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SUMMABILITY OF FIRST INTEGRALS OF A RESONANT HAMILTONIAN SYSTEM

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1. Motivation

Hamiltonian. For $n \geq 2$ let $q = (q_1, \ldots, q_n) \in \mathbb{R}^n$, $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$). For a Hamiltonian function $H = H(q, p)$ we consider a Hamiltonian system

$$\dot{q} = \nabla_p H, \quad \dot{p} = -\nabla_q H, \tag{1.1}$$

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or a Hamiltonian vector field

$$\chi_H := \{H, \cdot\} = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \right), \tag{1.2}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket.

ϕ is called the **first integral** of χ_H if $\chi_H \phi = 0$. Eq. (1.1) is said to be **C^ω-Liouville integrable** if there exist first integrals, $\exists \phi_j \in C^\omega$ $(j = 1, \ldots, n)$ being functionally independent (on an open dense set) and Poisson commuting, i.e., $\{\phi_j, \phi_k\} = 0$, $\{H, \phi_k\} = 0$. If $\exists \phi_j \in C^\infty$ $(j = 1, \ldots, n)$, then we say **C^∞-Liouville integrable**.

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In the paper

Integrable geodesic flows with positive topological entropy. Invent Math. **140** *(3), 639-650 (2000)*

Bolsinov and Taimanov showed that there exists a Hamiltonian related with geodesic flow on a Riemannian manifold which is C^∞-integrable and not C^ω-integrable. Then in the paper

Analytic-non-integrability of an integrable analytic Hamiltonian system. Differ. Geom. Appl. **22**, *287-296 (2005)*

Gorni, G. and Zampieri, G. showed that the following Hamiltonian has similar properties in some neighborhood of the origin of $(q_1, q_2, p_1, p_2) \in \mathbb{R}^4$

$$H = -q_2 p_2 \partial_{q_1} r + (r^2 + q_2 \partial_{q_2} r) p_1,$$

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Resonances. For the Hamiltonian function H we may assume that the Taylor expansion at the origin starts from terms of order 2. Let λ_j $(j = 1, 2, \ldots, n)$ be the eigenvalues of the bilinear form H_2 corresponding to $H = H_2 + H_3 + \cdots$, where H_j is homogeneous degree j. In G-Z's example we have $H_2 = 0$, namely the Hamiltonian has a resonance dimension 2.

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We expect that Taimanov' theorem could be understood from the singular structure of Hamiltonian ODE (1.1). Because n–parameter family of solutions of (1.1) corresponds to an n functionally independent first integrals by implicit function theorem **we study singularity sturucture of (1.1) from the viewpoint of nonintegrable structure.**

We study (1.1) with Hamiltonians with **1− resonance**, $\lambda_1 = 0$. The case with resonance dimension ≥ 2 will be a future problem. (See also a jointwork with W. Balser in Math. Z. 2011.)

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2. C^ω nonintegrablility

Hamiltonians. We consider the Hamiltonian $H := H_0 + H_1$ where

$$H_0 = q_1^{2\sigma} p_1 + \sum_{j=2}^{n} \lambda_j q_j p_j, \tag{2.1}$$

$$H_1 = \sum_{j=2}^{n} q_j^2 B_j(q_1, q_1^{2\sigma} p_1, q), \ \ q = (q_2, \ldots , q_n), \tag{2.2}$$

where $B_j(q_1, s, t)$ are holomorphic at the origin with respect to $(q_1, s, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1}$. (This Hamiltonian is similar to the one studied by Taimanov.)

Monodromy condition.

(M) *For $k = 2, 3, \ldots, n$ the equation*

$$q_1^{2\sigma}\frac{dv}{dq_1} + 2\lambda_k v = B_k(q_1, 0, 0) \tag{2.3}$$

has no analytic solution v at the origin.

We have (cf. Lemma 6 of [3])

(M) is equivalent to that *the monodoromy of an analytic continuation of the solution along a path encircling the origin does not vanish.* (open condition) Then we have

Theorem 1. Assume (M) and suppose that λ_j $(j = 2, \ldots, n)$ are linearly independent over $\mathbb{Z}$. Then χ_H has no C^ω first integral which is functionally independent of H. Especially χ_H is not C^ω-Liouville integrable.

