On the Exponential Parts of Singular Linear Differential Equations

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Introduction

Singular Differential Systems

$\mathrm{K} = \mathbb{C}((x)) = \mathbb{C}[[x]][x^{-1}]$ field formal Laurent series.

$$[A] \qquad \frac{d\mathbf{y}}{dx} = A(x)\mathbf{y},$$

$A(x)$ an $n \times n$ matrix with entries in K:

$$A(x) = x^{-p} \sum_{i=0}^{\infty} A_i x^i, \quad A_0 \neq 0, p = -\mathrm{val}(A) \in N^*$$

Gauge transformation: $\mathbf{y} = T\tilde{\mathbf{y}},\ T \in \mathrm{GL}(n, \mathrm{K})$, leads to

$$[B] \quad \frac{d\tilde{\mathbf{y}}}{dx} = B\tilde{\mathbf{y}}, \quad B = T[A] := T^{-1}AT - T^{-1}T'.$$

Systems $[A]$ and $[B]$ are called equivalent.

Exponential Parts

$\overline{\mathbb{K}} = \bigcup_{m \in \mathbb{N}^*} \mathbb{C}((x^{1/m}))$ field of formal Puiseux series in x.

Definition: $w \in \overline{\mathbb{K}}$ is an exponential part of $[A]$ if there exists a formal solution of the form $\exp\left(\int w\right) \mathbf{u}$ with $\mathbf{u} \in \overline{\mathbb{K}}^n \setminus \{0\}$, i.e. $(\frac{d}{dx} - A)(\mathbf{u}) = w\mathbf{u}$.

Remark. If w is an Exp. Part for $[A]$, so is $w + f'/f$, $\forall f \in \overline{\mathbb{K}}^*$

Definition: $w_1, w_2 \in \overline{\mathbb{K}}$ are *equivalent* if $w_1 - w_2 = f'/f$ for some $f \in \overline{\mathbb{K}}^*$.
Notation: $w_1 \sim w_2$.

Facts Modulo $\sim$

- Each system has n exponential parts (not necessarily distinct).
- Exponential parts are invariants under gauge transformations over $\overline{\mathbb{K}}$.
  Equivalent systems have the same exponential parts.

Exp. Parts and Formal Solutions

$[A]$ has a formal fundamental matrix solution of the form

$$\Phi(x^{1/s})x^{\Lambda}\exp\left(Q(x^{-1/s})\right)$$

$s \in \mathbb{N}^*$, $\Phi \in \mathrm{GL}(n, \mathbb{C}((x^{1/s})))$,
$Q(x^{-1/s}) = \mathrm{diag}\left(q_1(x^{-1/s}), \dots, q_n(x^{-1/s})\right)$ where the q_i's are polynomials in $x^{-1/s}$ over $\mathbb{C}$ without constant term,
Λ is a constant matrix commuting with Q.

- If w is an exponential part, then

$$w \sim \frac{dq}{dx} + \lambda/x$$

where $q \in \{q_1, \dots, q_n\}$ and λ is an eigenvalue of Λ.

Classification of singularities

- Q is invariant under all gauge transformations $T \in \mathrm{GL}(n, \overline{\mathrm{K}})$.
- When $Q(x^{-1/s}) \not\equiv 0$, the origin is called an *irregular singular point* of the system [A]. In this case the elements of $Q(x^{-1/s})$ determine the main asymptotic behavior of actual solutions as $x \to 0$ in sectors of sufficiently small angular opening (Asymptotic existence theorem (cf. Wasow)).
- If $Q(x^{-1/s}) \equiv 0$, the point $x = 0$ is called *regular singular* point of [A]. In this case $s = 1$ and the formal series $\Phi(x)$ converges whenever the series for $A(x)$ does.
- Existence of Formal Solutions goes back to Turritin, Hukuhara, Levelt, Balser-Jurkat-Lutz
- Efficient computation of Q [Barkatou 1997], [Pflügel 2000], [Barkatou-Pflügel 2007]

