $f(t,x)$ at $t=0$ by

$$ord_t(f,V) = \min\{\, k \in \mathbb{N}\,;\ (\partial_t^k f)(0,x) \not\equiv 0 \text{ on } V\}.$$

Under the conditions $\mathrm{a}_1) \sim \mathrm{a}_4)$ we set

$$k_{j,\alpha} = ord_t\Big(\frac{\partial F}{\partial z_{j,\alpha}}(t,x,Du(t,x)), V\Big).$$

And we suppose

$$\mathrm{a}_5) \qquad \begin{cases} k_{j,\alpha} \geq \gamma - m + j, & \text{if } |\alpha| = 0, \\ k_{j,\alpha} \geq \gamma - m + j + 1, & \text{if } |\alpha| > 0. \end{cases}$$

3.3. Main theorem (1)

The sufficient condition for the time regularity is as follows:

Theorem 3.2 (Gevrey regularity in time). Suppose the conditions a$_1$) $\sim$ a$_5$): then, we have

$$u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T]\times V)$$

for any $s \geq \max\{s_0, s_1, s_2\}$ where

$$s_0 = 1 + \max\left[\, 0,\ \max_{|\alpha|>0}\left(\frac{j+\sigma|\alpha|-m}{\min\{k_{j,\alpha}-\gamma+m-j,\ m-j\}}\right)\right].$$

3.4. In the case of examples

Example 1. In the case of KdV equation:

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = 0, \quad u(0,x) = \varphi(x) \text{ on } \mathbb{T}$$

we have $\gamma = 0$, $m = 1$, $L = 3$ and $s_0 = 3\sigma$.

3.4. In the case of examples

Example 1. In the case of KdV equation:

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = 0, \quad u(0,x) = \varphi(x) \text{ on } \mathbb{T}$$

we have $\gamma = 0$, $m = 1$, $L = 3$ and $s_0 = 3\sigma$.

Example 2. In the case

$$(t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t,x)$$

we have $\gamma = 2$, $m = 2$, $L = 2$ and

$$s_0 = 1 + \frac{2\sigma - 2}{\min\{k, 2\}} = \begin{cases} \sigma, & \text{if } k \geq 2, \\ 2\sigma - 1, & \text{if } k = 1. \end{cases}$$

3.4. In the case of examples

Example 1. In the case of KdV equation:

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = 0, \quad u(0,x) = \varphi(x) \text{ on } \mathbb{T}$$

we have $\gamma = 0$, $m = 1$, $L = 3$ and $s_0 = 3\sigma$.

Example 2. In the case

$$(t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t,x)$$

we have $\gamma = 2$, $m = 2$, $L = 2$ and

$$s_0 = 1 + \frac{2\sigma - 2}{\min\{k, 2\}} = \begin{cases} \sigma, & \text{if } k \geq 2, \\ 2\sigma - 1, & \text{if } k = 1. \end{cases}$$

Example 3. In the case

$$\partial_t^2 u - t^k \partial_x^2 u = b(t)\partial_t u + c(t)\partial_x u$$

we have $\gamma = 0$, $m = 2$, $L = 2$ and $s_0 = \sigma$.

3.5. Formal solution

If we have a solution $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$, by the formal Taylor expansion at $t=0$ we have a formal solution

$$\hat{u}(t,x) = \sum_{k=0}^{\infty} u_k(x) t^k \in \mathcal{E}^{\{\sigma\}}(V)[[t]].$$

3.5. Formal solution

If we have a solution $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$, by the formal Taylor expansion at $t=0$ we have a formal solution

$$\hat{u}(t,x) = \sum_{k=0}^{\infty} u_k(x) t^k \in \mathcal{E}^{\{\sigma\}}(V)[[t]].$$

We will write $\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\};V)$ if the following property holds: for any compact subset K of V there are $C>0$ and $h>0$ such that

$$\max_{x \in K} |\partial_x^\alpha u_k(x)| \le C h^{k+|\alpha|} k!^{s-1} |\alpha|!^\sigma, \quad \forall (k,\alpha) \in \mathbb{N} \times \mathbb{N}^n.$$

3.5. Formal solution

If we have a solution $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$, by the formal Taylor expansion at $t=0$ we have a formal solution

$$\hat{u}(t,x) = \sum_{k=0}^{\infty} u_k(x) t^k \in \mathcal{E}^{\{\sigma\}}(V)[[t]].$$

We will write $\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\};V)$ if the following property holds: for any compact subset K of V there are $C>0$ and $h>0$ such that

$$\max_{x \in K} |\partial_x^\alpha u_k(x)| \le C h^{k+|\alpha|} k!^{s-1} |\alpha|!^\sigma, \quad \forall (k,\alpha) \in \mathbb{N} \times \mathbb{N}^n.$$

By Theorem 3.2 we have the result:
$\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\};V)$ for any $s \ge \max\{s_0, s_1, s_2\}$.

