Semi-classical orthogonal polynomials and the Painlevé equations

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GOOD PRACTICE
SCHEME

Outline

1. Introduction
2. Orthogonal polynomials
3. Some properties of the **fourth Painlevé equation**

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q} \qquad \mathrm{P_{IV}}$$

4. Relationship between Painlevé equations and orthogonal polynomials
   - **Semi-classical orthogonal polynomials**

$$\begin{aligned} &\omega(x;t) = x^\nu \exp(-x^2 + tx), && x \in \mathbb{R}^+, \quad t \in \mathbb{R}, \quad \nu > -1 \\ &\omega(x;t) = x^\nu \exp(-x^2 + tx), && x, t \in \mathbb{R}, \quad \nu > -1 \\ &\omega(x;t) = |x|^{2\nu-1} \exp\left(-\tfrac{1}{4}x^4 - tx^2\right), && x, t \in \mathbb{R}, \quad \nu > 0 \end{aligned}$$

   - **Discrete orthogonal polynomials**

$$\omega(k;t) = \frac{t^k}{(\beta)_k\, k!}, \qquad \omega(k;t) = \frac{(\alpha)_k\, t^k}{(\beta)_k\, k!}$$

5. Some more examples
6. Conclusions

References

- **P A Clarkson & K Jordaan**, “The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation", *Constr. Approx.*, to appear [arXiv:1301.4134]
- **P A Clarkson**, “Recurrence coefficients for discrete orthonormal polynomials and the Painlevé equations", *J. Phys. A* **46** (2013) 185205

Some History

- The relationship between semi-classical orthogonal polynomials and integrable equations dates back to **Shohat [1939]**, **Freud [1976]**, **Bonan & Nevai [1984]**.
- Subsequently **Fokas, Its & Kitaev [1991, 1992]** identified these integrable equations as **discrete Painlevé equations**.
- **Magnus [1995]** considered the **Freud weight**

$$\omega(x;t) = \exp\left(-\tfrac{1}{4}x^4 - tx^2\right), \qquad x,t \in \mathbb{R},$$

and showed that the coefficients in the three-term recurrence relation can be expressed in terms of solutions of

$$q_n(q_{n-1} + q_n + q_{n+1}) + 2tq_n = n$$

which is discrete $\mathrm{P_I}$ ($\mathrm{dP_I}$), as shown by **Bonan & Nevai [1984]**, and

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}z^2} = \frac{1}{2q_n}\left(\frac{\mathrm{d}q_n}{\mathrm{d}z}\right)^2 + \frac{3}{2}q_n^3 + 4zq_n^2 + 2(z^2 + \tfrac{1}{2}n)q_n - \frac{n^2}{2q_n}$$

which is $\mathrm{P_{IV}}$ with $A = -\frac{1}{2}n$ and $B = -\frac{1}{2}n^2$.

- **Filipuk, van Assche & Zhang [2012]** commented

> "*We note that for classical orthogonal polynomials (Hermite, Laguerre, Jacobi) one knows these recurrence coefficients explicitly in contrast to non-classical weights*".

Painlevé Equations

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = 6q^2 + z \qquad \mathrm{P_I}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = 2q^3 + zq + A \qquad \mathrm{P_{II}}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{Aq^2+B}{z} + Cq^3 + \frac{D}{q} \qquad \mathrm{P_{III}}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2-A)q + \frac{B}{q} \qquad \mathrm{P_{IV}}$$

$$\begin{aligned}\frac{\mathrm{d}^2q}{\mathrm{d}z^2} &= \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{(q-1)^2}{z^2}\left(Aq + \frac{B}{q}\right) \\ &\quad + \frac{Cq}{z} + \frac{Dq(q+1)}{q-1}\end{aligned} \qquad \mathrm{P_V}$$

$$\begin{aligned}\frac{\mathrm{d}^2q}{\mathrm{d}z^2} &= \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-z}\right)\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{q-z}\right)\frac{\mathrm{d}q}{\mathrm{d}z} \\ &\quad + \frac{q(q-1)(q-z)}{z^2(z-1)^2}\left\{A + \frac{Bz}{q^2} + \frac{C(z-1)}{(q-1)^2} + \frac{Dz(z-1)}{(q-z)^2}\right\}\end{aligned} \qquad \mathrm{P_{VI}}$$

with A, B, C and D arbitrary constants.

Painlevé σ-Equations

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - 2\sigma = 0 \qquad \mathrm{S_I}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}\beta^2 \qquad \mathrm{S_{II}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2} - \frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - 2\sigma\right) + 4z\vartheta_\infty\frac{\mathrm{d}\sigma}{\mathrm{d}z} = z^2\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - 2\sigma + 2\vartheta_0\right) \qquad \mathrm{S_{III}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S_{IV}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - \left[2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - z\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right]^2 + 4\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \kappa_j\right) = 0 \qquad \mathrm{S_V}$$