Example: $y''(z) = zy(z)$ (Airy Equation)

$z = \infty$ is a singular point at ∞

$$x = \frac{1}{z} \Rightarrow x^5 y'' + 2x^4 y' - y = 0$$

$$\Rightarrow \mathbf{y}' = \begin{pmatrix} 0 & 1 \\ \frac{1}{x^5} & -\frac{2}{x} \end{pmatrix} \mathbf{y}$$

Formal Solution Matrix: $Y = \Phi(x) U x^J U^{-1} e^{Q(x^{-1/2})}$

$$\Phi(x) = \ldots, \quad U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{3}{4} \end{pmatrix}$$

$$Q = \begin{pmatrix} -\frac{2}{3x^{3/2}} & 0 \\ 0 & \frac{2}{3x^{3/2}} \end{pmatrix}$$

Hence we have two exponential parts:

$$w_1 = \frac{1}{x^{5/2}} + \frac{1}{4x}, \quad w_2 = -\frac{1}{x^{5/2}} - \frac{3}{4x}$$

Hukuhara-Turritin's Normal Form

Any differential system $[A]$ is equivalent over $\overline{\mathbb{K}}$ to a system $[B]$ of the form:

$$B = \Gamma_1 x^{-k_1-1} + \Gamma_2 x^{-k_2-1} + \cdots + \Gamma_m x^{-k_m-1} + (\Gamma_0 + N)x^{-1}$$

where

(a) $k_1 > k_2 > \cdots > k_m > 0$ are rational numbers,

(b) the Γ_j's and N are constant matrices,

(c) the Γ_j's are diagonal and commute with N.

The exponential parts of $[A]$ are given by:

$$w_i \sim \Gamma_1^{(i)} x^{-k_1-1} + \Gamma_2^{(i)} x^{-k_2-1} + \cdots + \Gamma_m^{(i)} x^{-k_m-1} + \Gamma_0^{(i)} x^{-1}$$

where $\Gamma_j = \mathrm{diag}(\Gamma_j^{(1)}, \ldots, \Gamma_j^{(n)})$

What's the point of knowing the Exp. Parts?

- Local analysis near a singularity:
  - Asymptotic behavior of actual solutions
  - Summability problems,
  - Stokes multipliers, etc.
- Local data are useful for solving global problems:
  - Computing exponential solutions of systems with coefficients in $\mathbb{C}(x)$ or in an extension of $\mathbb{C}(x)$
  - Factorisation problems, Computing eigenrings
  - Computing Hom(A,B)
  - Testing the equivalence of two systems with rational function coefficients

The problem I want to discuss

To describe the relations that may exist between Exponential Parts of $[A]$ and Eigenvalues of A.

- **Application:** To calculate Exp. Parts of $[A]$ (or parts of them) from Eigenvalues of A.

- **Interest:** Eigenvalues are easier to compute than Exp. Parts.
  - Eigenvalues of A are roots in $\overline{\mathbb{K}}$ of the algebraic **scalar equation** $\chi_A(\lambda) = \det(A - \lambda I_n) = 0$.
    - Can be computed using Newton-Puiseux algorithm
  - The leading terms of the eigenvalues can be found by using *tropical* calculations.

How the exp-parts of $[A]$ do relate to the eigenvalues of A?

$\mathrm{Expp}(A)$ = set of Exp. Parts of $[A]$.

$\mathrm{Spec}(A)$ = spectrum of A.

Observation: Each Exp. Part of $[A]$ is an eigenvalue of $A + H$ for some matrix H in $\overline{\mathrm{K}}$ with $\mathrm{val}(H) > -1$.

Consequence: For all $w \in \mathrm{Expp}(A)$,

$$\max_{\alpha \in \mathrm{Spec}(A)} \mathrm{val}(w - \alpha) > k := \mathrm{val}(A) + \frac{-1 - \mathrm{val}(A)}{n}.$$

Exp. Parts and Eigenvalues of A do agree up to the order k.

