Introduction to middle convolution for differential equations with irregular singularities

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References

Middle convolution and Heun's equation, *SIGMA*, 2009, 040

Introduction to middle convolution for differential equations with irregular singularities, arXiv:1002.2535

Middle convolution for systems of linear differential equations with irregular singularities, in preparation.

We study middle convolution for systems of linear differential equations with irregular singular points;

$$\frac{dY}{dz} = \left(-\sum_{j=1}^{m_0} A_j^{(0)} z^{j-1} + \sum_{i=1}^{r} \sum_{j=0}^{m_i} \frac{A_j^{(i)}}{(z-t_i)^{j+1}} \right) Y,\ Y \in \mathbb{C}^n$$

Middle convolution =
Euler's integral transformation $\int_C f(w)(z-w)^{\mu} dw$
+ trasformation of vector spaces and matrices.

First we consider systems of Fuchsian differential equations;

$$\frac{dY}{dz} = \left(\frac{A_1}{z-t_1} + \frac{A_2}{z-t_2} + \cdots + \frac{A_r}{z-t_r} \right) Y.$$

1 Deligne-Simpson problem (DSP)

$C_0, \ldots, C_r$: conjugacy classes of $n \times n$ matrix.
$C_i = \{P_i^{-1} D_i P_i \mid P_i \in GL(n)\}$, D_i : Jordan normal forms

Additive Deligne-Simpson problem (aDSP):

Find a condition for existence of (or solution for) $A_0, \ldots, A_r$ s.t. irreducible and

$$A_0 + A_1 + \cdots + A_r = 0, \; A_i \in C_i.$$

System of Fuchsian differential equations

$$\frac{dY}{dz} = \left(\frac{A_1}{z - t_1} + \frac{A_2}{z - t_2} + \cdots + \frac{A_r}{z - t_r} \right) Y. \tag{1}$$

The residue matrix about $z = t_i$: A_i.
The residue matrix about $z = \infty$: $A_0 = -(A_1 + \cdots + A_r)$.

Symbol $(\mathbf{m}, \underline{\lambda})$

$(\mathbf{m}_i, \underline{\lambda}_i) \Leftrightarrow$ **Conjugacy class** C_i (Jordan normal form)
$\underline{\lambda}_i = (\lambda_{i,1}, \ldots, \lambda_{i,n_i}) \in \mathbb{C}^{n_i}$: eigenvalues
$\mathbf{m}_i = (m_{i,1}, \ldots, m_{i,n_i}) \in (\mathbb{Z}_{\geq 1})^{n_i}$: multiplicities
$(m_{i,1} + \cdots + m_{i,n_i} = n,\ m_{i,1} \geq \cdots \geq m_{i,n_i})$,

If $\lambda_{i,1}, \ldots, \lambda_{i,n_i}$ are mutually distinct, then

$$C_i = \begin{pmatrix} \lambda_{i,1} I_{m_{i,1}} & & & \\ & \lambda_{i,2} I_{m_{i,2}} & & \\ & & \ddots & \\ & & & \lambda_{i,n_i} I_{m_{i,n_i}} \end{pmatrix} \Leftrightarrow (\mathbf{m}_i, \underline{\lambda}_i),$$

Conjugacy classse $(C_1, \ldots C_r, C_0) \Leftrightarrow (\mathbf{m}, \underline{\lambda})$
$\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_r, \mathbf{m}_0),\ \underline{\lambda} = (\underline{\lambda}_1, \ldots, \underline{\lambda}_r, \underline{\lambda}_0)$.

Index of rigidity

$\mathbf{A} = (A_0, A_1, \ldots, A_r)$: $n \times n$ matrices.

Define

$$\mathrm{idx}(\mathbf{A}) = \sum_{i=0}^{r} \dim Z(A_i) - (r-1)n^2,$$

$Z(A_i) = \{X \in \mathbb{C}^{n\times n} \mid A_i X = X A_i\}$,

$\dim Z(A_i) = \sum_{j=1}^{n_i} (m_{i,j})^2, \quad (A_i \sim (\mathbf{m}_i, \underline{\lambda}_i))$.

$\mathbf{A}$: irred. $\Rightarrow$ $\mathrm{idx}(\mathbf{A}) \le 2$, even.

- $\mathrm{idx}(\mathbf{m})$ is defined by $\mathrm{idx}(\mathbf{A})$.

If aDSP has a solution, then

the number of accessory parameters $= 2-$ (index of rigidity).

Addition

Addition w.r.t. the parameter $(\mu_1, \ldots, \mu_r) \in \mathbb{C}^r$:

$$(A_1, \ldots, A_r) \ \Rightarrow \ (A_1 + \mu_1 I_n, \ldots, A_r + \mu_r I_n)$$

$$A_0 \Rightarrow A_0 - (\mu_1 + \cdots + \mu_r) I_n.$$

On Fuchsian system (1), it corresponds to

$$Y \mapsto (z - t_1)^{\mu_1} \ldots (z - t_r)^{\mu_r} Y.$$

Index of rigidity is preserved by addition.

2 Middle convolution

Middle convolution was introduced by Katz (1996) in "Rigid local systems".

We explain the version for Fuchsian differential systems given by Dettweiler and Reiter (2000,2007).

Given data: $n, r \in \mathbb{Z}_{\geq 1}$, A_1, A_2, $\ldots$, A_r: $n \times n$ matrices.

Fuchsian differential system

$$\frac{dY}{dz} = \left(\frac{A_1}{z-t_1} + \frac{A_2}{z-t_2} + \cdots + \frac{A_r}{z-t_r}\right) Y.$$

$\nu \in \mathbb{C}$, **convolution matrices** $B_1, B_2, \ldots, B_r \in \mathbb{C}^{nr \times nr}$

$$B_1 = \begin{pmatrix} A_1 + \nu I_n & A_2 & \ldots & A_r \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ A_1 & A_2 + \nu I_n & \ldots & A_r \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

$$\ldots\ldots \qquad , \quad B_r = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ A_1 & A_2 & \ldots & A_r + \nu I_n \end{pmatrix},$$

$$\frac{dU}{dz} = \left(\frac{B_1}{z - t_1} + \frac{B_2}{z - t_2} + \cdots + \frac{B_r}{z - t_r} \right) U.$$

Proposition 1. *[Dettweiler & Reiter]*
Let Y be a solution of

$$\frac{dY}{dz} = \left(\frac{A_1}{z - t_1} + \frac{A_2}{z - t_2} + \cdots + \frac{A_r}{z - t_r} \right) Y \quad (Y : \text{size } n).$$

Then the function U of size nr defined by

$$U = \begin{pmatrix} U_1(w) \\ \vdots \\ U_r(w) \end{pmatrix}, \quad U_j(w) = \int_C \frac{Y(w)}{(w - t_j)} (z - w)^{\nu} dw$$

is a solution of $\dfrac{dU}{dz} = \left(\dfrac{B_1}{z - t_1} + \dfrac{B_2}{z - t_2} + \cdots + \dfrac{B_r}{z - t_r} \right) U.$

C: an appropriate contour
(e.g. Pochhammer contour $[\alpha_z, \alpha_{t_i}]$ around z and t_i in w-plane).