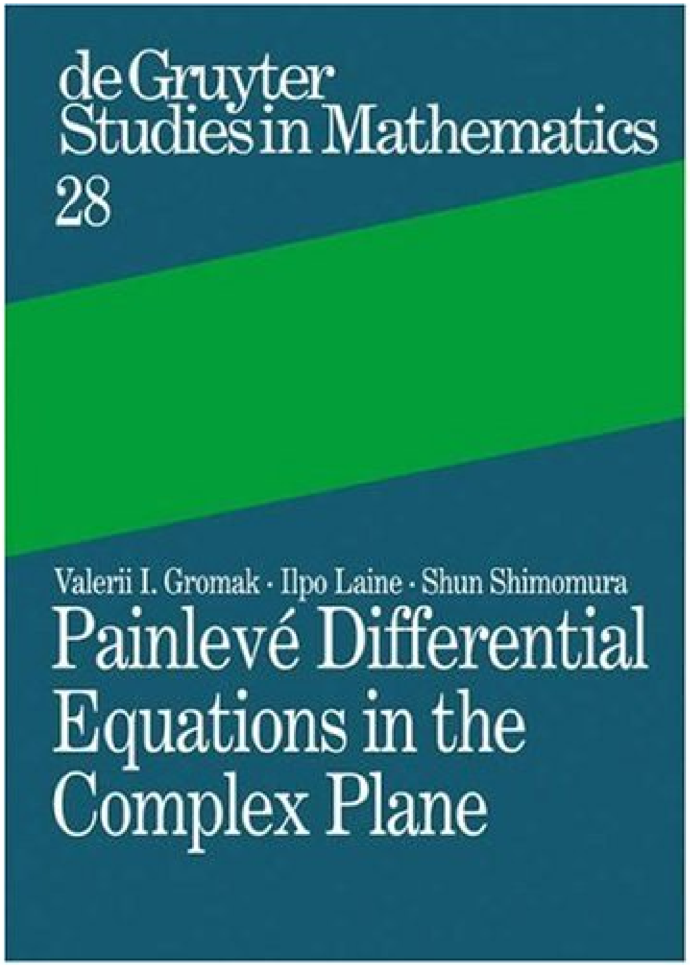
$$\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left[z(z-1)\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right]^2 + \left[\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left\{2\sigma - (2z-1)\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right\} + \kappa_1\kappa_2\kappa_3\kappa_4\right]^2 = \prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \kappa_j^2\right) \qquad \mathrm{S_{VI}}$$

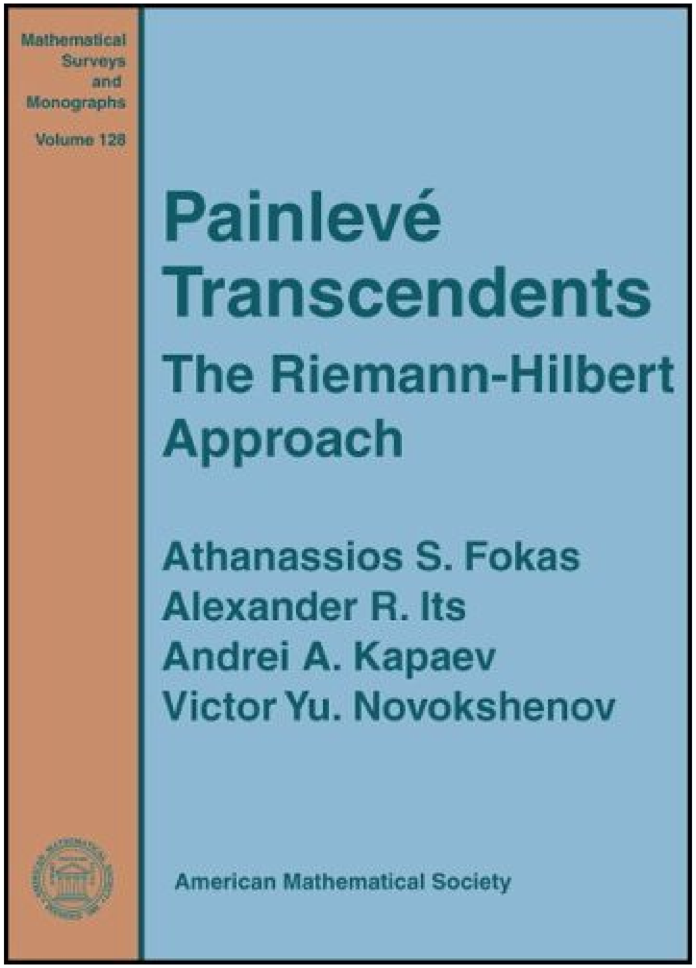
where β, $\vartheta_0, \vartheta_\infty$ and $\kappa_1, \ldots, \kappa_4$ are arbitrary constants.

Theorem

The Painlevé equations are special functions

Proof





Motivation

Monic orthogonal polynomials satisfy the three-term recurrence relation of the form

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

- Recurrence coefficients for the weight

  $$\omega(x;t) = \exp\left(\tfrac{1}{3}x^3 + tx\right), \qquad x^3 < 0$$

  are expressed in terms of solutions of $\mathrm{P_{II}}$ (**Magnus [1995]**).
- Recurrence coefficients for the weight

  $$\omega(x;t) = x^{\nu-1}\exp(-x - t/x), \qquad x \in \mathbb{R}, \quad \nu > 0$$

  are expressed in terms of solutions of $\mathrm{P_{III}}$ (**Chen & Its [2010]**).
- Recurrence coefficients for the weight

  $$\omega(x;t) = x^{\nu-1}\exp\left(-x^2 + tx\right), \qquad x \in \mathbb{R}^+, \quad \nu > 0$$

  are expressed in terms of solutions of $\mathrm{P_{IV}}$ (**Filipuk, van Assche & Zhang [2011]**).
- Recurrence coefficients for the weights

  $$\begin{aligned}
  \omega(x;t) &= (1-x)^a(1+x)^b\exp(-tx), && x \in [-1,1], && a,b>0\\
  \omega(x;t) &= x^a(1-x)^b\exp(-t/x), && x \in [0,1], && a,b>0\\
  \omega(x;t) &= x^{a-1}(x+t)^{b-1}\mathrm{e}^{-x}, && x \in \mathbb{R}^+, && a,b>0
  \end{aligned}$$

  are expressed in terms of solutions of $\mathrm{P_V}$ (**Basor, Chen & Ehrhardt [2010]**, **Chen & Dai [2010]**, **Chen & McKay [2012]**, **Forrester & Witte [2007]**).

Orthogonal Polynomials

- Monic orthogonal polynomials
- Semi-classical orthogonal polynomials

Monic Orthogonal Polynomials

Let $P_n(x)$, $n = 0, 1, 2, \ldots$, be the **monic orthogonal polynomials** of degree n in x, with respect to the positive weight $\omega(x)$, such that

$$\int_a^b P_m(x)P_n(x)\,\omega(x)\,\mathrm{d}x = h_n\delta_{m,n}, \quad h_n > 0, \qquad m, n = 0, 1, 2, \ldots$$

One of the important properties that orthogonal polynomials have is that they satisfy the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

where the recurrence coefficients are given by

$$\alpha_n = \frac{\widetilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\widetilde{\Delta}_n}{\Delta_n}, \qquad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}$$

with

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}, \qquad \widetilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}$$

and $\mu_k = \displaystyle\int_a^b x^k\,\omega(x)\,\mathrm{d}x$ are the **moments** of the weight $\omega(x)$.