BUT Property above is of no use when the order of Exp. Parts exceeds k.

Goal: Give easily checkable sufficient conditions on A in order that Exponential parts and Eigenvalues have same leading terms.

Simple Examples

• **Systems associated with a scalar differential equation**:

$$A = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{pmatrix}, \quad a_i \in \mathrm{K}.$$

Leading Terms (L. T.) of Eigenvalues of A are equal to L.T. of Exp. Parts of $[A]$.

- **The matrix A consists in a single term**:

$$A = A_0 x^{-p}, \qquad A_0 \in \mathbb{C}^{n\times n}.$$

Exp. Parts of $[A]$ are equal to the eigenvalues of A:

$$\alpha x^{-p} \qquad \alpha \in spec(A_0).$$

- **The leading matrix A_0 is not nilpotent**:

$$A = A_0 x^{-p} + O(x^{-p+1}).$$

If A_0 is not nilpotent then to each eigenvalue α of A_0 corresponds an exponential part of $[A]$ of the form

$$\alpha x^{-p} + O(x^{-p+1}).$$

In particular, Exp. Parts and Eigenvalues have the **same** order $-p = \mathrm{val}(A)$.

The Nilpotent Case: A_0 is nilpotent

1. Eigenvalues and Exp. Parts have order bigger than $\mathrm{val}(A)$.
2. They do not necessarily have the same order.

Example

$$A = \begin{pmatrix} -x^{-2} & -1 \\ -1 + x^{-4} & x^{-2} \end{pmatrix}$$

$$\mathrm{val}(A) = -4 \qquad A_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Exp. Parts of [A]:

$$w_1 = \sqrt{2}x^{-3/2} - 5/4\, x^{-1}, \quad w_2 = -\sqrt{2}x^{-3/2} - 5/4\, x^{-1}$$

Eigenvalues of A:

$$\lambda_1 = 1, \quad \lambda_2 = -1.$$

Consider the equivalent matrix

$$B := T[A] = \begin{pmatrix} -\frac{2+x^3}{x} & \frac{2+x^3}{x} \\ -\frac{x^4-1}{x^2} & x^2 \end{pmatrix} \quad \text{where} \quad T = \begin{pmatrix} x^2 & -x^2 \\ 0 & 1 \end{pmatrix}.$$

Exp. Parts of $[B]$:

$$w_1 = \sqrt{2}x^{-3/2} - 5/4\,x^{-1}, \quad w_2 = -\sqrt{2}x^{-3/2} - 5/4\,x^{-1}$$

Eigenvalues of B:

$$\alpha_1 = \sqrt{2}x^{-3/2} - x^{-1} + O\left(x^{-1/2}\right), \quad \alpha_2 = -\sqrt{2}x^{-3/2} - x^{-1} + O\left(x^{-1/2}\right)$$

Note that

$$\mathrm{val}(B) = -2 \qquad B_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Comparison of Eigenvalues and Exp. Parts

Proposition Suppose that $\mathrm{val}(A) \leq -1$. Then for each $w \in \mathrm{Expp}(A)$

$$\max_{\alpha \in spec(A)} \mathrm{val}(w - \alpha) > \mathrm{val}(A) + \frac{-1 - \mathrm{val}(A)}{n}.$$

In other words,

$$\forall w \in \mathrm{Expp}(A)\ \exists \alpha \in \mathrm{Spec}(A): \quad w = \alpha + o(x^{\mathrm{val}(A) + \frac{-1-\mathrm{val}(A)}{n}}).$$

Corollary Let $w \in \mathrm{Expp}(A)$ satisfies $\mathrm{val}(w) - \mathrm{val}(A) \leq \frac{-1 - \mathrm{val}(A)}{n}$ then $\mathrm{LT}(w) = \mathrm{LT}(\alpha)$ for some $\alpha \in \mathrm{Spec}(A)$.