We are going to pick up an irreducible part by a quotient.

$$\mathcal{K}_1 = \begin{pmatrix} \mathrm{Ker}(A_1) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathcal{K}_r = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{Ker}(A_r) \end{pmatrix},$$

$$\mathcal{K} = \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_r, \quad \mathcal{L}(\nu) = \mathrm{Ker}(B_1) \cap \cdots \cap \mathrm{Ker}(B_r).$$

We denote matrices B_k on $\mathbb{C}^{nr}/(\mathcal{K} + \mathcal{L}(\nu)) \simeq \mathbb{C}^m$ by $\tilde{B}_k$.

Middle convolution

$$mc_\nu(A_1, A_2, \ldots, A_r) = (\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_r)$$

Proposition 2. *[DR] Assume that* $\mathbf{A} = (A_1, \ldots, A_r)$ *is irred.*
(i) $mc_\nu(\mathbf{A})$*: irred. and* $\mathrm{idx}(mc_\nu(\mathbf{A})) = \mathrm{idx}(\mathbf{A})$*, i.e.*
the index of rigidity is preserved by middle convolution.
(ii) $mc_{\nu+\mu}(\mathbf{A}) = mc_\nu(mc_\mu(\mathbf{A}))$ *and* $mc_0(\mathbf{A}) = \mathbf{A}$*.*
In particular $mc_{-\nu} \circ mc_\nu = \mathrm{id}$*.*

3 Fuchsian differential system of size 2×2

Three singularities $\{0, 1, \infty\}$

We use A_∞, A_0, A_1: 2×2 matrices instead of A_0, A_1, A_2.

Assumption

$$\text{The eigenvalues of } A_0\text{: } 0,\ \theta_0$$
$$\text{The eigenvalues of } A_1\text{: } 0,\ \theta_1$$
$$A_\infty = -(A_0 + A_1) = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

- aDSP for $\mathbf{m} = (1, 1;\ 1, 1;\ 1, 1)$, $\underline{\lambda} = (0, \theta_0;\ 0, \theta_1;\ \kappa_1, \kappa_2)$

Set

$$A_0 = \begin{pmatrix} u_0 + \theta_0 & -w_0 \\ u_0(u_0+\theta_0)/w_0 & -u_0 \end{pmatrix}, \ A_1 = \begin{pmatrix} u_1 + \theta_1 & -w_1 \\ u_1(u_1+\theta_1)/w_1 & -u_1 \end{pmatrix},$$

If $\theta_0+\theta_1+\kappa_1+\kappa_2=0(\Leftrightarrow \sum \mathrm{tr} A_i = 0)$, then we have solutions:

$$\begin{matrix} w_0 = k \\ w_1 = -k \end{matrix}, \ u_0 = \frac{\kappa_2(\kappa_2+\theta_1)}{2\kappa_2+\theta_0+\theta_1} = \frac{\kappa_2(\kappa_2+\theta_1)}{\kappa_2-\kappa_1}, \ u_1 = \frac{\kappa_2(\kappa_2+\theta_0)}{\kappa_2-\kappa_1},$$

k is absorbed by diagonal conjugation.

Hence aDSP has a unique solution. **(Rigid)**

On this case,
$\mathrm{idx}(\mathbf{A}) = \dim Z(A_\infty) + \dim Z(A_0) + \dim Z(A_1) - (2-1)2^2 = 2.$

2×2 Fuchsian system with singularities $\{0, 1, \infty\}$

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1}\right) Y, \quad Y = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}. \tag{2}$$

$y = y_1(z)$ satisfies Gauss hypergeometric differential equation

$$z(1-z)y'' + (\gamma - (\alpha+\beta+1)z)y' - \alpha\beta y = 0,$$

$$\gamma = 1 - \theta_0, \ \{\alpha, \beta\} = \{\kappa_1, \kappa_2 + 1\}.$$

• **aDSP for** A_0**,** A_1**,** $A_\infty \in \mathbb{C}^{2\times 2}$
$\Leftrightarrow$ **Gauss hypergeometric differential equation; rigid**

Other examples of rigid differential equations (idx$= 2$**)**
Jordan-Pochhammer differential equation,
Generalized hypergeometric differential equation.

Middle convolution for 2×2 Fuchsian system with three singularities

The convolution matrices are 4×4

$$B_0 = \begin{pmatrix} A_0 + \nu I_2 & A_1 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ A_0 & A_1 + \nu I_2 \end{pmatrix}.$$

Since A_0 and A_1 have 0-eigenvalues,

$$\dim \mathcal{K}_0 = \dim \mathcal{K}_1 = 1.$$

$$\nu \neq 0, \kappa_1, \kappa_2 \Rightarrow \mathcal{L}(\nu) = \{0\},$$

$$\nu = \kappa_1, \kappa_2 \Rightarrow \dim \mathcal{L}(\nu) = 1.$$

We only consider the case $\nu = \kappa_1$.

The size of $mc_{\kappa_1}(\mathbf{A})$ is 1.