Further Properties

- The Hankel determinant

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-2} \end{vmatrix}, \qquad \mu_k = \int_a^b x^k \, \omega(x) \, \mathrm{d}x$$

also has the integral representation

$$\Delta_n = \frac{1}{n!} \int_a^b \overset{n}{\cdots} \int_a^b \prod_{\ell=1}^{n} \omega(x_\ell) \prod_{1 \le j < k \le n} (x_j - x_k)^2 \, \mathrm{d}x_1 \dots \mathrm{d}x_n, \qquad n \ge 1$$

- The monic polynomials $P_n(x)$ can be uniquely expressed as

$$P_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}$$

- The normalization constants can be expressed as

$$h_n = \frac{\Delta_{n+1}}{\Delta_n}, \qquad h_0 = \Delta_1 = \mu_0$$

Example — Hermite polynomials

Hermite polynomials are orthogonal with respect to the weight

$$\omega(x) = \exp(-x^2), \qquad x \in \mathbb{R}$$

In this case

$$\mu_{2k} = \int_{-\infty}^{\infty} x^{2k} \exp(-x^2)\,\mathrm{d}x = \frac{\sqrt{\pi}\,(2k)!}{2^{2k}\,k!}, \qquad \mu_{2k+1} = \int_{-\infty}^{\infty} x^{2k+1} \exp(-x^2)\,\mathrm{d}x = 0$$

so

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} = \left(\tfrac{1}{2}\right)^{n(n-1)/2} \prod_{k=1}^{n-1} (k!), \qquad \widetilde{\Delta}_n = 0$$

and therefore

$$\alpha_n = 0, \qquad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2} = \tfrac{1}{2}n$$

which gives the three-term recurrence relation

$$P_{n+1}(x) = xP_n(x) - \tfrac{1}{2}nP_{n-1}(x)$$

where

$$P_n(x) = 2^{-n}H_n(x)$$

with $H_n(x)$ the **Hermite polynomial**.

Example — Associated Laguerre polynomials

Associated Laguerre polynomials are orthogonal with respect to the weight

$$\omega(x) = x^{\nu} \exp(-x), \qquad x \in \mathbb{R}^{+}, \quad \nu > -1$$

In this case

$$\mu_k = \int_0^{\infty} x^{k+\nu} \exp(-x)\,\mathrm{d}x = \Gamma(k+\nu+1)$$

so

$$\Delta_n = \prod_{j=1}^{n} (j-1)!\,\Gamma(\nu+j), \qquad \widetilde{\Delta}_n = n(n+\nu) \prod_{j=1}^{n} (j-1)!\,\Gamma(\nu+j)$$

and therefore

$$\alpha_n = \frac{\widetilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\widetilde{\Delta}_n}{\Delta_n} = 2n+1+\nu, \qquad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2} = n(n+\nu)$$

which gives the three-term recurrence relation

$$P_{n+1}(x) = (x-2n-1-\nu)P_n(x) - n(n+\nu)P_{n-1}(x)$$

where

$$P_n(x) = (-1)^n n! L_n^{(\nu)}(x)$$

with $L_n^{(\nu)}(x)$ the **associated Laguerre polynomial**.

Semi-classical Orthogonal Polynomials

Consider the **Pearson equation** satisfied by the weight $\omega(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sigma(x)\omega(x)] = \tau(x)\omega(x)$$

- **Classical orthogonal polynomials**: $\sigma(x)$ and $\tau(x)$ are polynomials with $\deg(\sigma) \leq 2$ and $\deg(\tau) = 1$

	$\omega(x)$	$\sigma(x)$	$\tau(x)$
Hermite	$\exp(-x^2)$	1	$-2x$
Associated Laguerre	$x^\nu \exp(-x)$	x	$1+\nu-x$
Jacobi	$(1-x)^\alpha(1+x)^\beta$	$1-x^2$	$\beta-\alpha-(2+\alpha+\beta)x$

- **Semi-classical orthogonal polynomials**: $\sigma(x)$ and $\tau(x)$ are polynomials with either $\deg(\sigma) > 2$ or $\deg(\tau) > 1$

	$\omega(x)$	$\sigma(x)$	$\tau(x)$
semi-classical Laguerre	$x^\nu \exp(-x^2+tx)$	x	$1+\nu+tx-2x^2$
Freud	$\exp(-\frac{1}{4}x^4-tx^2)$	1	$-2tx-x^3$

If the weight has the form

$$\omega(x;t) = \omega_0(x)\exp(tx)$$

where $\omega_0(x)$ is a classical weight with finite moments and $\displaystyle\int_{-\infty}^{\infty} x^k \omega_0(x)\exp(tx)\,\mathrm{d}x$ exists for all $k \geq 0$. Then:

- the recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ satisfy the **Toda system**

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = \beta_{n+1} - \beta_n, \qquad \frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n(\alpha_n - \alpha_{n-1})$$

- the kth moment is given by

$$\mu_k(t) = \int_{-\infty}^{\infty} x^k \omega_0(x)\exp(tx)\,\mathrm{d}x = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\int_{-\infty}^{\infty} \omega_0(x)\exp(tx)\,\mathrm{d}x\right) = \frac{\mathrm{d}^k \mu_0}{\mathrm{d}t^k}$$