Problem: It may happen that the difference

$$\min_{w \in \mathrm{Expp}(A)} \mathrm{val}(w) - \mathrm{val}(A)$$

be too big.

Solution: Replace A by an equivalent matrix B such that

$$\min_{w \in \mathrm{Expp}(B)} \mathrm{val}(w) - \mathrm{val}(B)$$

is as small as possible.

How? By using Moser reduction algorithm.

Moser Reduced Systems

Moser rank: $m(A) = \max\left(0, -\mathrm{val}(A) - 1 + \frac{rank(A_0)}{n}\right)$.

Moser invariant: $\mu(A) = \min\{m(T[A]) \mid T \in GL(n, \mathrm{K})\}$

Definition. $[A]$ is said to be Moser-reduced if $m(A) = \mu(A)$.

Theorem. [Moser 1960] If $\mathrm{val}(A) < -1$ then A is Moser-reduced iff the polynomial

$$\Theta_A(\lambda) := x^{rank(A_0)} \det(\lambda I - A_0/x - A_1)_{|x=0} \not\equiv 0.$$

Moser Algorithm: It transforms a given system into an equivalent Moser-reduced one.

Minimal Order of Exponential Parts

$$\kappa(A) = \min_{w \in \mathrm{Expp}(A)} \mathrm{val}(w).$$

- $\kappa(A) \geq \mathrm{val}(A)$ with equality iff A_0 is non-nilpotent.

- Let $D = \sum_{i=0}^{n} c_i \partial^i$, $(c_i \in \mathrm{K}, c_n = 1)$ be any operator obtained by transforming $[A]$ to a scalar equation of nth order. Then

$$\kappa(A) = \min_{0 \leq i \leq n-1} \frac{\mathrm{val}(c_i)}{n-i}.$$

An estimation of Minimal Order of Exponential Parts

Theorem Suppose A Moser-reduced and A_0 nilpotent. Then

$$\mathrm{val}(A) + \frac{1}{n-d} \leq \kappa(A) \leq \mathrm{val}(A) + 1 - \frac{r}{n-d}$$

where $r = rank(A_0)$ and $d = deg\Theta_A(\lambda)$.

Corollary With the assumptions and notation above, if $n - d - r = 1$ then

$$\kappa(A) = \mathrm{val}(A) + \frac{1}{n-d}$$

Example

$$A(x) = \frac{1}{x^4}\begin{bmatrix} 0 & 0 & x & 0 \\ 1 & -x^2 & x^2 & -x^2 \\ 0 & 1 & x^2 & 0 \\ x^2 & x^2 & 0 & -x^2 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is nilpotent and has rank $r = 2$.

$\Theta_A(\lambda) = \lambda$ is not zero and has degree $d = 1$.

The above theorem tells us:

$$-4 + 1/3 = \mathrm{val}(A) + \frac{1}{n-d} \leq \kappa(A) \leq \mathrm{val}(A) + 1 - \frac{r}{n-d} = -4 + 1 - 2/3.$$

Hence $\kappa(A) = -11/3$.

Minimal Order of Eigenvalues

Definition:

$$\rho(A) = \min_{\alpha \in \mathrm{Spec}(A)} \mathrm{val}(\alpha).$$

- $\rho(A) \geq \mathrm{val}(A)$ with equality iff A_0 is non-nilpotent.
- Let $\det(A - \lambda I_n) = \sum_{i=0}^{n} a_i \lambda^i$ be the characteristic polynomial of A. Then

$$\rho(A) = \min_{0 \leq i \leq n-1} \frac{\mathrm{val}(a_i)}{n-i}.$$

Theorem Suppose A Moser-reduced and A_0 nilpotent. Then

$$\mathrm{val}(A) + \frac{1}{n-d} \leq \rho(A) \leq \mathrm{val}(A) + 1 - \frac{r}{n-d}.$$

where $r = rank(A_0)$ and $d = deg\Theta_A(\lambda)$.