3.6. Main theorem (2)

But, in the case of formal solutions we have more:

Theorem 3.3 (Maillet type theorem). Suppose the conditions $\mathrm{a}_1) \sim \mathrm{a}_5)$: then, the formal Taylor expansion $\hat{u}(t,x)$ satisfies $\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\};V)$ for any $s \geq \max\{s_0^*, s_1, s_2\}$ with

$$s_0^* = 1 + \max\left[0, \max_{|\alpha|>0} \left(\frac{j + \sigma|\alpha| - m}{k_{j,\alpha} - \gamma + m - j} \right) \right].$$

3.6. Main theorem (2)

But, in the case of formal solutions we have more:

Theorem 3.3 (Maillet type theorem). Suppose the conditions $\mathsf{a}_1) \sim \mathsf{a}_5)$: then, the formal Taylor expansion $\hat{u}(t,x)$ satisfies $\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\};V)$ for any $s \geq \max\{s_0^*, s_1, s_2\}$ with

$$s_0^* = 1 + \max\left[\,0,\ \max_{|\alpha|>0}\left(\frac{j+\sigma|\alpha|-m}{k_{j,\alpha}-\gamma+m-j}\right)\right].$$

We note that $s_0^* \leq s_0$ holds: in general, the time regularity in the case of formal solutions is better than the case of actual solutions.

3.7. Example

Example. Let $\sigma > 1$. We consider

$$(t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t,x) \in \mathcal{E}^{\{1,\sigma\}}([0,T] \times \mathbb{R}):$$

let $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ be the unique solution and let $\hat{u}(t,x)$ be the formal Taylor expansion of $u(t,x)$ at $t = 0$. Then we have:

3.7. Example

Example. Let $\sigma > 1$. We consider

$$(t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t,x) \in \mathcal{E}^{\{1,\sigma\}}([0,T] \times \mathbb{R}) :$$

let $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ be the unique solution and let $\hat{u}(t,x)$ be the formal Taylor expansion of $u(t,x)$ at $t = 0$. Then we have:

$$(1) \qquad u(t,x) \in \begin{cases} \mathcal{E}^{\{\sigma,\sigma\}}([0,T] \times \mathbb{R}), & \text{if } k \geq 2, \\ \mathcal{E}^{\{2\sigma-1,\sigma\}}([0,T] \times \mathbb{R}), & \text{if } k = 1. \end{cases}$$

3.7. Example

Example. Let $\sigma > 1$. We consider

$$(t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t,x) \in \mathcal{E}^{\{1,\sigma\}}([0,T] \times \mathbb{R}) :$$

let $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ be the unique solution and let $\hat{u}(t,x)$ be the formal Taylor expansion of $u(t,x)$ at $t=0$. Then we have:

$$(1) \qquad u(t,x) \in \begin{cases} \mathcal{E}^{\{\sigma,\sigma\}}([0,T] \times \mathbb{R}), & \text{if } k \geq 2, \\ \mathcal{E}^{\{2\sigma-1,\sigma\}}([0,T] \times \mathbb{R}), & \text{if } k = 1. \end{cases}$$

(2) $\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\}, \mathbb{R})$ for

$$s = 1 + \frac{2\sigma - 2}{k} \begin{cases} < \sigma, & \text{if } k \geq 3, \\ = \sigma, & \text{if } k = 2, \\ = 2\sigma - 1, & \text{if } k = 1. \end{cases}$$

§4. Necessity of the condition

4.1. Fuchsian case (1)

Let us consideer the Fuchasian partial differential equation:

$$C(t\partial_t)u = F\big(t, x, \Theta u\big) \tag{4.1}$$

where $\Theta u = \{(t\partial_t)^j \partial_x^\alpha u\}_{j+|\alpha| \leq L, j<m}$ and
$C(\lambda) = \lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0$.