• Since $\mu_k(t) = \dfrac{\mathrm{d}^k\mu_0}{\mathrm{d}t^k}$, then $\Delta_n(t)$ and $\widetilde{\Delta}_n(t)$ can be expressed as Wronskians

$$\Delta_n(t) = \begin{vmatrix} \mu_0(t) & \mu_1(t) & \dots & \mu_{n-1}(t) \\ \mu_1(t) & \mu_2(t) & \dots & \mu_n(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(t) & \mu_n(t) & \dots & \mu_{2n-2}(t) \end{vmatrix} = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \dots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right)$$

$$\begin{aligned} \widetilde{\Delta}_n(t) &= \begin{vmatrix} \mu_0(t) & \mu_1(t) & \dots & \mu_{n-2}(t) & \mu_n(t) \\ \mu_1(t) & \mu_2(t) & \dots & \mu_{n-1}(t) & \mu_{n+1}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1}(t) & \mu_n(t) & \dots & \mu_{2n-3}(t) & \mu_{2n-1}(t) \end{vmatrix} \\ &= \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \dots, \frac{\mathrm{d}^{n-2}\mu_0}{\mathrm{d}t^{n-2}}, \frac{\mathrm{d}^{n}\mu_0}{\mathrm{d}t^{n}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \dots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right) \end{aligned}$$

$$\Rightarrow \qquad \boxed{\frac{\widetilde{\Delta}_n(t)}{\Delta_n(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\ln\mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \dots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right)}$$

• the Hankel determinant $\Delta_n(t)$ satisfies the **Toda equation**

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t) = \frac{\Delta_{n-1}(t)\Delta_{n+1}(t)}{\Delta_n^2(t)}$$

Some Properties of the Fourth Painlevé Equation and the Fourth Painlevé σ-Equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q} \qquad \mathrm{P_{IV}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S_{IV}}$$

- **Hamiltonian Representation**
- **Parabolic Cylinder Function Solutions**

Hamiltonian Representation of P_{IV}

P_{IV} can be written as the **Hamiltonian system**

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathcal{H}_{\mathrm{IV}}}{\partial p} = 4qp - q^2 - 2zq - 2\vartheta_0$$

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathcal{H}_{\mathrm{IV}}}{\partial q} = -2p^2 + 2pq + 2zp - \vartheta_\infty$$

where $\mathcal{H}_{\mathrm{IV}}(q,p,z;\vartheta_0,\vartheta_\infty)$ is the Hamiltonian defined by

$$\mathcal{H}_{\mathrm{IV}}(q,p,z;\vartheta_0,\vartheta_\infty) = 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q$$

Eliminating p then q satisfies

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \tfrac{3}{2}q^3 + 4zq^2 + 2(z^2 + \vartheta_0 - 2\vartheta_\infty - 1)q - \frac{2\vartheta_0^2}{q}$$

which is P_{IV} with $A = 1 - \vartheta_0 + 2\vartheta_\infty$ and $B = -2\vartheta_0^2$, whilst eliminating q then p satisfies

$$\frac{\mathrm{d}^2p}{\mathrm{d}z^2} = \frac{1}{2p}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + 6p^3 - 8zp^2 + 2(z^2 - 2\vartheta_0 + \vartheta_\infty + 1)p - \frac{\vartheta_\infty^2}{2p}$$

and letting $p = -\tfrac{1}{2}q$ gives P_{IV} with $A = 2\vartheta_0 - \vartheta_\infty - 1$ and $B = -2\vartheta_\infty^2$.

Theorem (**Okamoto [1986]**)

The function

$$\sigma(z;\vartheta_0,\vartheta_\infty) = \mathcal{H}_{\rm IV} \equiv 2qp^2 - (q^2+2zq+2\vartheta_0)p + \vartheta_\infty q$$

where q and p satisfy the Hamiltonian system

$$\frac{\mathrm{d}q}{\mathrm{d}z} = 4qp - q^2 - 2zq - 2\vartheta_0, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -2p^2 + 2pq + 2zp - \vartheta_\infty \qquad \mathrm{H}_{\rm IV}$$

satisfies the second-order, second-degree equation

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S}_{\rm IV}$$

Conversely, if $\sigma(z;\vartheta_0,\vartheta_\infty)$ is a solution of $\mathrm{S}_{\rm IV}$, then

$$q(z;\vartheta_0,\vartheta_\infty) = \frac{\sigma'' - 2z\sigma' + 2\sigma}{2(\sigma' + 2\vartheta_\infty)}, \qquad p(z;\vartheta_0,\vartheta_\infty) = \frac{\sigma'' + 2z\sigma' - 2\sigma}{4(\sigma' + 2\vartheta_0)}$$

are solutions of the Hamiltonian system $\mathrm{H}_{\rm IV}$.

Parabolic Cylinder Function Solutions of $\mathrm{P_{IV}}$

Theorem

Suppose $\tau_{\nu,n}(z;\varepsilon)$ *is given by*

$$\tau_{\nu,n}(z;\varepsilon) = \mathcal{W}\left(\varphi_\nu(z;\varepsilon), \varphi_\nu'(z;\varepsilon), \ldots, \varphi_\nu^{(n-1)}(z;\varepsilon)\right), \qquad n \geq 1$$

where $\tau_{\nu,0}(z;\varepsilon) = 1$ *and* $\varphi_\nu(z;\varepsilon)$ *satisfies*

$$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\varphi_\nu = 0, \qquad \varepsilon^2 = 1$$

Then solutions of $\mathrm{P_{IV}}$

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q}$$

are given by

$$q^{[1]}_{\nu,n}(z) = -2z + \varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu,n+1}(z;\varepsilon)}{\tau_{\nu,n}(z;\varepsilon)}, \qquad (A_1, B_1) = \big(\varepsilon(2n-\nu), -2(\nu+1)^2\big)$$

$$q^{[2]}_{\nu,n}(z) = \varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu,n+1}(z;\varepsilon)}{\tau_{\nu+1,n}(z;\varepsilon)}, \qquad (A_2, B_2) = \big(-\varepsilon(n+\nu), -2(\nu-n+1)^2\big)$$

$$q^{[3]}_{\nu,n}(z) = -\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu+1,n}(z;\varepsilon)}{\tau_{\nu,n}(z;\varepsilon)}, \qquad (A_3, B_3) = \big(\varepsilon(2\nu-n+1), -2n^2\big)$$