The differential equation after applying mc_{κ_1} is

$$\frac{d\tilde{y}(z)}{dz} = \left(\frac{\theta_0 + \kappa_1}{z} + \frac{\theta_1 + \kappa_1}{z-1}\right)\tilde{y}(z), \tag{3}$$

Solutions are written as

$$\tilde{y}(z) = C z^{\theta_0+\kappa_1}(z-1)^{\theta_1+\kappa_1}$$

Thinking of $mc_{-\kappa_1} mc_{\kappa_1} = \mathrm{id}$, we apply $mc_{-\kappa_1}$ to Eq.(3).

$$\frac{dW}{dz} = \left(\frac{1}{z}\begin{pmatrix} \theta_0 & \theta_1+\kappa_1 \\ 0 & 0 \end{pmatrix} + \frac{1}{z-1}\begin{pmatrix} 0 & 0 \\ \theta_0+\kappa_1 & \theta_1 \end{pmatrix}\right) W,$$

The integral representation of solution given by Proposition 1 is

$$W = \begin{pmatrix} \int_C \frac{1}{w} w^{\theta_0+\kappa_1}(w-1)^{\theta_1+\kappa_1}(z-w)^{-\kappa_1}dw \\ \int_C \frac{1}{w-1} w^{\theta_0+\kappa_1}(w-1)^{\theta_1+\kappa_1}(z-w)^{-\kappa_1}dw \end{pmatrix}.$$

By diagonalizing the matrix about $z=\infty$, we have

$$\tilde{W} = \begin{pmatrix} \frac{\kappa_2+\theta_0}{\kappa_1-\kappa_2} & -\frac{k}{\kappa_2} \\ -\frac{\kappa_2+\theta_1}{\kappa_1-\kappa_2} & -\frac{k}{\kappa_2} \end{pmatrix}^{-1} W, \quad \frac{d\tilde{W}}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1}\right)\tilde{W}$$

and we recover Eq.(2).

It follows from taking the first component of the integral representation of $\tilde{W}$ that the functions

$$y(z) = \int_C w^{\alpha-\gamma}(w-1)^{\gamma-\beta-1}(z-w)^{-\alpha}dw$$

are solutions of Gauss hypergeometric differential equation.

$$2\times 2 \underset{mc}{\to} \underset{\text{solution}}{1\times 1} \underset{mc^{-1}}{\to} 2\times 2, \quad \text{integral representation}$$

$2{\times}2$ system with four singularities $\{0,1,t,\infty\}$

Assumption

$$\begin{gathered}\text{The eigenvalues of } A_0\text{: } 0,\ \theta_0\\ \text{The eigenvalues of } A_1\text{: } 0,\ \theta_1\\ \text{The eigenvalues of } A_t\text{: } 0,\ \theta_t\end{gathered}$$

$$A_\infty = -(A_0 + A_1 + A_t) = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \quad \kappa_1 - \kappa_2 = \theta_\infty,$$

- aDSP for $\mathbf{m} = (1,1;1,1;1,1;1,1)$, $\underline{\lambda} = (0,\theta_0;0,\theta_1;0,\theta_t;\kappa_1,\kappa_2)$.

idx($\mathbf{m}$) $= 0 \Rightarrow$ # of accessory parameters $= 2$.
Accessory parameters λ, μ.

The elements of A_0, A_1 and A_t are determined uniquely by fixing $\lambda(\notin \{0,1,t,\infty\})$, μ, k.

We denote Fuchsian differential system

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}\right) Y$$

by $D_Y(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu; k)$.
Painlevé VI is obtained by monodromy preserving deformation.

By eliminating $y_2(z)$, we have

$$\begin{aligned}
\frac{d^2 y_1(z)}{dz^2} &+ \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t} - \frac{1}{z-\lambda}\right) \frac{dy_1(z)}{dz} \\
&\quad + \left(\frac{\kappa_1(\kappa_2+1)}{z(z-1)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)} - \frac{t(t-1)H}{z(z-1)(z-t)}\right) y_1(z) = 0, \\
H &= \frac{1}{t(t-1)}[\lambda(\lambda-1)(\lambda-t)\mu^2 - \{\theta_0(\lambda-1)(\lambda-t) + \theta_1\lambda(\lambda-t) \\
&\qquad + (\theta_t - 1)\lambda(\lambda-1)\}\mu + \kappa_1(\kappa_2+1)(\lambda-t)],
\end{aligned}$$

which we denote by $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$.

$z = \lambda$: apparent singularity with exponents $0, 2$.

By suitable limits $\lambda \to 0, 1, t, \infty$ from

$$\frac{d^2y_1(z)}{dz^2} + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t} - \frac{1}{z-\lambda}\right)\frac{dy_1(z)}{dz} + \left(\frac{\kappa_1(\kappa_2+1)}{z(z-1)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)} - \frac{t(t-1)H}{z(z-1)(z-t)}\right)y_1(z) = 0,$$

we obtain **Heun's differential equation**:

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)}y = 0,$$

$(\gamma + \delta + \epsilon = \alpha + \beta + 1)$.

To obtain precise statement, the space of initial conditions for Painlevé VI appears naturally (T. *SIGMA* 2009).

Middle convolution for 2×2 Fuchsian system with four singularities

We consider middle convolution for $D_Y(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu; k)$

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} \right) Y, \quad Y = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}.$$

The convolution matrices are 6×6.

The rank of Fuchsian differential system of $mc_\nu(\mathbf{A})$:

$$\nu \neq 0, \kappa_1, \kappa_2 \Rightarrow \text{rank} = 3, \quad \nu = 0 \Rightarrow mc_0 = id.$$

$$\nu = 0, \kappa_1, \kappa_2 \Rightarrow \text{rank} = 2,$$

If $\nu = \kappa_1$, then $mc_{\kappa_1}(\mathbf{A})$ was calculated by Filipuk (*Kumamoto J. Math.* 2006) and a relationship to Bäcklund transformation of Painlevé VI was studied.
We can also calculate the integral transformation explicitly (c.f. T. *JMAA* 2008)).

For the case $\nu = \kappa_2$, we have

Theorem 3. *[T. SIGMA 2009] Set*

$$\tilde{\lambda} = \lambda - \frac{\kappa_2}{\mu - \frac{\theta_0}{\lambda} - \frac{\theta_1}{\lambda - 1} - \frac{\theta_t}{\lambda - t}}, \qquad \tilde{\mu} = \frac{\kappa_2 + \theta_0}{\tilde{\lambda}} + \frac{\kappa_2 + \theta_1}{\tilde{\lambda} - 1} + \frac{\kappa_2 + \theta_t}{\tilde{\lambda} - t} + \frac{\kappa_2}{\lambda - \tilde{\lambda}}.$$

If $Y = {}^t(y_1(z), y_2(z))$ *satisfies* $D_Y(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu; k)$, *then the function* $\tilde{Y} = {}^t(\tilde{y}_1(z), \tilde{y}_2(z))$ *defined by*

$$\tilde{y}_1(z) = \int_C \frac{dy_1(w)}{dw} (z - w)^{\kappa_2} dw,$$

$$\tilde{y}_2(z) = \frac{\kappa_2 \lambda (\lambda - 1)(\lambda - t)}{k(\lambda - \tilde{\lambda})} \int_C \left\{ \frac{\frac{dy_1(w)}{dw} - \mu y_1(w)}{\lambda - w} + \frac{\mu}{\kappa_1} \frac{dy_1(w)}{dw} \right\} (z - w)^{\kappa_2} dw.$$

satisfy $D_Y(\kappa_2 + \theta_0, \kappa_2 + \theta_1, \kappa_2 + \theta_t, \kappa_2 + \theta_\infty; \tilde{\lambda}, \tilde{\mu}; k)$.