3. **Singular integrability**

Introduction of log-exponential series As for the general reference we refer

W. Balser, Existence and structure of complete formal solutions of non-linear meromorphic systems of ordinary differential equations. *Asymptot. Anal., 15 (1997).*

O. Costin, Asymptotics and Borel summability, *Chapman Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 141 (2009).*

We change the notation a little bit in the following in order to indicate the resonance variable q_1 and p_1. We write the variables in the form

$$(q_1, q_2, q_3, \cdots, q_n) = (q_1, q), \quad (p_1, p_2, p_3, \cdots, p_n) = (p_1, p).$$

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Construction of formal first integrals. Let H_0 and H_1 be given by (2.1) and (2.2), respectively. Define $H = H_0 + H_1$.

Assume that $H_1 = \sum_{j=2}^n q_j^2 B_j$ satisfy

$$B_j = B_j(q_1, q_1^{2\sigma} p_1, q) = B_{j,0}(q_1, q) + q_1^{2\sigma} p_1 B_{j,1}(q_1, q), \tag{3.1}$$

where $2 \le j \le n$, $B_{j,0}$ and $B_{j,1}$ are analytic at $q_1 = 0$, $q = 0$. Moreover, we suppose

$$\lambda_j \ (j = 2, 3, \ldots, n) \text{ are linearly independent over } \mathbb{Z}. \tag{3.2}$$

Set $p_1^0 = q_1^{2\sigma} p_1$ and

$$E_c \equiv E_c(q_1) = \exp\left(\frac{c q_1^{-2\sigma+1}}{(2\sigma - 1)}\right). \tag{3.3}$$

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where $2 \le j \le n$, $B_{j,0}$ and $B_{j,1}$ are analytic at $q_1 = 0$, $q = 0$. Moreover, we suppose

$$\lambda_j \ (j = 2, 3, \ldots, n) \text{ are linearly independent over } \mathbb{Z}. \tag{3.5}$$

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$$E_c \equiv E_c(q_1) = \exp\left(\frac{c {q_1}^{-2\sigma+1}}{(2\sigma - 1)}\right). \tag{3.6}$$

We will construct a formal first integral v in the form

$$v = \sum_{\alpha \ge 0} v^{(\alpha)}(q_1, p_1, q, p) E^{\alpha}, \tag{3.7}$$

where $E^{\alpha} = E_{\lambda_2}^{\alpha_2} \cdots E_{\lambda_n}^{\alpha_n}$, $q = (q_2, \ldots, q_n)$, and $v^{(\alpha)}(q_1, p_1, q, p)$ is a formal power series of q_1, q, p_1 and p. We say that v is the formal integral of the Hamiltonian vector field χ_H if $\chi_H v = 0$ as a formal power series.

By definition we have, for $\mathcal{L} := \{H_0, \cdot\}$ and $R := \{H_1, \cdot\}$,

$$\mathcal{L} \;=\; q_1^{2\sigma}\frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1}p_1\frac{\partial}{\partial p_1} + \sum_{j=2}^{n}\lambda_j\left(q_j\frac{\partial}{\partial q_j} - p_j\frac{\partial}{\partial p_j}\right), \tag{3.8}$$

$$R = \sum_{j=2}^{n}\left(-2q_jB_j\frac{\partial}{\partial p_j} + q_j^2(\partial_{p_1}B_j)\frac{\partial}{\partial q_1} - q_j^2(\partial_{q_1}B_j)\frac{\partial}{\partial p_1} - q_j^2\nabla_q B_j\cdot\frac{\partial}{\partial p}\right). \tag{3.9}$$

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$$R = \sum_{j=2}^{n}\left(-2q_jB_j\frac{\partial}{\partial p_j} + q_j^2(\partial_{p_1}B_j)\frac{\partial}{\partial q_1} - q_j^2(\partial_{q_1}B_j)\frac{\partial}{\partial p_1} - q_j^2\nabla_qB_j\cdot\frac{\partial}{\partial p}\right). \tag{3.11}$$

By using the formula

$$\partial_{p_1}B_j = B_{j,1}q_1^{2\sigma}, \quad q_1^{2\sigma}(\partial/\partial q_1)E^\alpha = -(\sum_{j=2}^{n}\lambda_j\alpha_j)E^\alpha = -\langle\lambda,\alpha\rangle E^\alpha,$$

we have

$$\begin{aligned}\mathcal{L}(v^{(\alpha)}E^{\alpha}) \;=\; & E^{\alpha}\left(q_1^{2\sigma}\frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1}p_1\frac{\partial}{\partial p_1}\right.\\ & \left. +\; \sum_{j=2}^{n}\lambda_j\left(q_j\frac{\partial}{\partial q_j} - p_j\frac{\partial}{\partial p_j} - \alpha_j\right)\right)v^{(\alpha)},\end{aligned} \tag{3.12}$$

and

$$R(v^{(\alpha)}E^{\alpha}) = E^{\alpha}\left(-\langle\lambda,\alpha\rangle\sum_{j=2}^{n}q_j^2B_{j,1} + R\right)v^{(\alpha)}. \tag{3.13}$$