Parabolic Cylinder Function Solutions of S_{IV}

Theorem

Suppose $\tau_{\nu,n}(z;\varepsilon)$ *is given by*

$$\tau_{\nu,n}(z;\varepsilon) = \mathcal{W}\left(\varphi_\nu(z;\varepsilon), \varphi_\nu'(z;\varepsilon), \ldots, \varphi_\nu^{(n-1)}(z;\varepsilon)\right), \qquad n \geq 1$$

where $\tau_{\nu,0}(z;\varepsilon) = 1$ *and* $\varphi_\nu(z;\varepsilon)$ *satisfies*

$$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\varphi_\nu = 0, \qquad \varepsilon^2 = 1$$

Then solutions of S_{IV}

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0$$

are given by

$$\sigma_{\nu,n}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_{\nu,n}(z;\varepsilon), \qquad (\vartheta_0, \vartheta_\infty) = \big(\varepsilon(\nu - n + 1), -\varepsilon n\big)$$

$$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\varphi_\nu = 0, \qquad \varepsilon^2 = 1 \tag{$*$}$$

- **If** $\nu \notin \mathbb{Z}$

$$\varphi_\nu(z;\varepsilon) = \begin{cases} \left\{C_1 D_\nu(\sqrt{2}\,z) + C_2 D_\nu(-\sqrt{2}\,z)\right\} \exp\left(\tfrac{1}{2}z^2\right), & \text{if} \quad \varepsilon = 1 \\ \left\{C_1 D_{-\nu-1}(\sqrt{2}\,z) + C_2 D_{-\nu-1}(-\sqrt{2}\,z)\right\} \exp\left(-\tfrac{1}{2}z^2\right), & \text{if} \quad \varepsilon = -1 \end{cases}$$

- **If** $\nu = n \in \mathbb{Z}$**, with** $n \geq 0$

$$\varphi_n(z;\varepsilon) = \begin{cases} C_1 H_n(z) + C_2 \exp(z^2)\dfrac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{\operatorname{erfi}(z)\exp(-z^2)\right\}, & \text{if} \quad \varepsilon = 1 \\ C_1 H_n(\mathrm{i}z) + C_2 \exp(-z^2)\dfrac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{\operatorname{erfc}(z)\exp(z^2)\right\}, & \text{if} \quad \varepsilon = -1 \end{cases}$$

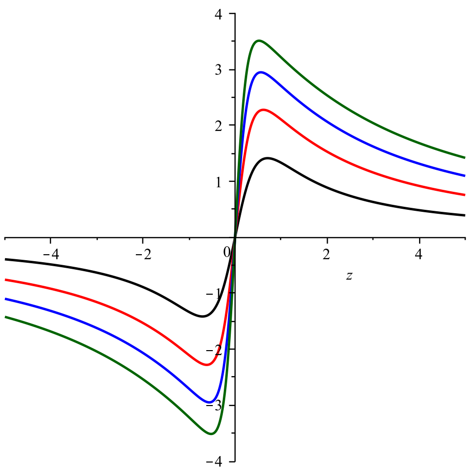
- **If** $\nu = -n \in \mathbb{Z}$**, with** $n \geq 1$

$$\varphi_{-n}(z;\varepsilon) = \begin{cases} C_1 H_{n-1}(\mathrm{i}z)\exp(z^2) + C_2\dfrac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left\{\operatorname{erfc}(z)\exp(z^2)\right\}, & \text{if} \quad \varepsilon = 1 \\ C_1 H_{n-1}(z)\exp(-z^2) + C_2\dfrac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left\{\operatorname{erfi}(z)\exp(-z^2)\right\}, & \text{if} \quad \varepsilon = -1 \end{cases}$$

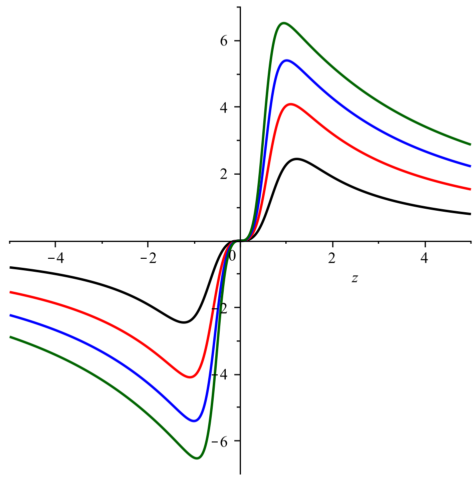
with C_1 and C_2 arbitrary constants, $D_\nu(\zeta)$ the **parabolic cylinder function**, $H_n(z)$ the **Hermite polynomial**, $\operatorname{erfc}(z)$ the **complementary error function** and $\operatorname{erfi}(z)$ the **imaginary error function**.

Plots of Bounded Rational Solutions of S_{IV}

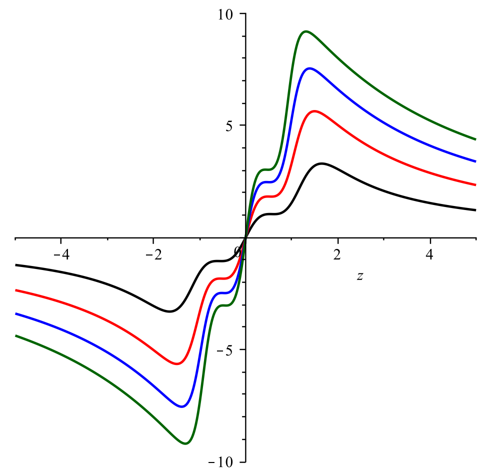
$$\sigma_{m,n}(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \mathcal{W}\big(H_m(z), H_{m+1}(z), \ldots, H_{m+n-1}(z)\big)$$



