Nonlinear Cauchy Problems with Small Analytic Data

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A Simple Example

Cauchy problem of ODE:

$$\frac{d^2}{dt^2}u = 6u^2, \quad u(0) = \varepsilon^2, \quad u'(0) = 2\varepsilon^3.$$

Solution:

$$u(t) = \frac{\varepsilon^2}{(1-\varepsilon t)^2}.$$

Blow up at $t = 1/\varepsilon$.

If ε and the Cauchy data are SMALL, the solution exists for a LONG TIME,

because the nonlinear term is small and does not cause a trouble then.

Known Results

- Many results in the $\mathcal{C}^\infty$ category about nonlinear wave equations. see Hörmander's lecture note
- Gourdin-Mechab...Kirchhoff eq., m-th order equations, real analytic solutions
- D'Ancona-Spagnolo, Gourdin-Gramchev...hyperbolic eq., real analytic solutions
- Y...Proceedings of AMS 2006

  We employ the method of Gourdin-Mechab.

  It goes back (at least) to Wagschal.

  – Peter Lax's series and Banach algebra, fixed point

  – no hyperbolicity assumption (in the spirit of Cauchy-Kowalevsky)

Result 1 (sketched): small data⇒long life

$P(\partial_t, \partial_x) = \sum_{j=1}^{n} p_j \partial_t \partial_j + \sum_{k=1}^{n} \sum_{j=1}^{k} p_{jk} \partial_j \partial_k$ constant coeff.

$$(\mathrm{CP1}) \begin{cases} (\partial_t^2 - P(\partial_t, \partial_x))u = f_1(t; \underbrace{u}_{X}; \underbrace{\partial_t u, \nabla u}_{Y}; \underbrace{\nabla \partial_t u, \nabla^2 u}_{Z}), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases}$$

f_1 is a bounded function in $t \in \mathbb{R}$,

$$f_1(t; X; Y; Z) = \sum_{L \geq 4} a_{\alpha\beta\gamma}(t) X^\alpha Y^\beta Z^\gamma, \quad L = \alpha + 2|\beta| + 3|\gamma|.$$

EX. $u^4,\ \partial_t u \cdot \partial_1 u,\ u \cdot \partial_1 \partial_2 u$

If $\sup_{x\in\Omega} |\partial^\alpha \varphi| \leq \varepsilon^{|\alpha|+1} |\alpha|!$, $\sup_{x\in\Omega} |\partial^\alpha \psi| \leq \varepsilon^{|\alpha|+2} |\alpha|!$,

then a real analytic solution exists for $t \leq \mathrm{const.}\varepsilon^{-1}$.

Result 1 (comlex version)

(CP1c): the complex version of (CP1)
f_1 is independent of t.

If the Cauchy data are small in the sense that

$$\sup_{x\in U} |\partial^\alpha \varphi| \le \varepsilon^{|\alpha|+1}|\alpha|!$$

$$\sup_{x\in U} |\partial^\alpha \psi| \le \varepsilon^{|\alpha|+2}|\alpha|!,$$

then a holomorphic solution uniquely exists for
$|t| \le \mathrm{const.}\varepsilon^{-1}$.

Result 2 (sketched): small data⇒long life

$P = P(\partial_x) = \sum_{k=1}^{n} \sum_{j=1}^{k} p_{jk} \partial_j \partial_k$ free from ∂_t.

$$(\mathrm{CP2}) \begin{cases} \big(\partial_t^2 - P(\partial_x)\big)u = f_2(t, \underbrace{u}_{X}, \underbrace{\partial_t u}_{Y}, \underbrace{\nabla u}_{Z}, \underbrace{\nabla \partial_t u}_{\Theta}, \underbrace{\nabla^2 u}_{\Xi}), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases}$$

$$f_2(t, X, Y, Z, \Theta, \Xi) = \sum_{L_1 \geq 2, L_2 \geq 2} a_{\alpha\beta\gamma\lambda\mu}(t) X^\alpha Y^\beta Z^\gamma \Theta^\lambda \Xi^\mu,$$

$L_1 = \alpha + |\gamma| + |\mu|,\ L_2 = \beta + |\gamma| + 2|\lambda| + 2|\mu|.$

f_2 is a bounded function in t.

If $\sup_{x\in\Omega} |\partial^\alpha \varphi| \leq \varepsilon^{|\alpha|+1} |\alpha|!$, $\sup_{x\in\Omega} |\partial^\alpha \psi| \leq \varepsilon^{|\alpha|+1} |\alpha|!$

then a real analytic solution exists for $t \leq \mathrm{const.}\varepsilon^{-1}$.