It follows that if v is a formal first integral of χ_H, then every $v^{(\alpha)}$ satisfies

$$\left(q_1^{2\sigma}\frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1}p_1\frac{\partial}{\partial p_1} + \sum_{j=2}^{n}\lambda_j\left(q_j\frac{\partial}{\partial q_j} - p_j\frac{\partial}{\partial p_j} - \alpha_j\right)\right)v^{(\alpha)} \tag{3.14}$$
$$+ \left(-\sum_{j=2}^{n}\langle\lambda,\alpha\rangle q_j^2 B_{j,1} + R\right)v^{(\alpha)} = 0.$$

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Expand $v^{(\alpha)}$ into the formal power series

$$v^{(\alpha)} = \sum_{\nu,k,\ell} v^{(\alpha)}_{\nu,k,\ell}(q_1)p_1^{\nu}p^k q^{\ell}, \tag{3.16}$$

then, insert the expansion into (3.15) and compare the coefficients of $p_1^{\nu}p^k q^{\ell}$. One can easily see that the first term of the left-hand side of (3.15) yields

$$\left(q_1^{2\sigma}\frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1}\nu + \lambda\cdot(\ell - k - \alpha)\right)v^{(\alpha)}_{\nu,k,\ell}(q_1). \tag{3.17}$$

Hence we obtain the recurrence relation

$$\left(q_1^{2\sigma}\frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1}\nu + \lambda\cdot(\ell - k - \alpha)\right) v^{(\alpha)}_{\nu,k,\ell}(q_1) = F, \tag{3.18}$$

where F denotes terms which appear from the second term of the left-hand side of (3.15). (We omit the detailed inductive arguments.) In this way, one can construct the formal first integral. In fact we have

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$$\left(q_1^{2\sigma}\frac{\partial}{\partial q_1} - 2\sigma q_1^{2\sigma-1}\nu + \lambda\cdot(\ell - k - \alpha)\right) v_{\nu,k,\ell}^{(\alpha)}(q_1) = F, \tag{3.19}$$

where F denotes terms which appear from the second term of the left-hand side of (3.15). (We omit the detailed inductive arguments.) In this way, one can construct the formal first integral. In fact we have

Theorem 2. Assume (3.4), (3.5). Then the Hamiltonian system with the Hamiltonian $H = H_0 + H_1$ given either by (2.1)-(2.2) has $2(n-1)$ functionally independent formal first integrals of the form (3.7) being a polynomial of p_1 and p.

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where F denotes terms which appear from the second term of the left-hand side of (3.15). (We omit the detailed inductive arguments.) In this way, one can construct the formal first integral. In fact we have

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Remark. We have $\mathbf{2}(n-\mathbf{1})+\mathbf{1} \geq n$ if $n \geq 1$. ($\mathbf{2}$ =1+1= formal power series + exponential series, $\mathbf{1}$, resonance dimension; $\mathbf{1}$, First integral H). In case of time dependent Hamiltonians (after some reductions) we need $2(n-1)+1 \geq n+1$ for the integrability. This explains that the exponential term is natural in the resonance case.

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where F denotes terms which appear from the second term of the left-hand side of (3.15). (We omit the detailed inductive arguments.) In this way, one can construct the formal first integral. In fact we have

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Summability of formal integrals. We will study the summability of (3.7) with (3.16) constructed in Theorem 2. For every α in (3.16) we shall show the $(2\sigma-1)$- summability of $v^{(\alpha)}$. We define the set of singular directions

$$(3.22)\quad S_0 := \{z \in \mathbb{C}; \exists \nu \geq 0,\ k \geq 0,\ \ell \geq 0,\ \alpha \geq 0$$
$$(2\sigma-1)z^{2\sigma-1} + \lambda \cdot (\ell - \alpha - k) = 0;\ v^{(\alpha)}_{\nu,k,\ell} \neq 0, \ell - \alpha - k \geq 0\} \setminus 0.$$