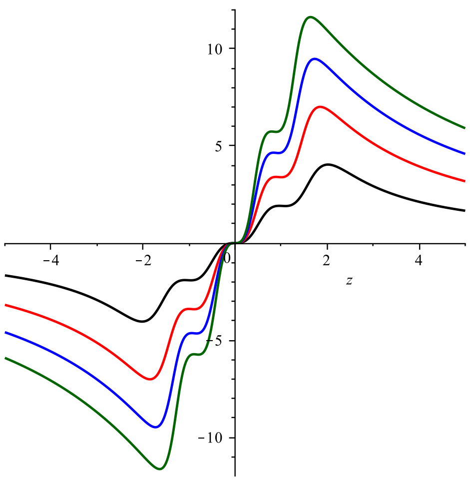
$\sigma_{1,2j}(z), \quad j = 1,2,3,4$



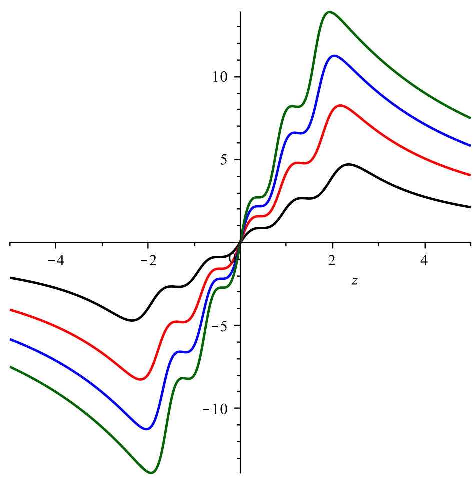
$\sigma_{2,2j}(z), \quad j = 1,2,3,4$



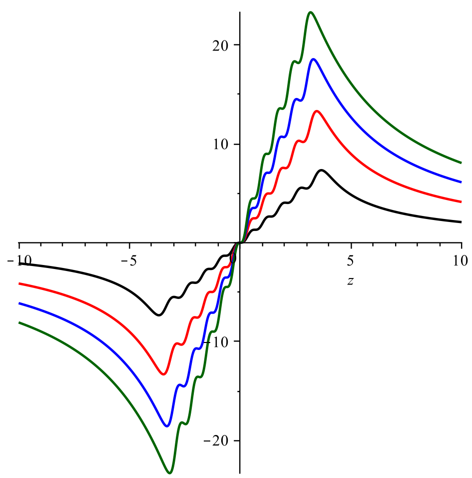
$\sigma_{3,2j}(z), \quad j = 1,2,3,4$



$\sigma_{4,2j}(z), \quad j = 1,2,3,4$



$\sigma_{5,2j}(z), \quad j = 1,2,3,4$

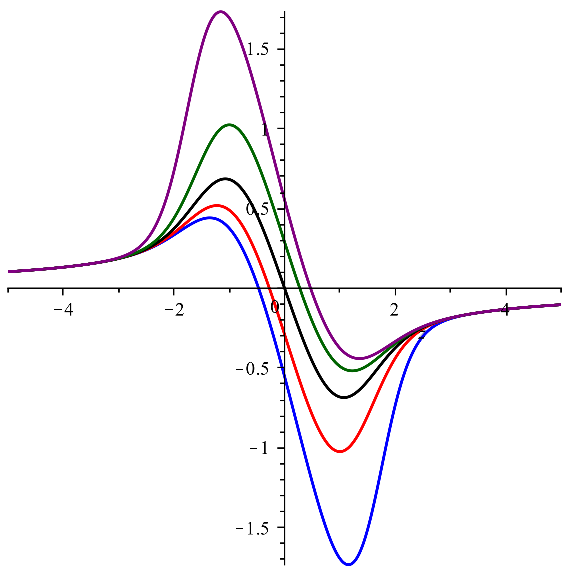


$\sigma_{10,2j}(z), \quad j = 1,2,3,4$

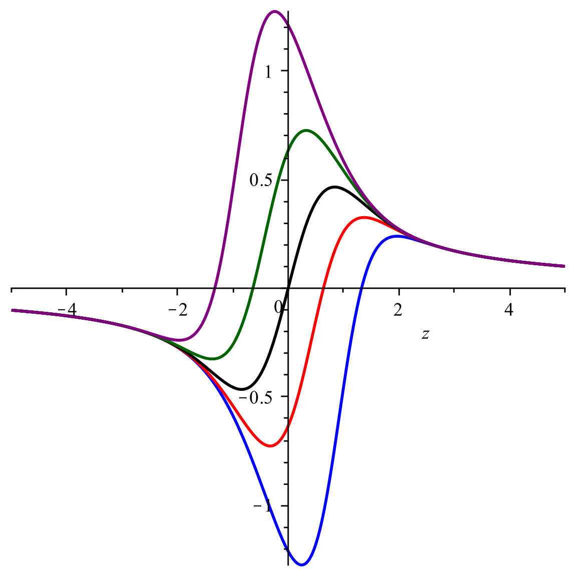
Plots of Bounded Special Function Solutions of S_{IV}

$$\sigma_{\nu,n}(z) = -2nz + \frac{\mathrm{d}}{\mathrm{d}z}\ln \mathcal{W}\big(\varphi_\nu, \varphi_\nu', \ldots, \varphi_\nu^{(n-1)}\big)$$

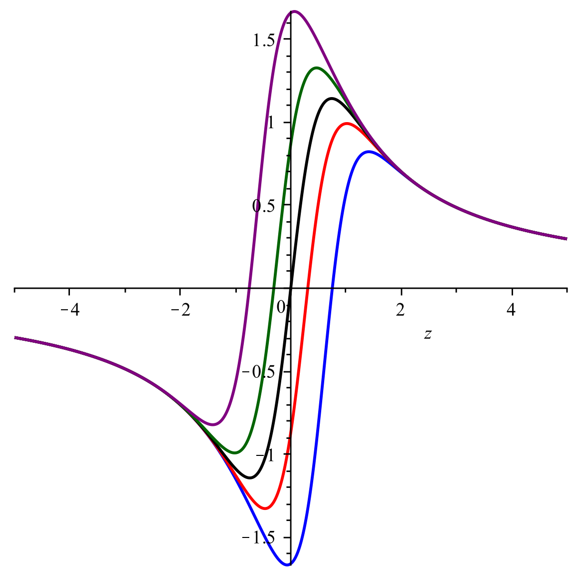
$$\varphi_\nu(z) = \Big\{C_1 D_{-\nu}\big(\sqrt{2}\,z\big) + C_2 D_{-\nu}\big(-\sqrt{2}\,z\big)\Big\}\exp\left(\tfrac{1}{2}z^2\right), \quad C_1C_2>0,\ \ \nu>0$$