4.1. Fuchsian case (1)

Let us consideer the Fuchasian partial differential equation:

$$C(t\partial_t)u = F\big(t, x, \Theta u\big) \tag{4.1}$$

where $\Theta u = \{(t\partial_t)^j \partial_x^\alpha u\}_{j+|\alpha|\le L, j<m}$ and $C(\lambda) = \lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0$. We suppose:

- $\mathbf{b}_1$) $c_i \ge 0 \ \ (i = 0, 1, \ldots, m-1)$;
- $\mathbf{b}_2$) $F(t, x, z) \gg 0$ (at $(t, x, z) = (0, 0, 0)$), and
$$\liminf_{|\beta|\to\infty} \left(\frac{F^{(1,\beta,0)}(0,0,0)}{|\beta|!^\sigma} \right)^{1/|\beta|} > 0;$$
- $\mathbf{b}_3$) $u(t, x)$ is a solution satisfying $u(0, x) = 0$, and
$$\frac{\partial F}{\partial z_{j,\alpha}}(t, x, \Theta u)\Big|_{t=0} \equiv 0 \ \text{ on } V \text{ for any } (j, \alpha).$$

4.2. Fuchsian case (2)

In this case, our indices s_0 and s_0^* are written as

$$s_0 = 1 + \max\Big[\, 0,\ \max_{|\alpha|>0} \Big(\frac{j+\sigma|\alpha|-m}{\min\{q_{j,\alpha},\ m-j\}} \Big) \Big],$$

$$s_0^* = 1 + \max\Big[\, 0,\ \max_{|\alpha|>0} \Big(\frac{j+\sigma|\alpha|-m}{q_{j,\alpha}} \Big) \Big]$$

with $q_{j,\alpha} = ord_t((\partial F/\partial z_{j,\alpha})(t,x,\Theta u(t,x)),V)$.

4.2. Fuchsian case (2)

In this case, our indices s_0 and s_0^* are written as

$$s_0 = 1 + \max\Big[\,0,\ \max_{|\alpha|>0}\Big(\frac{j+\sigma|\alpha|-m}{\min\{q_{j,\alpha},\ m-j\}}\Big)\,\Big],$$

$$s_0^* = 1 + \max\Big[\,0,\ \max_{|\alpha|>0}\Big(\frac{j+\sigma|\alpha|-m}{q_{j,\alpha}}\Big)\,\Big]$$

with $q_{j,\alpha} = ord_t((\partial F/\partial z_{j,\alpha})(t,x,\Theta u(t,x)),V)$.
Then we have the expression

$$\frac{\partial F}{\partial z_{j,\alpha}}(t,x,\Theta u(t,x)) = a_{j,\alpha}(x)t^{q_{j,\alpha}} + O(t^{q_{j,\alpha}+1})$$

for some $a_{j,\alpha}(x) \gg 0$ (at $x=0$). We set

$$\Lambda(+) = \{(j,\alpha) \in \Lambda\,;\ a_{j,\alpha}(0) > 0, |\alpha| > 0\}.$$

4.3. Necessity of the condition in Fuchsian case

Then, we have the necessity of the condition:

Theorem 4.1 (Fuchsian case). If $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ or $\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\}; V)$ holds for some $s \geq 1$, we have

$$s \geq 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda(+)} \Big(\frac{j+\sigma|\alpha|-m}{q_{j,\alpha}}\Big)\Big].$$

4.3. Necessity of the condition in Fuchsian case

Then, we have the necessity of the condition:

Theorem 4.1 (Fuchsian case). If $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ or $\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\};V)$ holds for some $s \geq 1$, we have

$$s \geq 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda(+)} \Big(\frac{j+\sigma|\alpha|-m}{q_{j,\alpha}}\Big)\Big].$$

Recall that the sufficient condition is $s \geq s_0$ or $s \geq s_0^*$ with

$$s_0 = 1 + \max\Big[\,0,\ \max_{|\alpha|>0}\Big(\frac{j+\sigma|\alpha|-m}{\min\{q_{j,\alpha},\ m-j\}}\Big)\,\Big],$$

$$s_0^* = 1 + \max\Big[\,0,\ \max_{|\alpha|>0}\Big(\frac{j+\sigma|\alpha|-m}{q_{j,\alpha}}\Big)\,\Big].$$

4.4. Non-singular case (1)

Let us consider the initial value problem

$$(4.2)\quad \begin{cases} \partial_t^m u = F(t,x,Du) \ \text{ on } [0,T]\times V, \\ \partial_t^i u\big|_{t=0} = \varphi_i(x) \ \text{ on } V, \quad i=0,1,\ldots,m-1, \end{cases}$$

where $Du = \{\partial_t^j \partial_x^\alpha u\}_{j+|\alpha|\le L, j<m}$.