For a neighborhood Ω_0 of the origin and the convex cone Ω_1 with vertex at the origin, we define $\Sigma_0 := \Omega_0 \cup \Omega_1$. Then we assume that there exists Σ_0 such that the closure $\overline{S_0}$ satisfies

$$(3.23)\qquad \overline{S_0} \cap \Sigma_0 = \emptyset.$$

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For a neighborhood Ω_0 of the origin and the convex cone Ω_1 with vertex at the origin, we define $\Sigma_0 := \Omega_0 \cup \Omega_1$. Then we assume that there exists Σ_0 such that the closure $\overline{S_0}$ satisfies

$$\overline{S_0} \cap \Sigma_0 = \emptyset. \tag{3.25}$$

Theorem 3. Assume (3.4), (3.5) and (3.25). Let v be a formal first integral given in Theorem 2 which is a polynomial in p and p_1. Then, for each $\alpha \geq 0$ in (3.7) $v^{(\alpha)}$ is $(2\sigma-1)$-summable in every direction of Ω_1 with respect to q_1 such that for every $\xi \in \Omega_1$ there exists a neighborhood V_0 of the origin $q = 0$ such that $v^{(\alpha)}$ is analytic in $q \in V_0$ and $(2\sigma-1)$-summable with respect to q_1 in the direction ξ.

Before proving Theorem 3 we give a corollary, in which we have global existence in q and p.

Corollary 4. Suppose (3.5). Assume

$$B_j = B_{j,0}(q_1, q), \quad 2 \le j \le n, \tag{3.26}$$

where $B_{j,0}$ is analytic at $q_1 = 0$ and polynomial in $q = (q_2, \ldots, q_n)$. Let $v = \sum_{\alpha \ge 0} v^{(\alpha)} E^\alpha$ be the formal first integral as in Theorem ?? which is a polynomial in p and p_1. Then the set of singular directions S_0 is a finite set, and for each α $v^{(\alpha)}$ is a polynomial in q and $(2\sigma - 1)$-summable with respect to q_1. More precisely, for every $\xi \notin S_0$ $v^{(\alpha)}$ is $(2\sigma - 1)$-summable with respect to q_1 in the direction ξ.

4. C^∞-integrability

Let $v = \sum_{\alpha \geq 0} v^{(\alpha)} E^\alpha$ be the first integral given by (3.7). By Theorem 3 every $v^{(\alpha)}$ is $(2\sigma - 1)$-summable in every direction of $\Omega_1 \equiv \Omega_1(v^{(\alpha)})$. Hence we write the summed one with the same letter for the sake of simplicity. We define

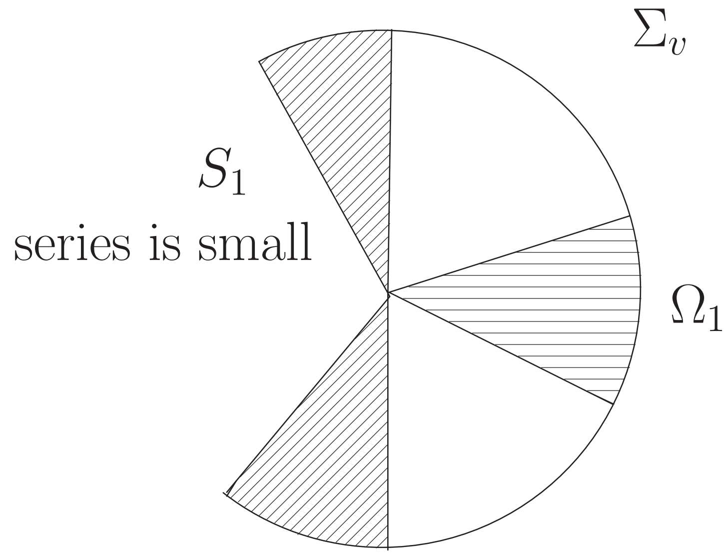
$$\Sigma_v = \left\{ z \in \mathbb{C};\ |\arg z - \arg \xi| < \frac{\pi}{2(2\sigma - 1)},\ \xi \in \Omega_1 \right\}. \tag{4.1}$$

Let e_j $(j = 2, 3, \ldots, n)$ be the j-th unit vector.For direction θ, let $R_\theta := \{t\theta; t > 0\}$ be a ray. Then we have