Corollary 4. *[Novikov 2007] If* $y_1(z)$ *satisfies* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$, *then*

$$\tilde{y}(z) = \int_C y_1(w)(z - w)^{\kappa_2 - 1} dw,$$

satisfies $D_{y_1}(\kappa_2 + \theta_0, \kappa_2 + \theta_1, \kappa_2 + \theta_t, \kappa_2 + \theta_\infty; \tilde{\lambda}, \tilde{\mu})$.

By taking a suitable limit of λ and μ, we have integral transformation of Heun's equation, which was essentially obtained by Kazakov and Slavyanov (1996) by another method.

Theorem 5. *Set*

$$\mu=\alpha,\ \gamma'=\gamma+1-\alpha,\ \delta'=\delta+1-\alpha,\ \epsilon'=\epsilon+1-\alpha,$$
$$\alpha'=2-\alpha,\ \beta'=-\alpha+\beta+1,\ q'=q+(1-\alpha)(\epsilon+\delta t+(\gamma-\alpha)(t+1)).$$

Let $v(w)$ be a solution of Heun's differential equation

$$\frac{d^2v}{dw^2}+\left(\frac{\gamma'}{w}+\frac{\delta'}{w-1}+\frac{\epsilon'}{w-t}\right)\frac{dv}{dw}+\frac{\alpha'\beta'w-q'}{w(w-1)(w-t)}v=0.$$

Then the function $y(z)=\int_C v(w)(z-w)^{-\mu}dw$ satisfies

$$\frac{d^2y}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-t}\right)\frac{dy}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-t)}y=0.$$

Integral transformation: $\alpha', \beta', \gamma', \delta', \epsilon', q' \Rightarrow \alpha, \beta, \gamma, \delta, \epsilon, q$.

We can obtain new solutions by using known solutions of different parameters.

• Finite-gap solutions ($\gamma', \delta', \epsilon', \beta' - \alpha' \in \mathbb{Z} + \frac{1}{2}$, q': general)
$\Rightarrow$ The case $\gamma, \delta, \epsilon, \alpha + 1/2, \beta + 1/2 \in \mathbb{Z}$, q: general.

• Polynomial-type solutions $\Rightarrow$
The case that one of the singularities $\{0, 1, t, \infty\}$ is apparent (non-logarithmic). (T., arXiv:1008.4007)

4 Simplification by middle convolution

$(A_0, A_1, \ldots, A_r)$: given.

To simplify the system of diff'l equations, we want to decrease the size of matrices by addition and middle convolution.

By addition, we adjust the dimensions of $\mathcal{K}_i (\simeq \mathrm{Ker} A_i)$ $(i = 1, \ldots, r)$ to be maximum. We choose ν s.t. the rank of $A_0 - \nu I$ is minimum, and we apply middle convolution mc_ν.

In some cases we cannot decrease the size. (terminal cases)

Theorem 6. *[Oshima (c.f. Kostov)]*
Assume $(A_0, A_1, \ldots, A_r)$ *is irreducible.*

By applying addition and middle convolution repeatedly, we can reduce to the terminal pattern $\mathbf{m}$ *described as follows:*

(1) $idx = 2 \Rightarrow \mathbf{m} = (1)$.

(2) $idx = 0 \Rightarrow \exists d \in \mathbb{Z}_{\geq 1}$

$$\mathbf{m} = \begin{cases} (d,d;\ d,d;\ d,d;\ d,d) & D_4^{(1)} \text{ case, } (r=3) \\ (d,d,d;\ d,d,d;\ d,d,d) & E_6^{(1)} \text{ case, } (r=2) \\ (2d,2d;\ d,d,d,d;\ d,d,d,d) & E_7^{(1)} \text{ case, } (r=2) \\ (3d,3d;\ 2d,2d,2d;\ d,d,d,d,d,d) & E_8^{(1)} \text{ case, } (r=2) \end{cases}$$

(3) If $idx < 0$*, then the number of the terminal patterns* $\mathbf{m}$ *is finite for each idx. If* $idx = -2$*, then we have 13 patterns.*

Note that Theorem 6 does not assert solvability of DSP.

Crawley-Boevey solved aDSP by using data of a root system determined by $(\mathbf{m}, \underline{\lambda})$.

$\mathrm{idx} = 0 \Rightarrow d = 1$ is the condition for solvability (irreducibility) of aDSP.

$D_4^{(1)}$: 2×2, sing. $\{0, 1, t, \infty\}$, $E_6^{(1)}$: 3×3, sing. $\{0, 1, \infty\}$

5 Middle convolution for linear differential system with irregular singularities

We consider middle convolution for

$$\frac{dY}{dz} = \left(-\sum_{j=1}^{m_0} A_j^{(0)} z^{j-1} + \sum_{i=1}^{r} \sum_{j=0}^{m_i} \frac{A_j^{(i)}}{(z - t_i)^{j+1}} \right) Y, \; Y \in V = \mathbb{C}^n$$

$m_i = 0 \Rightarrow z = t_i$ is regular singularity.

Kawakami (*IMRN* 2010) considered it for the case $m_1, \dots, m_r$: arbitrary, $m_0 = 0$ by using (generalized) Okubo normal form.

Yamakawa (*Math. Ann.* 2011) studied it for the case $m_0 \leq 1$ by applying symplectic geometry (Harnad duality).

Boalch, Hiroe, Oshima . . .

