Okubo's hypergeometric system

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Hypergeometric Systems

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Until further notice, assume that all λ_j are distinct. Split $A_1 = \Lambda' + A$, with

$$\Lambda' \;=\; \mathrm{diag}\,[\lambda_1', \ldots, \lambda_n']\,, \qquad A \;=\; \begin{bmatrix} 0 & a_{12} & \ldots & a_{1n} \\ a_{21} & 0 & \ldots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & 0 \end{bmatrix}$$

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Are the solutions of the hypergeometric system, or the entries in their Stokes multipliers, special functions in the weak sense? Reinhard and myself believe so!

Stokes multipliers

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$$V \;=\; \begin{bmatrix} 0 & v_{12} & \ldots & v_{1n} \\ v_{21} & 0 & \ldots & v_{2n} \\ \vdots & & \ddots & \vdots \\ v_{n1} & v_{n2} & \ldots & 0 \end{bmatrix} \;=\; V(\Lambda, \Lambda', A)\,.$$

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For fixed Λ and Λ', V is an entire function of the entries in A, hence is an entire map from the $n(n-1)$-dimensional complex vector space into itself. At "most" points, this map is locally injective [1].

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For fixed Λ and Λ', V is an entire function of the entries in A, hence is an entire map from the $n(n-1)$-dimensional complex vector space into itself. At "most" points, this map is locally injective [1]. Roughly speaking, it is also surjective, due to the Birkhoff reduction theorem [2, 4].

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Proof: For every pair (j, k) with $j \neq k$, $1 \leq j, k \leq n$, we can find a permutation matrix P so that

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Prenormalizations: From now on, let $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda'_1 = 0$.

This situation can be made to hold by means of some elementary transformations which do not change the Stokes multipliers!

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Assume that the above prenormalizations hold. Let α, β be so that

$$\alpha + \beta \;=\; \lambda_2', \quad \alpha\,\beta \;=\; -a_{12}\,a_{21}\,.$$

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Then we have

$$v \;:=\; v_{21} \;=\; 2\pi i\,\mathrm{e}^{-i\pi\lambda_2'}\;\gamma\,, \qquad \gamma \;=\; \frac{a_{21}}{\Gamma(1+\alpha)\,\Gamma(1+\beta)}\,.$$

The number γ in this formula is the relevant quantity in the asymptotic of the coefficients of a formal solution of the confluent system.

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Under the additional assumption of

$$|\lambda_j| > 1\,, \qquad 3 \le j \le n\,,$$

the function $v = v_{21}$ is holomorphic in $\lambda_3, \ldots, \lambda_n$, and remains bounded at infinity. Hence it may be expanded into a power series in the variables $w_j := \lambda_j^{-1}$.

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In this article, you also find a representation of the Stokes function as a quotient of two functions which both are holomorphic for $|w_j| < 1$, and whose coefficients can also be computed recursively.

A system of difference equations

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In [3], the following system of linear difference equations plays an important role:

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Due to the assumption $\lambda_1 = 0$, it is a system in dimension $n-1$.

The system has a formal solution which is one-summable, and the Stokes function can be expressed explicitly in terms of the sum of this formal solution!

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Open question: Who is going to write this book?

References

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