Theorem 5. Assume (3.4), (3.5) and (3.25). Then we have
(i) Suppose $\Omega_1(v^{(\alpha)}) \neq \emptyset$. Then there exists an $\varepsilon_0 > 0$ and a sector $S_1 \subset \Sigma_v$ such that the summed $v = v^{(\alpha)}$ in Theorem ?? is holomorphic and is the first integral of χ_H in the domain

$$q_1 \in \Sigma_v,\ |q_1| < \varepsilon_0,\ p_1 \in \mathbb{C},\ p_j \in \mathbb{C},\ |q_j| < \varepsilon_0,\ j = 2, \dots, n \tag{4.2}$$

as well as is C^∞ at $q_1 = 0$ when $q_1 \in S_1$, $q_1 \to 0$.
(ii) Assume either the Poincaré condition or that $\exists v^{(e_j)}$ and $\exists v^{(2e_j)}$ for which S_0 are finite set. Set $v = v^{(e_j)}$ or $v = v^{(2e_j)}$ and let Σ_v and $S_1 \subset \Sigma_v$ be given in (i) and let $\theta \in S_1$. Then we have $\Omega_1(v) \neq \emptyset$, and v is extended as a C^∞ first integral with respect to q_1 on $R_\theta \cup -R_\theta \cup \{0\}$ being analytic in $q \in \mathbb{R}^{n-1}$ at $q = 0$. Moreover, there exists a neighborhood of the origin U in $\mathbb{R}$ such that χ_H is C^∞–integrable when $q_1 \in (R_\theta \cup -R_\theta \cup \{0\}) \cap U$, $p_1, p_j, q_j \in \mathbb{R}$, $|q_j| < \varepsilon_0$ $(j \geq 2)$.

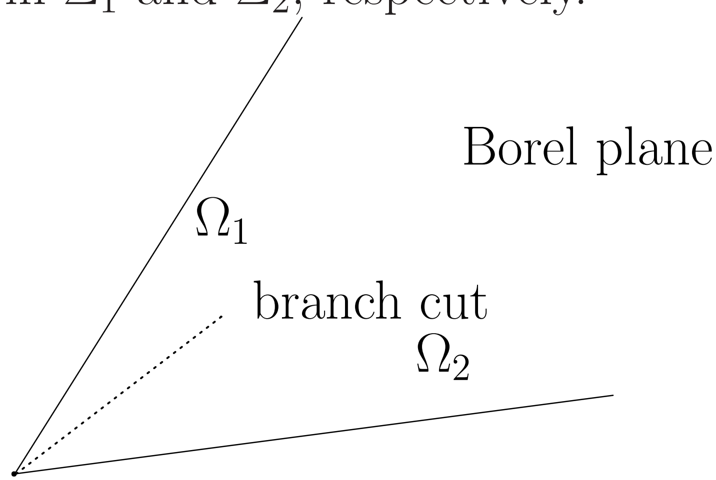

Σ_v
S_1
series is small
Ω_1

5. Connection of first integrals -example -

In this section we study first integrals obtained in Corollary 4. We assume $\sigma = 1$. We recall that the set of singular directions S_0 is a finite set. Let Ω_1 and Ω_2 be the adjacent sectors in the Borel plane and let Σ_1 and Σ_2 be the corresponding sectors in q_1 plane. Let

$$\phi := (\phi_1, \phi_2, \ldots, \phi_n), \quad \psi := (\psi_1, \psi_2, \ldots, \psi_n)$$

be summed first integals in Σ_1 and Σ_2, respectively.



By what we have proved before, ϕ and ψ are polynomials of p_1, p, q in each sector. (**cf. normalized n paremeter solutions in Balser's talk**) We recall that these Borel summed system of first integrals are constructed by the Laplace transforms of the corresponding first integrals $\hat{\phi}$ and $\hat{\psi}$ on the path C_1 and C_2.

Clearly, every direction in S_0 is the Stokes direction. We first consider (local) connection of solutions which do not contain exponential factors. ($\alpha = 0$.) In the intersecting sector $\Sigma_1 \cap \Sigma_2$ we shall look for monodromy

$$\phi(x) = \psi(x) + m(x). \tag{5.1}$$

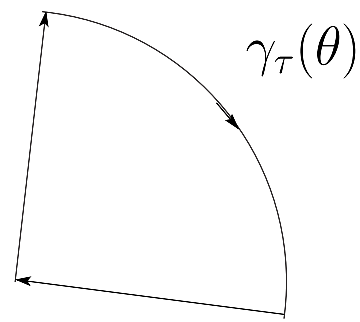
In order to study $m(x)$ we use moment Borel-Laplace summability method (for PDE). (See Balser's book for the definition and some properties. We note that we need to extend the moment functions with singularities at the origin.)