In this talk, we introduce middle convolution including the case $m_0 \geq 2$ directly. (T. arXiv:1002.2535)

Convolution

Write $\mathbf{A} = (A_{m_0}^{(0)}, \dots, A_1^{(0)}, A_{m_1}^{(1)}, \dots, A_0^{(r)})$, $(A_j^{(i)} \in \mathrm{End}(V))$.
Set $V' = V^{\oplus M} = \mathbb{C}^{nM}$, $M = r + \sum_{i=0}^{r} m_i$.
We define $c_\mu(\mathbf{A}) = \tilde{\mathbf{A}} = (\tilde{A}_{m_0}^{(0)}, \dots, \tilde{A}_1^{(0)}, \tilde{A}_{m_1}^{(1)}, \dots, \tilde{A}_0^{(r)})$,
$(\mu \in \mathbb{C}$, $\tilde{A}_j^{(i)} \in \mathrm{End}(V') = \mathbb{C}^{nM \times nM})$ as follows:

$$\tilde{A}_{m_0}^{(0)} = \begin{pmatrix} A_{m_0}^{(0)} & \cdots & A_2^{(0)} & A_1^{(0)} & A_{m_1}^{(1)} & \cdots & A_0^{(1)} & A_{m_2}^{(2)} & \cdots\cdots & A_0^{(r)} \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \end{pmatrix},$$

$$\tilde{A}_j^{(0)} = \begin{pmatrix} \left.\begin{matrix} \mu I & & \\ & \ddots & \\ & & \mu I \end{matrix}\right\}_{m_0-j} & & & & & & & \\ A_{m_0}^{(0)} \; \cdots \; A_2^{(0)} \; A_1^{(0)} & A_{m_1}^{(1)} & \cdots & A_0^{(1)} & A_{m_2}^{(2)} & \cdots\cdots & A_0^{(r)} \\ & & & & & & \end{pmatrix},$$

$$\tilde{A}^{(i)}_{m_i} = \begin{pmatrix} & & & & \left.\begin{matrix} \\ \\ \end{matrix}\right\} {\scriptstyle m_0+(m_1+1)+\cdots+(m_{i-1}+1)} & & & \\ A^{(0)}_{m_0} & \cdots & A^{(i)}_{m_i} & \cdots & A^{(i)}_1 & A^{(i)}_0 + \mu I & A^{(i+1)}_{m_{i+1}} & \cdots & A^{(r)}_0 \\ & & & & & & & & \\ & & & & & & & & \end{pmatrix},$$

$$\tilde{A}^{(i)}_{j} = \begin{pmatrix} & & & & & \left.\begin{matrix} \\ \\ \end{matrix}\right\} {\scriptstyle m_0+(m_1+1)+\cdots+(m_{i-1}+1)} & & & \\ & & \mu I & & & & & & \\ & & & \ddots & & \left.\begin{matrix} \\ \\ \end{matrix}\right\} {\scriptstyle m_i-j} & & & \\ & & & & \mu I & & & & \\ A^{(0)}_{m_0} & \cdots & A^{(i)}_{m_i} & \cdots & A^{(i)}_1 & A^{(i)}_0 + \mu I & A^{(i+1)}_{m_{i+1}} & \cdots & A^{(r)}_0 \\ & & & & & & & & \\ & & & & & & & & \end{pmatrix}.$$

Euler's integral tranformation

Convolution of matrices corresponds to Euler's integral tranformation for solutions of lienar differential system.

Proposition 7. *Assume that* $Y = \begin{pmatrix} y_1(z) \\ \vdots \\ y_n(z) \end{pmatrix}$ *satisfies*

$$\frac{dY}{dz} = \left(- \sum_{j=1}^{m_0} A_j^{(0)} z^{j-1} + \sum_{i=1}^{r} \sum_{j=0}^{m_i} \frac{A_j^{(i)}}{(z - t_i)^{j+1}} \right) Y.$$

The function U defined by

$$U = \begin{pmatrix} U^{(0)}_{m_0}(z) \\ \vdots \\ U^{(0)}_1(z) \\ U^{(1)}_{m_1}(z) \\ \vdots \\ U^{(r)}_0(z) \end{pmatrix}, \qquad \underset{(j=1,\dots m_0)}{U^{(0)}_j(z)} = -\begin{pmatrix} \int_\gamma w^{j-1} y_1(w)(z-w)^\mu dw \\ \vdots \\ \int_\gamma w^{j-1} y_n(w)(z-w)^\mu dw \end{pmatrix}, \qquad \underset{(i\neq 0)}{U^{(i)}_j(z)} = \begin{pmatrix} \int_\gamma \frac{y_1(w)}{(w-t_i)^{j+1}}(z-w)^\mu dw \\ \vdots \\ \int_\gamma \frac{y_n(w)}{(w-t_i)^{j+1}}(z-w)^\mu dw \end{pmatrix}$$

satisfies

$$\frac{dU}{dz} = \left(-\sum_{j=1}^{m_0} \tilde{A}^{(0)}_j x^{j-1} + \sum_{i=1}^{r}\sum_{j=0}^{m_i} \frac{\tilde{A}^{(i)}_j}{(x-t_i)^{j+1}} \right) U.$$

for appropriate contours γ.

Subspaces on convolution

We define subspaces of $V' = V^{\oplus M} = \mathbb{C}^{nM}$ as follows:

$$\underset{(i=1,\dots,r)}{\mathcal{K}^{(i)}} = \left\{ \begin{pmatrix} \vdots \\ 0 \\ v^{(i)}_{m_i} \\ v^{(i)}_{m_i-1} \\ \vdots \\ v^{(i)}_0 \\ 0 \\ \vdots \end{pmatrix} \middle| \begin{pmatrix} A^{(i)}_{m_i} & A^{(i)}_{m_i-1} & \cdots & A^{(i)}_0 \\ 0 & A^{(i)}_{m_i} & \cdots & A^{(i)}_1 \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & A^{(i)}_{m_i} \end{pmatrix} \begin{pmatrix} v^{(i)}_{m_i} \\ v^{(i)}_{m_i-1} \\ \vdots \\ v^{(i)}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\},$$

$$\mathcal{K} = \bigoplus_{i=1}^{r} \mathcal{K}^{(i)}, \quad \mathcal{L}(0) = \left\{ \begin{pmatrix} v_{m_0}^{(0)} \\ \vdots \\ v_1^{(0)} \\ v_{m_1}^{(1)} \\ \vdots \\ v_0^{(r)} \end{pmatrix} \middle| \sum_{i=0}^{r} \sum_{j=\delta_{i,0}}^{m_i} A_j^{(i)} v_j^{(i)} = 0 \right\},$$

$$\begin{array}{l} \mathcal{L}(\mu) = \\ (\mu \neq 0) \end{array} \left\{ \begin{pmatrix} v_{m_0}^{(0)} \\ \vdots \\ v_1^{(0)} \\ v_{m_1}^{(1)} \\ \vdots \\ v_0^{(r)} \end{pmatrix} \middle| \begin{array}{l} v_j^{(i)} = 0 \ \ (i \neq 0, j \neq 0), \\ v_0^{(1)} = \cdots = v_0^{(r)} = -\ell, \\ \begin{pmatrix} A_{m_0}^{(0)} & \ldots & A_1^{(0)} & A_0^{(0)} - \mu \\ 0 & A_{m_0}^{(0)} & \ldots & A_1^{(0)} \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & A_{m_0}^{(0)} \end{pmatrix} \begin{pmatrix} v_{m_0}^{(0)} \\ \vdots \\ v_1^{(0)} \\ \ell \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \end{array} \right\}.$$