∃complex version

Result 3

The nonlinear term can include the complex conjugate of the unknown function.
The general statement is complicated. We only give an example here.

$$(\partial_t^2 - \Delta)u = |\nabla u|^2 = \sum_{j=1}^{n} \partial_j u \, \partial_j \bar{u}.$$

If $\sup_{x \in \Omega} |\partial^\alpha \varphi| \leq \varepsilon^{|\alpha|+1} |\alpha|!$, $\sup_{x \in \Omega} |\partial^\alpha \psi| \leq \varepsilon^{|\alpha|+1} |\alpha|!$ then a real analytic solution exists for $t \leq \mathrm{const.}\varepsilon^{-1}$.

Result 4 and other topics

Result 4 We can deal with the case with linear part involving first order derivatives. Then the lifespan is the order of $\varepsilon^{-1/2}$.

Many other different cases could be studied.

- Gevrey class?
- Fuchsian PDE ($(t\partial_t)^j$ instead of ∂_t^j)?
- Goursat problem?
- m-th order equations?

Uniformly Analytic Functions

Ω: an open set $\subset \mathbb{R}^n_x$, $\quad x = (x_1, \ldots, x_n)$.

A function $\varphi(x) \in \mathcal{C}^\infty(\Omega)$ is said to be *uniformly analytic*

$$\exists C > 0, \forall \alpha \in \mathbb{N}^n, \sup_{x \in \Omega} |\partial^\alpha \varphi(x)| \le C^{|\alpha|+1} |\alpha|!.$$

$A(\Omega)$ is the totality of such functions.

$I_T =]-T, T[\subset \mathbb{R}_t$. $\qquad \Omega_T := I_T \times \Omega \subset \mathbb{R}_t \times \mathbb{R}^n_x$.

$u(t,x) \in \mathcal{C}(\Omega_T)$ is an element of $\mathcal{C}^k(T; A(\Omega))$ if:

(i) $\forall j \in \{0, \ldots, k\}, \forall \alpha \in \mathbb{N}^n, \partial_t^j \partial^\alpha u \in \mathcal{C}(\Omega_T)$,

(ii) $\forall T' \in]0, T[$, $\exists C = C_{T'} > 0$, $\forall j \in \{0, \ldots, k\}$, $\forall \alpha \in \mathbb{N}^n$,

$$\sup_{|t| \le T', x \in \Omega} |\partial_t^j \partial^\alpha u(t,x)| \le C^{|\alpha|+1} |\alpha|!.$$

Result 1 (detail): small data⇒long life

$$P(\partial_t,\partial_x)=\sum_{j=1}^{n}p_j\partial_t\partial_j+\sum_{k=1}^{n}\sum_{j=1}^{k}p_{jk}\partial_j\partial_k$$

$$(\mathrm{CP1})\begin{cases}\big(\partial_t^2-P(\partial_t,\partial_x)\big)u=f_1(t;\underbrace{u}_{X};\underbrace{\partial_t u,\nabla u}_{Y};\underbrace{\nabla\partial_t u,\nabla^2 u}_{Z}),\\ u(0,x)=\varphi(x),\ \partial_t u(0,x)=\psi(x),\end{cases}$$

f_1 is a bounded function in $t\in\mathbb{R}$.

$$f_1(t;X;Y;Z)=\sum_{L\geq 4}a_{\alpha\beta\gamma}(t)X^{\alpha}Y^{\beta}Z^{\gamma},\qquad L=\alpha+2|\beta|+3|\gamma|.$$

For $\exists\delta>0$ and $\exists\varepsilon_0>0$, we have —

$0<\forall\varepsilon\leq\varepsilon_0$:

If $\sup_{x\in\Omega}|\partial^{\alpha}\varphi|\leq\varepsilon^{|\alpha|+1}|\alpha|!$, $\sup_{x\in\Omega}|\partial^{\alpha}\psi|\leq\varepsilon^{|\alpha|+2}|\alpha|!$,

then a solution $u\in\mathcal{C}^2(T;A(\Omega))$ exists for $\boxed{T=\delta/\varepsilon}$.

Result 2 (detail): small data$\Rightarrow$long life

$P = P(\partial_x) = \sum_{k=1}^{n} \sum_{j=1}^{k} p_{jk}\partial_j\partial_k$ is free from ∂_t.

$$(\mathrm{CP2}) \begin{cases} \left(\partial_t^2 - P(\partial_x)\right)u = f_2(t, u, \partial_t u, \nabla u, \nabla\partial_t u, \nabla^2 u), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases}$$

$$f_2(t, X, Y, Z, \Theta, \Xi) = \sum_{L_1 \geq 2, L_2 \geq 2} a_{\alpha\beta\gamma\lambda\mu}(t) X^\alpha Y^\beta Z^\gamma \Theta^\lambda \Xi^\mu,$$

$L_1 = \alpha + |\gamma| + |\mu|,\ L_2 = \beta + |\gamma| + 2|\lambda| + 2|\mu|.$

f_2 is a bounded function in t.