Moment summability method Let $\tau \geq 1/2$ and $\nu \in \mathbb{N}$ be given. We define kernel functions of order τ, $e(x)$ and $E(x)$ ($x \in \mathbb{C}$) by

$$e(x) := \tau x^{-2\sigma\nu} exp\left(-x^{\tau}\right), \quad E(x) := \sum_{j \geq 2\sigma\nu} x^{-j} \Gamma(\frac{j - 2\sigma\nu}{\tau}). \tag{5.2}$$

Then the moment Borel transform and moment Laplace transform are defined, respectively, by

$$\mathcal{B}(f)(z) := -\frac{1}{2\pi i}\int_{\gamma_\tau(\theta)} E(z/t)f(t)\frac{dt}{t}, \tag{5.3}$$



$$\mathcal{L}(g)(t) := \int_0^{\infty(d)} e(z/t)g(z)\frac{dz}{z}, \tag{5.4}$$

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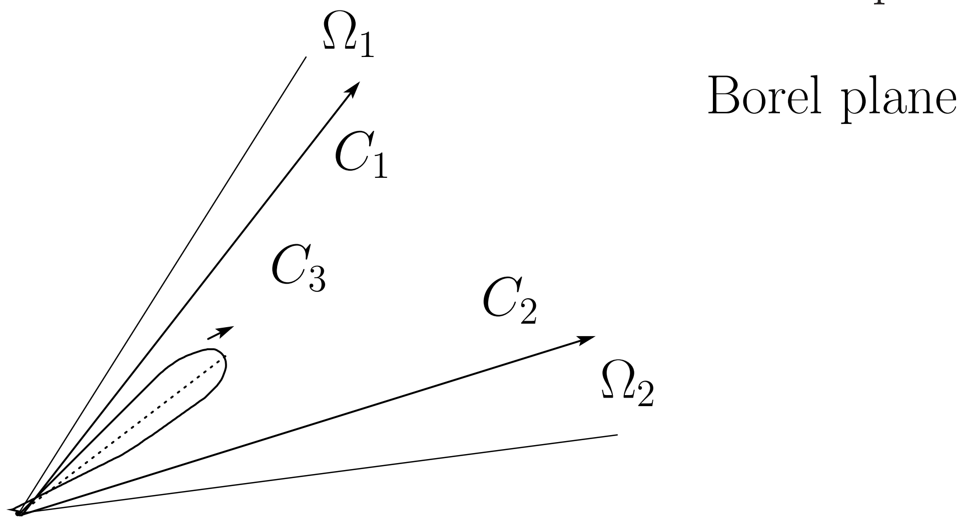
where the path of integration is the straight line in the direction d. In the following we take $\tau = 1$ and $\sigma = 1$ for simplicity.

Local monodromy For the sake of simplicity, let us, for the moment, assume that our moment Laplace transform behaves like a standard Laplace transform.



We note that $\phi(x)$ and $\psi(x)$ are constructed as the moment Laplace transforms along the paths C_1 and C_2, respectively. We deform the path C_1 such that $C_1 = C_3 + C_2$. It follows that

$$m(x) = \int_{C_3} e(z/t)\hat{\phi}(z, p_1, q, p)\frac{dz}{z}. \tag{5.5}$$

We have

Theorem 6. Assume that (M), (2.3) is verified. Then There exists an analytic vector function of one variable $\lambda(s)$ such that $m(x) = \lambda(H)$ in some neighborhood of the origin of $q_1 = 0$, $p_1 = 0$, $p = 0$, $q = 0$.

6. Proof of Theorem 3

In order to prove Theorem 3 we prepare a lemma. Let $\kappa > 0$ and $\mathcal{B}_\kappa$ denotes the Borel transform

$$(\mathcal{B}_\kappa f)(\zeta) = \frac{1}{2\pi i} \int_{\gamma_\kappa} t^\kappa f(t) \exp\left(\zeta^\kappa t^{-\kappa}\right) dt^{-\kappa}, \tag{6.1}$$

where γ_κ is an appropriate path of integration. Then, by simple computations we have

$$\mathcal{B}_\kappa(t^{\kappa+1}\frac{d}{dt} f)(\zeta) = \kappa \zeta^\kappa \mathcal{B}_\kappa(f)(\zeta) - \kappa \mathcal{B}_\kappa(t^\kappa f)(\zeta). \tag{6.2}$$