Proposition 8. *$\mathcal{K}$ and $\mathcal{L}(\mu)$ are $\langle \tilde{\mathbf{A}} \rangle$-invariant. Namely $\tilde{A}_j^{(i)} \mathcal{K} \subset \mathcal{K}$ and $\tilde{A}_j^{(i)} \mathcal{L}(\mu) \subset \mathcal{L}(\mu)$ for all i, j.*

Middle convolution

Convolution $c_\mu(\mathbf{A}) = \tilde{\mathbf{A}} = (\tilde{A}^{(0)}_{m_0}, \ldots, \tilde{A}^{(0)}_1, \tilde{A}^{(1)}_{m_1}, \ldots, \tilde{A}^{(r)}_0)$ is well-defined on $V'/(\mathcal{K} + \mathcal{L}(\mu))$.

$mc_\mu(\mathbf{A})$: middle convolution $\tilde{\mathbf{A}}$
on the space $mc_\mu(V) = V'/(\mathcal{K} + \mathcal{L}(\mu))$.

Proposition 9. V: *irred.* $\Rightarrow mc_\mu(V)$: *irred.*,
$V \simeq mc_0(V) \simeq mc_{-\mu}(mc_\mu(V))$.

Conjecture 1. V: *irred.* $\Rightarrow mc_{\mu_1+\mu_2}(V) \simeq mc_{\mu_2}(mc_{\mu_1}(V))$.

Addition

$$\overline{\mu} = (\mu^{(0)}_{m_0}, \ldots, \mu^{(0)}_1, \mu^{(1)}_{m_1}, \ldots, \mu^{(r)}_0) \in \mathbb{C}^M,$$

$$M_{\overline{\mu}}(\mathbf{A}) = \mathbf{A} + \overline{\mu} I_n = (A^{(0)}_{m_0} + \mu^{(0)}_{m_0} I_n, \ldots, A^{(r)}_0 + \mu^{(r)}_0 I_n)$$

On solutions of linear differential system,
addition corresponds to multiplying the function;

$$Y \mapsto \exp\left(-\sum_{j=1}^{m_0} \frac{\mu^{(0)}_j}{j} z^j - \sum_{i=1}^{r} \sum_{j=1}^{m_i} \frac{\mu^{(i)}_j}{j(z-t_i)^{j+1}} \right) \prod_{i=1}^{r} (z-t_i)^{\mu^{(i)}_0} Y.$$

Index of rigidity

$\mathbf{A} = (A^{(0)}_{m_0}, \dots, A^{(0)}_1, A^{(1)}_{m_1}, \dots, A^{(r)}_0)$, $(A^{(i)}_j \in \mathrm{End}(V))$. Set

$$A^{(i)} = \begin{pmatrix} A^{(i)}_{m_i} & A^{(i)}_{m_i-1} & \cdots & A^{(i)}_0 \\ 0 & A^{(i)}_{m_i} & \cdots & A^{(i)}_1 \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & A^{(i)}_{m_i} \end{pmatrix} \begin{array}{c} \in \mathrm{End}(V^{\oplus(m_i+1)}) \\ (i = 0, \dots, r) \end{array},$$

$$A^{(0)}_0 = -(A^{(1)}_0 + \dots A^{(r)}_0)$$

$$\mathcal{C}^{(i)} = \left\{ C^{(i)} = \begin{pmatrix} C^{(i)}_{m_i} & C^{(i)}_{m_i-1} & \cdots & C^{(i)}_0 \\ 0 & C^{(i)}_{m_i} & \cdots & C^{(i)}_1 \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & C^{(i)}_{m_i} \end{pmatrix} \middle| \, A^{(i)} C^{(i)} = C^{(i)} A^{(i)} \right\}.$$

Define the index of rigidity by

$$\mathrm{idx}(\mathbf{A}) = \sum_{i=0}^{r} \dim(\mathcal{C}^{(i)}) - \left(\left(\sum_{i=0}^{r} m_i \right) + r - 1 \right) (\dim(V))^2.$$

The condition $A^{(i)}C^{(i)} = C^{(i)}A^{(i)}$ is equivalent to

$$\sum_{l=0}^{k} \left(A^{(i)}_{m_i-l} C^{(i)}_{m_i-(k-l)} - C^{(i)}_{m_i-(k-l)} A^{(i)}_{m_i-l} \right) = 0, \quad k = 0, \ldots, m_i.$$

Proposition 10. *Index of rigidity is preserved by addition, i.e.* $\mathrm{idx}(M_{\overline{\mu}}(\mathbf{A})) = \mathrm{idx}(\mathbf{A})$.

Conjecture 2. *If V is irred., then the index of rigidity is preserved by middle convolution, i.e.* $\mathrm{idx}(mc_{\mu}(\mathbf{A})) = \mathrm{idx}(\mathbf{A})$.

The case $m_i = 1$

The condition $A^{(i)}C^{(i)} = C^{(i)}A^{(i)}$ for $m_i = 1$ is equivalent to

$$A_1C_1 = C_1A_1, \quad A_1C_0 - C_0A_1 + A_0C_1 - C_1A_0 = 0.$$

Here we ignore the superscript $^{(i)}$.