$\Rightarrow \exists\delta > 0$ and $\exists\varepsilon_0 > 0$: $\quad 0 < \forall\varepsilon \leq \varepsilon_0$:

If $\sup_{x\in\Omega} |\partial^\alpha \varphi| \leq \varepsilon^{|\alpha|+1}|\alpha|!, \quad \sup_{x\in\Omega} |\partial^\alpha \psi| \leq \varepsilon^{|\alpha|+1}|\alpha|!,$

then a solution $u \in \mathcal{C}^2(T; A(\Omega))$ exists for $\boxed{T = \delta/\varepsilon}$.

Banach algebra $\mathcal{G}_{T,\zeta}(\Omega) \subset \mathcal{C}^0(T; A(\Omega))$

$\theta(X) = K^{-1} \sum_{k=0}^{\infty} X^k/(k+1)^2, \quad K = 4\pi^2/3$

Peter Lax's series $\quad \theta^2 \ll \theta$ (majorant)

For $\zeta > 0$, a function $u(t,x)$ on Ω_T belongs to $\mathcal{G}_{T,\zeta}(\Omega)$ if:

for $\exists C > 0, \forall \alpha \in \mathbb{N}^n, \forall t \in I_T$,

$$\sup_{x \in \Omega} |\partial^\alpha u(t,x)| \le C \zeta^{|\alpha|} D^{|\alpha|} \theta(|t|/T).$$

The norm $\|u\|$ is the infimum of such C's.

$\mathcal{G}_{T,\zeta}(\Omega)$: a Banach algebra, a subspace of $\mathcal{C}^0(T; A(\Omega))$.

We shall construct $w := \partial_t^2 u \in \mathcal{G}_{T,\zeta}(\Omega) \subset \mathcal{C}^0(T; A(\Omega))$ and get $u \in \mathcal{C}^2(T; A(\Omega))$. We shall work in $\mathcal{G}_{T,\zeta}(\Omega)$.

(CP1) reduced to a fixed point problem.

$w(t,x) := \partial_t^2 u(t,x)$. Then $u = \partial_t^{-2} w + \varphi + t\psi$, $\partial_t^{-1} := \int_0^t$.

$Qu := (u; \partial_t u, \nabla u; \nabla \partial_t u, \nabla^2 u)$,

$$\mathcal{L}_1(w) := P(\underbrace{\partial_t^{-2} w + \varphi + t\psi}_{u}) + f_1(t; Q(\underbrace{\partial_t^{-2} w + \varphi + t\psi}_{u})).$$

$\mathrm{ord}\,(P\partial_t^{-2}) = \mathrm{ord}\,(Q\partial_t^{-2}) = 0$and, $P\varphi$etc. are small

$(\mathrm{CP1}) \Leftrightarrow w = \mathcal{L}_1(w)$. FIXED POINT w of $\mathcal{L}_1$.

We have only to prove: $\mathcal{L}_1$ is a CONTRACTION from a closed ball of $\mathcal{G}_{T,\zeta}(\Omega)$ to itself.
Here $T = \delta/\varepsilon$, $\zeta = 2e^2\varepsilon$ $(\varepsilon > 0, 0 < \delta < 1)$.

Estimates involving the Cauchy data

$t\psi$ in $\mathcal{L}_1$ can be estimated by watching the exponents of ε.

$$\sup_{x\in\Omega} |\psi| \le \boxed{\varepsilon^2}.$$

Multiplication by t, hence by T, makes $t\psi$ BIGGER.

If $\boxed{T = \mathrm{const}/\varepsilon}$,

ε remains after cancellation and $t\psi$ is small enough.

Because of $\sup_{x\in\Omega} |\partial^\alpha \psi| \le \varepsilon^{|\alpha|+2}|\alpha|!$, we have sufficiently many ε's for $\partial_j(t\psi)$, $\partial_j\partial_k(t\psi)$.