Let $c > 0$. We define $H_c(\Omega)$ as the set of all f which is holomorphic and of exponential growth of order c in Ω such that

$$\|f\|_c := \sup_{z\in\Omega} |f(z)e^{-cz^\kappa}| < \infty. \tag{6.3}$$

The space $H_c(\Omega)$ is a Banach space with the norm (6.6). We have

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The space $H_c(\Omega)$ is a Banach space with the norm (6.6). We have

Lemma 1. *Let $\lambda > 0$ be given. Then there exists $K_0 > 0$ such that*

$$\|\mathcal{B}_\kappa(t^\lambda f)\|_c \le K_0\|\mathcal{B}_\kappa(f)\|_c, \quad \mathcal{B}_\kappa(f) \in H_c(\Omega). \tag{6.5}$$

Here K_0 can be taken arbtrarily small if we take $c > 0$ sufficiently small.

For the proof we refer [1].

Let $c > 0$. We define $H_c(\Omega)$ as the set of all f which is holomorphic and of exponential growth of order c in Ω such that

$$\|f\|_c := \sup_{z\in\Omega} |f(z)e^{-cz^\kappa}| < \infty. \tag{6.6}$$

The space $H_c(\Omega)$ is a Banach space with the norm (6.6). We have

Lemma 2. *Let $\lambda > 0$ be given. Then there exists $K_0 > 0$ such that*

$$\|\mathcal{B}_\kappa(t^\lambda f)\|_c \le K_0\|\mathcal{B}_\kappa(f)\|_c, \quad \mathcal{B}_\kappa(f) \in H_c(\Omega). \tag{6.7}$$

Here K_0 can be taken arbtrarily small if we take $c > 0$ sufficiently small.

For the proof we refer [1].

Proof of Theorem 3. In view of the inductive definitions of $v^{(\alpha)}_{\nu,k,\ell}$'s with respect to ℓ, the first non-vanishing term $v^{(\alpha)}_{\nu,k,\ell}$ is a polynomial of q_1. Hence it is $(2\sigma-1)$-summable in q_1. Therefore it is sufficient to show, by induction, that if F in (3.21) is $(2\sigma-1)$- summable, then $v^{(\alpha)}_{\nu,k,\ell}$ is $(2\sigma-1)$- summable as well.

Set $\kappa = 2\sigma - 1$. In the following we omit the suffix (α) in $v^{(\alpha)}_{\nu,k,\ell}$ for the sake of simplicity. We define $\Omega = \Sigma_0$. Suppose that there exists an integer N

such that $\mathcal{B}_\kappa(v_{\nu,k,\mu}) \in H_c(\Omega)$ for all ν, k and μ, $|\mu| \leq N$. We want to show $\mathcal{B}_\kappa(v_{\nu,k,\ell}) \in H_c(\Omega)$, $|\ell| = N+1$. Let ζ be the dual variable of q_1. Let $\chi_\lambda(D)$ be defined by

$$\chi_\lambda(D)\mathcal{B}_\kappa(f)(\zeta) := \mathcal{B}_\kappa(q_1^\lambda f)(\zeta), \quad \mathcal{B}_\kappa(f) \in H_c(\Omega).$$

By Lemma 2 $\chi_\lambda(D)$ is a linear continuous operator on $H_c(\Omega)$. Moreover, by taking $c > 0$ sufficiently large, we may assume that the norm can be made sufficiently small.

We apply the $(2\sigma-1)$-Borel transform to both sides of (3.21) with respect to q_1. Then we have

(6.8)
$$\left((2\sigma-1)\zeta^{2\sigma-1} - (2\sigma(\nu+1)-1)\chi_{2\sigma-1}(D) + \lambda\cdot(\ell-k-\alpha)\right)\mathcal{B}_{2\sigma-1}(v^{(\alpha)}_{\nu,k,\ell}) = g(\zeta),$$

where $g(\zeta)$ is the partial Borel transform of F with respect to q_1. We shall show that $g(\zeta) \in H_c(\Omega)$. Indeed, in view of the definition of R in (3.11) F is the sum of products of some $v_{\nu',k',\mu}$ and holomorphic functions of q_1. This implies that their Borel transforms are in $H_c(\Omega)$. Hence we have the assertion.