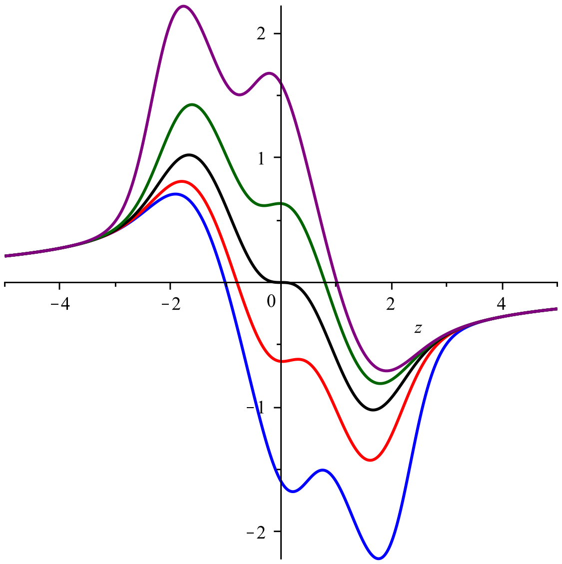
$\sigma_{1/2,1}(z)$



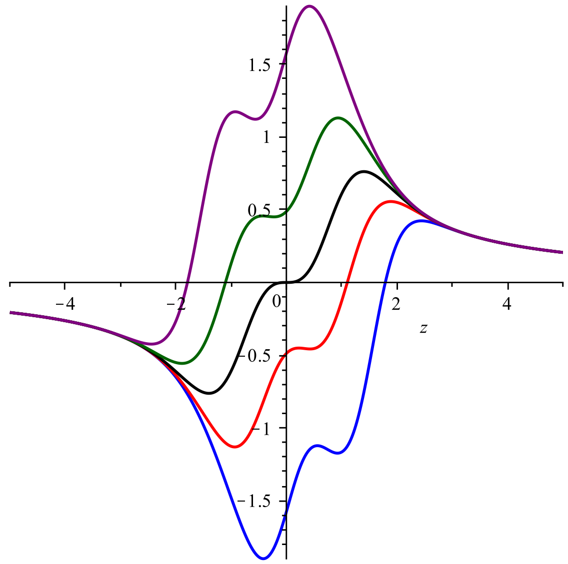
$\sigma_{3/2,1}(z)$



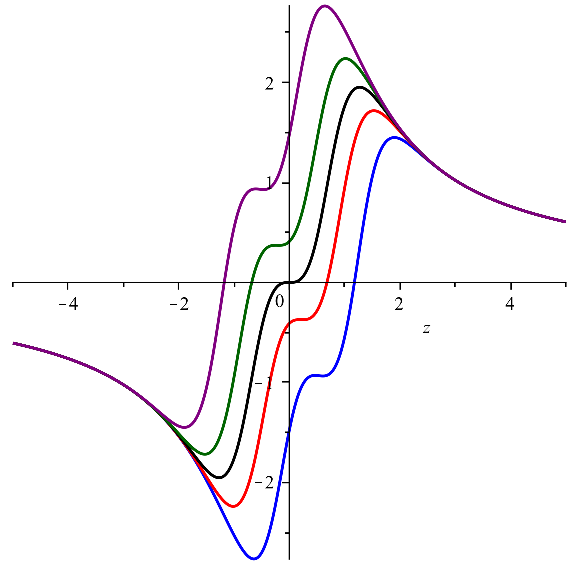
$\sigma_{5/2,1}(z)$



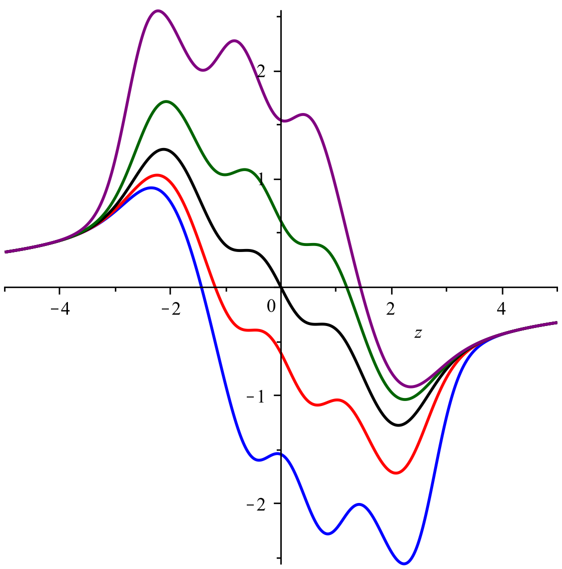
$\sigma_{1/2,2}(z)$



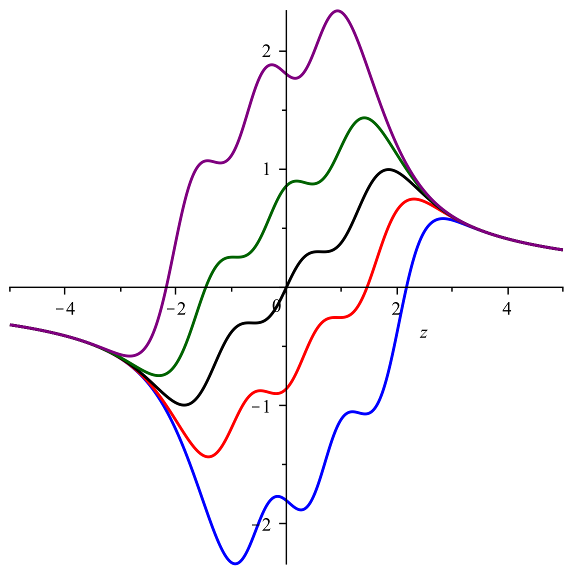
$\sigma_{3/2,2}(z)$



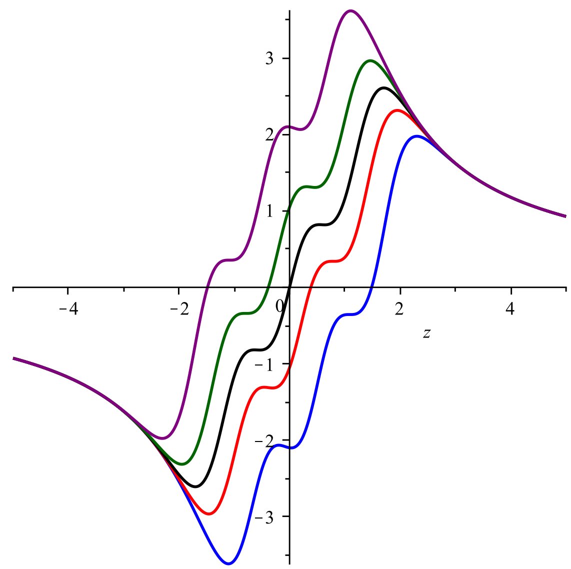
$\sigma_{5/2,2}(z)$



$\sigma_{1/2,3}(z)$



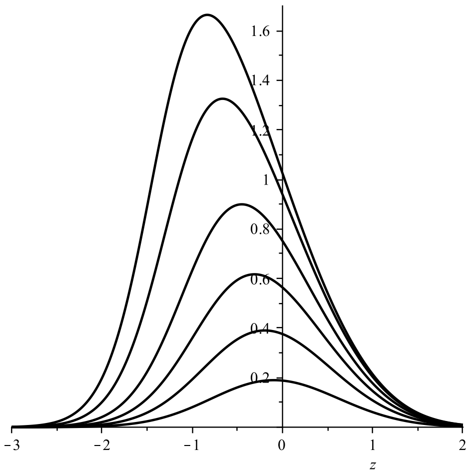
$\sigma_{3/2,3}(z)$



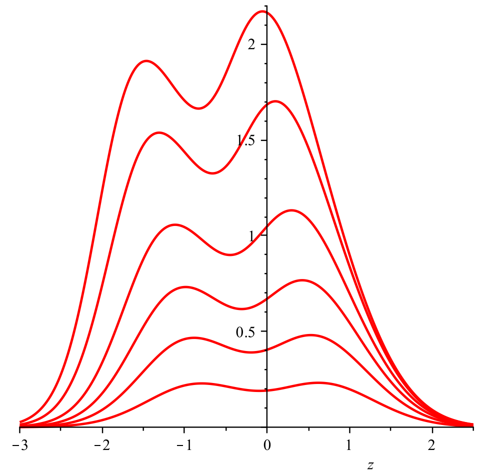
$\sigma_{5/2,3}(z)$

Plots of Error Function Solutions of S_{IV}