Main Results

Minimal Exponential Parts

Definition: $w \in \mathrm{Expp}(A)$ is minimal if $\mathrm{val}(w) = \kappa(A)$.

Theorem Let A be Moser-reduced, $r = rank(A_0)$, $d = deg\Theta_A(\lambda)$.
Suppose that

$$(\mathrm{C}) \qquad -\mathrm{val}(A) - 1 \geq (1 - \frac{r}{n-d})(r+1).$$

Then

(i) $\kappa(A) = \rho(A)$

(ii) If $w \in EP(A)$ is minimal then there exists $\alpha \in \mathrm{Spec}(A)$ such that $\mathrm{LT}(w) = \mathrm{LT}(\alpha)$.

Remarks

- Using a different method Hilali and Wazner (1986) showed that $\kappa(A) = \rho(A)$ under the **stronger condition**

$$-\mathrm{val}(A) - 1 \geq (r + 1).$$

That method **does not yield the result on leading terms**.

- Theorem above implies also the following old result (MB 1997): If A is Moser-reduced then L.T. of minimal Exp. Parts coincide with L.T. of minimal eigenvalues of A provided

$$-\mathrm{val}(A) - 1 > n - r.$$

- It is possible to come down to the case where the condition (C) is fullfiled.

Trick: Use a ramification $x = t^m$ with $m \geq \frac{n-r-d}{-\mathrm{val}(A)-2+r/(n-d)}$.

A Refined Result

Theorem Assume A Moser-reduced and let $r = rank(A_0)$.
If $-\mathrm{val}(A) - 1 \geq (r+1)$ then for each $w \in \mathrm{Expp}(A)$ satisfying $\mathrm{val}(w) \leq \mathrm{val}(A) + 1$ there exists $\alpha \in \mathrm{Spec}(A)$ such that $\mathrm{LT}(w) = \mathrm{LT}(\alpha)$.

Remark Using a result of Pflügel (2000) one can always *decouple* [A] into

$$A = A^{(1)} \oplus \cdots \oplus A^{(m)}$$

where each $A^{(i)}$ is Moser-reduced and all its Exp. Parts have order $\leq \mathrm{val}(A^{(i)}) + 1$.

An Example

$$A = \begin{pmatrix} -5\,x^{-2} & 5\,x^{-1} & -2\,x^{-1} & -9\,x^{-2} \\ 5\,x^{-3} & 3\,x^{-2} & 2\,x^{-2} & -4\,x^{-2} \\ 4\,x^{-1} & -6\,x^{-1} & -5\,x^{-2} & 2 \\ 2\,x^{-3}-2\,x^{-2} & -5\,x^{-1} & 3\,x^{-2} & -6\,x^{-2} \end{pmatrix}$$

$$n = 4, \quad \mathrm{val}(A) = -3, \quad r = rank(A_0) = 1, \quad d = deg\,\Theta_A(\lambda) = 2$$

$$\Theta_A(\lambda) := x^r \det\left(A_0/x + A_1 - \lambda I\right)_{|_{x=0}} = (\lambda - 3)(\lambda + 5).$$

A is Moser-reduced and $-\mathrm{val}(A) - 1 = 2 \geq 2 = r + 1$

L.T. of Exp. Parts with order ≤ -2 are equal to L. T. of Eigenvalues of A.