Estimate of $w = \partial_t^2 u$

We want to estimate

$$\mathcal{L}_1(w) := P(\partial_t^{-2} w + \varphi + t\psi) + f_1\big(t; Q(\partial_t^{-2} w + \varphi + t\psi)\big)\ .$$

$$Qu = (u; \partial_t u, \nabla u; \nabla \partial_t u, \nabla^2 u), \quad \mathrm{ord}\, P = 2.$$

$P\partial_t^{-2} w$, $Q\partial_t^{-2} w$ involve quantities
like $\partial_j \partial_k \partial_t^{-2} w$, $\partial_j \partial_t^{-2} w$, $\partial_j \partial_t^{-1} w$, $\partial_t^{-1} w$, $\partial_t^{-2} w$.

≪Wagschal≫ ⟨*0th order operators are bounded.*⟩
If $(k, \alpha) \in (-\mathbb{N}) \times \mathbb{N}^n$ satisfies $\mathrm{ord}\,(\partial_t^k \partial^\alpha) = k + |\alpha| \le 0$,
then $\partial_t^k \partial^\alpha$ is a bounded operator from $\mathcal{G}_{T,\zeta}(\Omega)$ to itself.
Its norm $\le C_{k,|\alpha|} T^{-k} \zeta^{|\alpha|}$ for $\exists C_{k,|\alpha|} > 0$.

CHOOSE A GOOD PAIR T, ζ.

Combining the previous estimates

$\exists C_1$, $\exists C_2$, $\exists \varepsilon$, $\exists \delta$ $(\varepsilon > 0, 0 < \delta < 1)$ such that

for a good pair $\boxed{\zeta = 2e^2\varepsilon, T = \delta/\varepsilon}$, we have

$$\|P\partial_t^{-2}w\| \le C_1\delta\|w\|, \quad \|P(\varphi + t\psi)\| \le C_2\varepsilon^3,$$
$$\|\partial_t^{-2}w + \varphi + t\psi\| \le C_{-2,0}\varepsilon^{-2}\|w\| + 2K\varepsilon,$$
$$\|\partial_t(\partial_t^{-2}w + \varphi + t\psi)\| \le C_{-1,0}\varepsilon^{-1}\|w\| + K\varepsilon^2,$$
$$\|\partial_j(\partial_t^{-2}w + \varphi + t\psi)\| \le 2e^2C_{-2,1}\varepsilon^{-1}\|w\| + 2K\varepsilon^2,$$
$$\|\partial_t\partial_j(\partial_t^{-2}w + \varphi + t\psi)\| \le 2e^2C_{-1,1}\|w\| + K\varepsilon^3,$$
$$\|\partial_j\partial_k(\underbrace{\partial_t^{-2}w + \varphi + t\psi}_{u})\| \le (2e^2)^2C_{-2,2}\|w\| + 6K\varepsilon^3$$

holds for $\forall w \in \mathcal{G}_{T,\zeta}(\Omega)$.

$\mathcal{L}_1$ maps a closed ball to itself.

$r := 2C_2\varepsilon^3/(1-2C_1\delta)$ order ε^3.

$B(r,T,\zeta) = B(r,\delta/\varepsilon, 2e^2\varepsilon) \subset \mathcal{G}_{T,\zeta}(\Omega)$ closed ball

$w \in B(r,T,\zeta)$ ($\|w\|$ is at most of order ε^3)

$u := \partial_t^{-2}w + \varphi + t\psi$.

$$f_1(X;Y;Z) = \sum_{L\geq 4} a_{\alpha\beta\gamma}X^\alpha Y^\beta Z^\gamma, \quad L = \alpha + 2|\beta| + 3|\gamma|.$$

$$X \leftrightarrow u,\ Y \leftrightarrow (\partial_t u, \nabla u),\ Z \leftrightarrow (\partial_t \nabla u, \nabla^2 u).$$

The estimates on the previous page implies:

$$\|X\| \leq C_4\varepsilon, \qquad \|Y\| \leq C_4^2\varepsilon^2, \qquad \|Z\| \leq C_4^3\varepsilon^3.$$

f_1 is of order ε^4 and is $< r/2$, together with the linear part.

$\mathcal{L}_1 : B(r,t,\zeta) \to B(r,t,\zeta)$

$\mathcal{L}_1$ is a contraction

Mean Value Theorem

$$\begin{aligned} & f_1(t; X'; Y'; Z') - f_1(t; X; Y; Z) \\ & = (X'-X,\, Y'-Y,\, Z'-Z) \cdot g_1 \\ & = (X'-X) \cdot g_1^X + (Y'-Y) \cdot g_1^Y + (Z'-Z) \cdot g_1^Z . \end{aligned}$$

g_1^X, g_2^Y, g_2^Z can be written in terms of derivatives of f_1.
$L = \alpha + 2|\beta| + 3|\gamma|$ becomes smaller, but we get good enough estimates.

Proof of Result 1 completed.
Other results are shown in a similar manner.

Thank you very much!