Assume that A_1 is semisimple. Then $\exists P \in GL(n)$,

$$P^{-1}A_1P = \begin{pmatrix} \lambda_1 I_{n_1} & 0 & \dots & 0 \\ 0 & \lambda_2 I_{n_2} & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_k I_{n_k} \end{pmatrix}, (\lambda_1, \dots, \lambda_k : \text{distinct}).$$

It follows from $A_1C_1 = C_1A_1$ that C_1 is written as

$$P^{-1}C_1P = \begin{pmatrix} C_1^{[1]} & 0 & \dots & 0 \\ 0 & C_1^{[2]} & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & C_1^{[k]} \end{pmatrix}.$$

Write

$$P^{-1}A_0P = \begin{pmatrix} A_0^{[1,1]} & \dots & A_0^{[1,k]} \\ A_0^{[2,1]} & \dots & A_0^{[2,k]} \\ \vdots & \ddots & \vdots \\ A_0^{[k,1]} & \dots & A_0^{[k,k]} \end{pmatrix}, \; P^{-1}C_0P = \begin{pmatrix} C_0^{[1,1]} & \dots & C_0^{[1,k]} \\ C_0^{[2,1]} & \dots & C_0^{[2,k]} \\ \vdots & \ddots & \vdots \\ C_0^{[k,1]} & \dots & C_0^{[k,k]} \end{pmatrix}.$$

It follows from $A_1C_0 - C_0A_1 + A_0C_1 - C_1A_0 = 0$ that

$$C_0^{[i,j]} = -\frac{A_0^{[i,j]}C_1^{[j]} - C_1^{[i]}A_0^{[i,j]}}{\lambda_i - \lambda_j} \; (i \neq j), \quad A_0^{[i,i]}C_1^{[i]} = C_1^{[i]}A_0^{[i,i]}.$$

Elements of $C_0^{[i,i]}$ are not restricted by relations.

If

$$A_0^{[l,l]} \sim \begin{pmatrix} \lambda_{l,1} I_{n_{l,1}} & 0 & \dots & 0 \\ 0 & \lambda_{l,2} I_{n_{l,2}} & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_{l,p_l} I_{n_{l,p_l}} \end{pmatrix}, \quad \lambda_{l,i} \neq \lambda_{l,j} \ (i \neq j),$$

then the dimension of solutions of $A_1C_1 = C_1A_1$ and $A_1C_0 - C_0A_1 + A_0C_1 - C_1A_0 = 0$ is

$$\sum_{l=1}^{k} \sum_{j=1}^{p_l} (n_{l,j})^2 + \sum_{l=1}^{k} (n_l)^2.$$

We denote the type of multiplicities of the matrices (A_1, A_0) by

$$(n_1, n_2, \dots, n_k) -$$
$$((n_{1,1}, \dots, n_{1,p_1}), (n_{2,1}, \dots, n_{2,p_2}), \dots, (n_{k,1}, \dots, n_{k,p_k})).$$

Unramified tuple $\langle A_m, A_{m-1}, \ldots A_0 \rangle$
$\langle A_m, A_{m-1}, \ldots A_0 \rangle$ is unramified $\Leftrightarrow_{def}$

$$\exists P = \begin{pmatrix} P_m & P_{m-1} & \ldots & P_0 \\ 0 & P_m & \ldots & P_1 \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & P_m \end{pmatrix},$$

$$P^{-1} \begin{pmatrix} A_m & A_{m-1} & \ldots & A_0 \\ 0 & A_m & \ldots & A_1 \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & A_m \end{pmatrix} P = \begin{pmatrix} D_m & D_{m-1} & \ldots & D_0 \\ 0 & D_m & \ldots & D_1 \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & D_m \end{pmatrix},$$

$D_m, \ldots, D_1, D_0$: mutually commuting, $D_m, \ldots, D_1$: diagonal.

Type of multiplicity for $\langle A_2, A_1, A_0 \rangle$

$$D_2 = \begin{pmatrix} \lambda_1 I_{n_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_p I_{n_p} \end{pmatrix},\ D_1 = \begin{pmatrix} \lambda_{1,1} I_{n_{1,1}} & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \lambda_{1,p_1} I_{n_{1,p_1}} & \vdots \\ 0 & \dots & 0 & \ddots \end{pmatrix},$$

$$D_0 = \begin{pmatrix} \lambda_{1,1,1} I_{n_{1,1,1}} & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \lambda_{1,1,p_{1,1}} I_{n_{1,1,p_{1,1}}} & \vdots \\ 0 & \dots & 0 & \ddots \end{pmatrix},$$

$(n_1 + \cdots + n_p = n, n_{k,1} + \cdots + n_{k,p_k} = n_k, n_{k,l,1} + \cdots + n_{k,l,p_{k,l}} = n_{k,l})$, then we write the type of multiplicity as

$$\begin{aligned} &(n_1, n_2, \dots, n_p) \\ &-((n_{1,1}, \dots, n_{1,p_1}), (n_{2,1}, \dots, n_{2,p_2}), \dots) \\ &-(((n_{1,1,1}, \dots, n_{1,1,p_{1,1}}), (n_{1,2,1}, \dots, n_{1,2,p_{1,2}}), \dots), \dots). \end{aligned}$$

We can define the type of multiplicity for $\langle A_m, \ldots, A_0 \rangle$ similarly.

On the case $m = 1$, this definition coincides to the previous one.

Proposition 11. *We assume that* $\langle A^{(i)}_{m_i}, \ldots, A^{(i)}_0 \rangle$ *is unramified for* $i = 0, \ldots, r$ *and* $\langle \mathbf{A} \rangle$ *is irreducible.*
(i) The index of rigidity can be written by using the type of multiplicities.
(ii) $mc_\mu(\mathbf{A})$ *is also irreducible, unramified and the index of rigidity is preserved by application of middle convolution, i.e.* $\mathrm{idx}(mc_\mu(\mathbf{A})) = \mathrm{idx}(\mathbf{A})$ *for all* $\mu \in \mathbb{C}$*.*

Proposition 12. *We assume that* $\langle A^{(i)}_{m_i}, \ldots, A^{(i)}_0 \rangle$ *is unramified for* $i = 0, \ldots, r$ *and* $\langle \mathbf{A} \rangle$ *is irreducible.*
(i) If $\mathrm{idx}(\mathbf{A}) = 2$*, then* $\mathbf{A}$ *is transformed to the rank one matrices by applying addition and middle convolution repeatedly.*
(ii) If $\mathrm{idx}(\mathbf{A}) = 0$*, then* $\mathbf{A}$ *is transformed to one of the following cases by applying middle convolution and addition repeatedly, where* $d \in \mathbb{Z}_{\geq 1}$.