We also note that the Borel transform of the differentiation $q_1^{2\sigma}(\partial/\partial q_1)$ in R is equal to $(2\sigma-1)\zeta^{2\sigma-1}-(2\sigma-1)\chi_{2\sigma-1}(D)$. In order to show that $\mathcal{B}_\kappa(v_{\nu,k,m+1}) \in H_c(\Omega)$ we may assume that $\ell-k-\alpha \neq 0$. Indeed, the number of terms satisfying $\ell - k - \alpha = 0$ is finite in view of the finiteness of k, and, by definition, the corresponding $v_{\nu,k,m+1}$ is a polynomial of q_1.

By (3.25) we see that $((2\sigma-1)\zeta^{2\sigma-1}+\lambda\cdot(\ell-k-\alpha))^{-1}$ exists for $\zeta \in \Omega$. Because one can make the norm of $\chi_{2\sigma-1}(D)$ arbitrarily small and ν runs in a finite set, it follows that $((2\sigma-1)\zeta^{2\sigma-1} - (2\sigma(\nu+1)-1)\chi_{2\sigma-1}(D) + \lambda\cdot(\ell-k-\alpha))^{-1}$ exists as a continuos operator on $H_c(\Omega)$. This proves that $\mathcal{B}_\kappa(v_{\nu,k,m+1}) \in H_c(\Omega)$ and its norm is bounded by constant times of $v_{\nu,k,\ell}$ for $\ell \leq N+1$ which are independent of ν, k and ℓ and $\zeta \in \Omega$. Hence we have proved the $(2\sigma-1)$ summability of every coefficient of our formal integral with respect to q_1 as desired. In view of the inductive estimate of $v_{\nu,k,\ell}$ with respect to $|\ell|$ we see that $v^{(\alpha)}$ is analytic with respect to q at the origin $q = 0$. $\square$

7. Proof of Theorem 5

Proof of (i). Let $v = v^{(\alpha)} E^\alpha$ be the summed first integral (3.7). We will show that every $v^{(\alpha)}$ is holomorphic in the domain (4.2). Because $v^{(\alpha)}$ is $(2\sigma-1)$-summable in every direction in Ω_1, $v^{(\alpha)}$ is holomorphic in Σ_v. Clearly, E^α is holomorphic in Σ_v. In order to show the smoothness we recall that every $v^{(\alpha)}$ is C^∞ when $q_1 \to 0$, $q_1 \in \Sigma_v$ because $v^{(\alpha)}$ has an asymptotic expansion. On the other hand, in view of

$$E^\alpha = \exp\left(\frac{q_1^{-2\sigma+1}}{2\sigma-1}\sum_{j=2}^{n}\lambda_j\alpha_j\right) \tag{7.1}$$

there exists a sum of sectors with opening $\pi/(2\sigma-1)$, on which E^α is bounded when $q_1 \to 0$. Because the opening of Σ_v is larger than $\pi/(2\sigma-1)$, it follows that there exists a sector $S_1 \subset \Sigma_v$ such that E^α is C^∞ when $q_1 \to 0$, $q_1 \in S_1$. Hence v is C^∞ when $q_1 \to 0$, $q_1 \in S_1$ as desired.

Proof of (ii). First we show that there exists Ω_1 such that $\Omega_1 \neq \emptyset$. The assertion is clear by definition if S_0 is a finite set. Suppose now that the Poincaré condition is verified. It follows that $\lambda \cdot (\ell - \alpha - k)$ is contained in some half plane in $\mathbb{C}$ for every ℓ, α, k with $\ell - \alpha - k \geq 0$. In view of the definition of S_0 one can choose Ω_1 which satisfies (3.25). Then, by (i), for $\alpha = e_j$ or $\alpha = 2e_j$ we have $2(n-1)$ C^∞ first integrals on $(R_\theta \cup -R_\theta \cup \{0\}) \cap U$, $p_1, p_j, q_j \in \mathbb{R}$, $|q_j| < \varepsilon_0$ with $U \subset \mathbb{R}$ being a neighborhood of the origin. For the sake of simplicity we denote these integrals with the same letter.

In view of the definition of S_1 and (7.1) we see that every derivative of E^α at $q_1 = 0$ when $q_1 \in R_\theta$, $q_1 \to 0$ vanishes, from which the same assertion holds for v. Hence, by defining $v = 0$ on $-R_\theta$, v can be extended as a smooth function on $R_\theta \cup -R_\theta \cup \{0\}$. In order to show the C^∞–integrability it remains to show that the $2(n-1) \geq n$ smooth first integrals are functionally independent almost everywhere. This is clear from the proof of Theorem 2 since we set $\alpha = e_j$ or $\alpha = 2e_j$. This ends the proof.

Thank you very much for your attention!

Very nice organization !!

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