Exponential Parts of $[A]$:

$$(x = t, w = 3\,t^{-2} + \frac{1937}{72}\,t^{-1}), (x = t, w = -5\,t^{-2} - \frac{33}{8}\,t^{-1}),$$

$$(x = -18\,t^2, w = \frac{1}{324}\,t^{-5} - \frac{11}{648}\,t^{-4} + \frac{173}{2592}\,t^{-3} + \frac{419}{648}\,t^{-2})$$

Eigenvalues of A:

$$(x = t, \lambda = 3\,t^{-2} + \frac{1937}{72}\,t^{-1} + O(1)),$$

$$(x = t, \lambda = -5\,t^{-2} - \frac{33}{8}\,t^{-1} + O(1)),$$

$$(x = -18\,t^2, \lambda = \frac{1}{324}\,t^{-5} - \frac{11}{648}\,t^{-4} + \frac{173}{2592}\,t^{-3} + \frac{205}{324}\,t^{-2} + O(t^{-1}))$$

Tropical Computation of Eigenvalues

A Tropical Excursion

- Tropical / min-plus algebra $\mathbb{R}_{min} := \mathbb{R} \cup \{\infty\}$ equipped with

$$a \oplus b = \min(a, b) \text{ and } a \otimes b = a + b$$

  The zero element is ∞ and the unit element is 0.
- A tropical polynomial function is a function of the form

$$P_n(t) = a_n t^n \oplus a_{n-1} t^{n-1} \oplus \cdots \oplus a_1 t \oplus a_0 = \min_{0 \le k \le n} \{kt + a_k\}.$$

- Tropical roots of a polynomial $P_n(t)$ are the points where the minimum of all monomials is achieved twice.
- A tropical polynomial function can be factored uniquely as

$$P_n(t) = a_n \prod_{1 \le k \le n} (t \oplus \alpha_k) = a_n + \sum_{1 \le k \le n} \min(t, \alpha_k)$$

  where $\alpha_1, \cdots, \alpha_n$ are the tropical roots.
- The tropical roots of $P_n(t)$ can be computed in $O(n)$ time.

The valuations of the roots of a polynomial in $\mathbb{C}((x))[\lambda]$

- Let $f(x,\lambda) = \sum_{0\le k\le n} f_k(x)\lambda^k \in \mathbb{C}((x))[\lambda]$
- Define the tropical polynomial

$$P(t) = \min_{0\le k\le n} \{kt + \mathrm{val}(f_k)\}$$

- The valuations of the roots of f coincide with the tropical roots of P.
  Example:

$$f(x,\lambda) = (x^2 + O(x^2)) + (-x^3 + O(x^3))\lambda + (x^5 + O(x^5))\lambda^2$$

  $P(t) = \min(2, 3+t, 5+2t)$ has two tropical roots: -1 and -2.

  The roots of $f(x,\lambda) = 0$ are:

$$\lambda_1(x) = x^{-1} + O(1) \ \text{ and } \ \lambda_2(x) = x^{-2} + O(t^{-1})$$

Tropical eigenvalues of a matrix $M \in M_n(\mathbb{R}_{min})$

- Tropical eigenvalues of $M \in M_n(\mathbb{R}_{min})$ are the roots of

$$p(t) = per(M \oplus tI)$$

where

$$per(M) = \min_{\sigma \in S_n} \sum_{i=1}^{n} M_{i\sigma(i)}$$

- The tropical eigenvalues can be computed in $O(n^4)$ by an algorithm by Burkarda&Butkovic (2003).
Example:

$$M = \begin{pmatrix} -1 & -2 & -1 \\ \infty & -1 & -1 \\ -2 & -2 & -1 \end{pmatrix}$$

M has one one tropical eigenvalue, $-5/3$ with multiplicity 3.

Theorem (Akian, Bapat, Gaubert 2004)
Let $A = (a_{ij}) \in M_n(\mathbb{C}((x)))$ and put $M := (\mathrm{val}(a_{ij}))$. Then, the valuations of the eigenvalues of A can be computed, in non-degenerate cases, by finding the tropical eigenvalues of M.