$$
\begin{aligned}
\textit{Four singularities}: \quad & \{(d,d),\ (d,d),\ (d,d),\ (d,d)\}, \\
\textit{Three singularities}: \quad & \{(d,d,d),\ (d,d,d),\ (d,d,d)\}, \\
& \{(2d,2d),\ (d,d,d,d),\ (d,d,d,d)\}, \\
& \{(3d,3d),\ (2d,2d,2d),\ (d,d,d,d,d,d)\}, \\
& \{(d,d)-((d),(d)),\ (d,d),\ (d,d)\},
\end{aligned}
$$

$$
\begin{aligned}
\textit{Two singularities}: \quad & \{(d,d)-((d),(d)),\ (d,d)-((d),(d))\},\\
& \{(d,d)-((d),(d))-(((d)),((d))),\ (d,d)\},\\
& \{(d,d,d)-((d),(d),(d)),\ (d,d,d)\},\\
& \{(d,d,d,d)-((d),(d),(d),(d)),\ (2d,2d)\},\\
& \{(2d,2d)-((d,d),(d,d)),\ (d,d,d,d)\},\\
& \{(3d,2d)-((d,d,d),(2d)),\ (d,d,d,d,d)\},\\
& \{(2d,2d,2d)-((d,d),(d,d),(d,d)),\ (3d,3d)\},\\
& \{(3d,3d,2d)-((d,d,d),(d,d,d),(2d)),\ (4d,4d)\},\\
& \{(5d,4d,3d)-((d,d,d,d,d),(2d,2d),(3d)),\ (6d,6d)\},\\
& \{(5d,4d)-((d,d,d,d,d),(2d,2d)),\ (3d,3d,3d)\},\\
& \{(3d,3d)-((d,d,d),(d,d,d)),\ (2d,2d,2d)\},\\
& \{(5d,3d)-((d,d,d,d,d),(3d)),\ (2d,2d,2d,2d)\},\\
& \{(4d,3d)-((2d,2d),(3d)),\ (d,d,d,d,d,d,d)\},
\end{aligned}
$$

$$
\begin{aligned}
\textit{One singularity}: \quad & \{(d,d)-((d),(d))-(((d)),((d)))-((((d))),(((d))))\},\\
& \{(d,d,d)-((d),(d),(d))-(((d)),((d)),((d)))\},\\
& \{(2d,2d)-((d,d),(d,d))-(((d),(d)),((d),(d)))\},\\
& \{(3d,2d)-((d,d,d),(2d))-(((d),(d),(d)),((d,d)))\},\\
& \{(4d,3d)-((2d,2d),(3d))-(((d,d),(d,d)),((d,d,d)))\},\\
& \{(5d,4d)-((3d,2d),(4d))-(((d,d,d),(2d)),((d,d,d,d)))\},\\
& \{(6d,4d)-((3d,3d),(4d))-(((d,d,d),(d,d,d)),((d,d,d,d))\},\\
& \{(7d,6d)-((4d,3d),(6d))-(((2d,2d),(3d)),((d,d,d,d,d,d)))\},\\
& \{(8d,6d)-((5d,3d),(6d))-(((d,d,d,d,d),(3d)),((2d,2d,2d)))\},\\
& \{(9d,6d)-((5d,4d),(6d))-(((d,d,d,d,d),(2d,2d)),((3d,3d)))\}.
\end{aligned}
$$

We conjecture that $d=1$ follows from irreducibility.

Deligne-Arinkin Theorem

Theorem 13. *[Arinkin, Compos. Math. 2010]*
If a system of differential equation is irreducible and rigid (i.e. $idx = 2$), then it is transformed to the rank one system by applying addition and Fourier-Laplace transformation repeatedly.

We do not need the assumption of unramification.

$$\text{Euler transformation} \sim \mathcal{L}^{-1} x^{\mu} \mathcal{L},$$
$$(\mathcal{L}\text{: Fourier-Laplace transformation})$$

Examples for the case $r = 1$, $m_0 = 0$, $m_1 = 1$, $z = 0$: irreg., $z = \infty$: reg., size: 2×2

(i) The eigenvalues of $A_1^{(1)}$ are distinct

Index of rigidity $= 2$.

By diagonalizing $A_1^{(1)}$ and applying additions, we may set

$$A_1^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}, \quad A_0^{(1)} = \begin{pmatrix} 0 & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

$$\frac{dY}{dz} = \left(\frac{A_1^{(1)}}{z^2} + \frac{A_0^{(1)}}{z} \right) Y \underset{z=\frac{1}{x}}{\Rightarrow} \text{Kummer's confluent hypergeometric differential equation}$$

$$A^{(1)} = \begin{pmatrix} A_1^{(1)} & A_0^{(1)} \\ 0 & A_1^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & a_{1,2} \\ 0 & \beta & a_{2,1} & a_{2,2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$$

$\Rightarrow \dim(\mathcal{K}) = \dim(\mathrm{Ker}(A^{(1)})) = 2.$

If $\mu(\neq 0)$ is an eigenvalue of $-A_0^{(1)}$, then $\dim(\mathcal{K} + \mathcal{L}(\mu)) = 3$ and we have a differential equation of rank 1 by middle convolution mc_μ.

By applying $mc_{-\mu}$ to the differential equation of rank 1, we obtain an integral representation of solutions to the differential system of rank 2.

(ii) $A_1^{(1)}$ is nilpotent

Index of rigidity $= 2$.

Set

$$A_1^{(1)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_0^{(1)} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix},$$

$$\begin{pmatrix} A_1^{(1)} + \beta I_2 & A_0^{(1)} + \alpha I_2 \\ 0 & A_1^{(1)} + \beta I_2 \end{pmatrix} = \begin{pmatrix} \beta & 1 & a_{1,1} + \alpha & a_{1,2} \\ 0 & \beta & a_{2,1} & a_{2,2} + \alpha \\ 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & \beta \end{pmatrix}$$

$\Rightarrow \dim(\mathcal{K}) \leq 1$ $(a_{2,1} \neq 0)$ for any choice of addition.

Hence $\dim(\mathcal{K}+\mathcal{L}(\mu)) \leq 2$ and the rank of differential equation cannot be deduced to one.

$a_{2,1} = 0 \Rightarrow$ reducible.

This case is not covered by Proposition 12.

Summary

- DSP, Fuchsian differential system, middle convolution.

- Index of rigidity is preserved.

- Examples: Gauss hypergeometric equation, Heun's equation.

- Simplification by addition and middle convolution.

- Middle convolution for linear differential system with irregular singularities.

Problems for linear differential system with irregular singularities

• Validity of definition the index of rigidity.
(Compatibility with the definition by Bloch-Esnault, Invariance of the index of rigidity by middle convolution)

• More examples. (conluent Heun equations . . .)

• Laplace transformation.
(we may treat the case that $A_1^{(1)}$ is nilpotent).

• Crawley-Boevey type theorem for linear differential system with irregular singularities.

We hope that middle convolution is also applied for several topics in mathematics and physics.

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