4.4. Non-singular case (1)

Let us consider the initial value problem

$$(4.2) \quad \begin{cases} \partial_t^m u = F(t,x,Du) \ \text{ on } [0,T] \times V, \\ \partial_t^i u\big|_{t=0} = \varphi_i(x) \ \text{ on } V, \quad i = 0,1,\ldots,m-1, \end{cases}$$

where $Du = \{\partial_t^j \partial_x^\alpha u\}_{j+|\alpha| \le L, j<m}$.
In this case, our indices s_0 and s_0^* are written as

$$s_0 = 1 + \max\Big[\, 0,\ \max_{|\alpha|>0}\Big(\frac{j+\sigma|\alpha|-m}{m-j}\Big)\Big],$$

$$s_0^* = 1 + \max\Big[\, 0,\ \max_{|\alpha|>0}\Big(\frac{j+\sigma|\alpha|-m}{k_{j,\alpha}+m-j}\Big)\Big].$$

where $k_{j,\alpha} = ord_t((\partial F/\partial z_{j,\alpha})(t,x,Du(t,x)),V)$.

4.4. Non-singular case (2)

We set $\Lambda = \{(j,\alpha)\,;\, j+|\alpha| \le L, j<m\}$,

$$\varphi_m(x) = F(0,x,\{\varphi_j^{(\alpha)}(x)\}_{(j,\alpha)\in\Lambda}), \text{ and}$$

$$a(x) = \frac{\partial F}{\partial t}\Big(0,x,\{\varphi_j^{(\alpha)}(x)\}_{(j,\alpha)\in\Lambda}\Big) + \sum_{(j,\alpha)\in\Lambda} \frac{\partial F}{\partial z_{j,\alpha}}\Big(0,x,\{\varphi_j^{(\alpha)}(x)\}_{(j,\alpha)\in\Lambda}\Big)\varphi_{j+1}^{(\alpha)}(x).$$

We assume:

c$_1$) $F(t,x,z) \gg 0$ (at $(t,x,z)=(0,0,p)$);

c$_2$) $\varphi_i(x) \gg 0$ (at $x=0$), $i=0,1,\ldots,m-1$;

c$_3$) $\liminf_{|\beta|\to\infty}\big(a^{(\beta)}(0)/|\beta|!^{\sigma}\big)^{1/|\beta|} > 0.$

4.6. Necessity of the condition in non-singular case

Let $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ be a solution of (4.2). We set $k_{j,\alpha} = ord_t((\partial F/\partial z_{j,\alpha})(t,x,Du(t,x)),V)$: then

$$\frac{\partial F}{\partial z_{j,\alpha}}(t,x,Du(t,x)) = a_{j,\alpha}(x)t^{k_{j,\alpha}} + O(t^{k_{j,\alpha}+1})$$

for some $a_{j,\alpha}(x) \gg 0$ (at $x=0$). We set

$$\Lambda(+) = \{(j,\alpha) \in \Lambda\,;\, a_{j,\alpha}(0) > 0, |\alpha| > 0\}.$$

Theorem 4.2 (Non-singular case). If $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ **or** $\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\};V)$ **holds for some** $s \geq 1$**, we have**

$$s \geq 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda(+)}\left(\frac{j+\sigma|\alpha|-m}{k_{j,\alpha}+m-j}\right)\right].$$

Theorem 4.2 (Non-singular case). If $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ or $\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\}; V)$ holds for some $s \geq 1$, we have

$$s \geq 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda(+)}\Big(\frac{j+\sigma|\alpha|-m}{k_{j,\alpha}+m-j}\Big)\right].$$

Recall that the sufficient condition is $s \geq s_0$ or $s \geq s_0^*$ with

$$s_0 = 1 + \max\Big[\,0,\ \max_{|\alpha|>0}\Big(\frac{j+\sigma|\alpha|-m}{m-j}\Big)\,\Big],$$

$$s_0^* = 1 + \max\Big[\,0,\ \max_{|\alpha|>0}\Big(\frac{j+\sigma|\alpha|-m}{k_{j,\alpha}+m-j}\Big)\,\Big].$$

where $k_{j,\alpha} = ord_t((\partial F/\partial z_{j,\alpha})(t,x,Du(t,x)), V)$.