Examples

Example 1

Let

$$A = \begin{pmatrix} -x^{-1} & 2/3\,\frac{x-1}{x^2} & \frac{16}{9}\,x^{-1} \\ 0 & -x^{-1} & 4/3\,x^{-1} \\ 9/4\,x^{-2} & 3/4\,\frac{1+x}{x^2} & -x^{-1} \end{pmatrix}.$$

The associated valuation matrix is

$$M = \begin{pmatrix} -1 & -2 & -1 \\ \infty & -1 & -1 \\ -2 & -2 & -1 \end{pmatrix}$$

It has has one tropical eigenvalue, $-5/3$ with multiplicity 3.
$A(x)$ has three eigenvalues of order $x^{-5/3}$.
The exponential parts of $\frac{d\mathbf{y}}{dx} = A(x)\mathbf{y}$

$$(x = -1/2\,t^3, w = 4\,t^{-5} - 10/3\,t^{-4} + 8/3\,t^{-3}).$$

Example 2

$$A = \begin{pmatrix} \frac{x-1}{x} & -\frac{x-1}{x^3} & -\frac{3+x}{x} & \frac{-2\,x^2+x^4+x^3+1}{x^2} \\ 0 & -\frac{2\,x+1}{x^2} & 0 & x^{-2} \\ \frac{x+1}{x} & -x^{-2} & -\frac{x+4}{x} & \frac{x^4+x^3+1}{x^2} \\ x^{-3} & 0 & 0 & -\frac{-1+3\,x}{x^2} \end{pmatrix}$$

The associated valuation matrix is

$$M = \begin{pmatrix} -1 & -3 & -1 & -2 \\ \infty & -2 & \infty & -2 \\ -1 & -2 & -1 & -2 \\ -3 & \infty & \infty & -2 \end{pmatrix}$$

The tropical tropical eigenvalues of M are:

$$[-\frac{8}{3}, 3], \quad [-1, 1]$$

The eigenvalues of $A(x)$ are of order:

$$x^{-8/3}, \quad x^{-1}$$

The exponential parts of $\frac{d\mathbf{y}}{dx} = A(x)\mathbf{y}$ are:

$$(x = t^3, w = t^{-8} + 1/3\, t^{-7} - \frac{1}{81}\, t^{-5} + \frac{82}{243}\, t^{-4} - 7/3\, t^{-3})$$

$$(x = t, w = -4\, t^{-1})$$

Example 3

$$A = \begin{pmatrix} -x^{-1} & \frac{8\,x^2+5-2\,x}{x^3} & \frac{1+2\,x^4+2\,x^3}{x^3} & -\frac{-1+8\,x^2}{x^3} \\ 0 & \frac{-1+3\,x}{x^2} & x+1 & -4\,x^{-1} \\ -1 & \frac{3\,x^3+x^4+1}{x^3} & \frac{x+1}{x^3} & \frac{2+x}{x^3} \\ \frac{1+x^5}{x^4} & -\frac{2\,x^2+1}{x} & 0 & \frac{x-1}{x} \end{pmatrix}$$

The associated valuation matrix is

$$M = \begin{pmatrix} -1 & -3 & -3 & -3 \\ \infty & -2 & 0 & -1 \\ 0 & -3 & -3 & -3 \\ -4 & -1 & \infty & -1 \end{pmatrix}$$

The tropical tropical eigenvalues of M are:

$$[-\frac{7}{2}, 2], \quad [-3, 1] \quad [-2, 1]$$

The eigenvalues of $A(x)$ are of order:

$$x^{-7/2}, \quad x^{-3}, \quad x^{-2}$$

The exponential parts of $\frac{d\mathbf{y}}{dx} = A(x)\mathbf{y}$ are:

$$(x = t^2, w = t^{-7} + t^{-6} - 1/2\, t^{-5} + 3/2\, t^{-4} - \frac{49}{8}\, t^{-3} + 9/4\, t^{-2})$$

$$(x = t, w = -t^{-3} - 2\, t^{-2} + 9\, t^{-1})$$

$$(x = t, -13\, w = -t^{-2} + t^{-1})$$

▶ The computation of tropical eigenvalues has been implemented in Maple by Georg Regenburger.

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