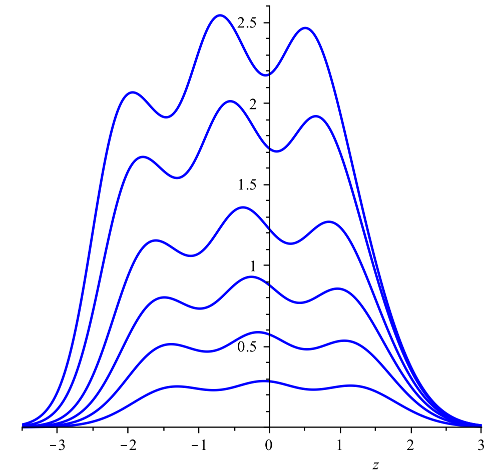
$$\sigma_{m,n} = \frac{\mathrm{d}}{\mathrm{d}z} \ln \mathcal{W}\big(\varphi_m, \varphi_m', \ldots, \varphi_m^{(n-1)}\big), \qquad \varphi_m = \exp(-z^2)\frac{\mathrm{d}^m}{\mathrm{d}z^m}\left\{C_1 + C_2\,\mathrm{erfc}(z)\right\}\exp(z^2)$$



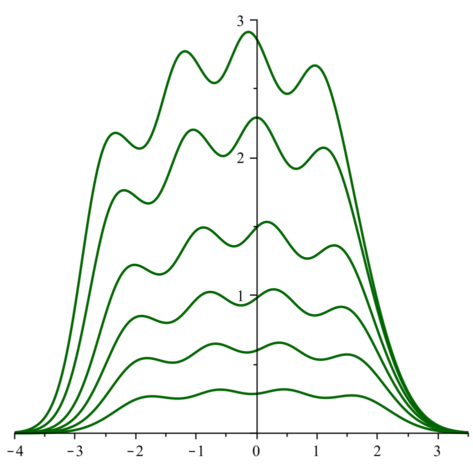
$\sigma_{1,0}(z)$



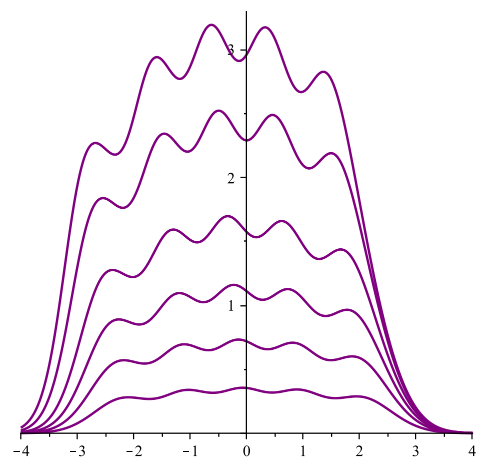
$\sigma_{2,1}(z)$



$\sigma_{3,2}(z)$



$\sigma_{4,3}(z)$



$\sigma_{5,4}(z)$

Semi-classical Laguerre Weight

$$\omega(x;t) = x^{\nu} \exp(-x^2 + tx), \qquad x \in \mathbb{R}^+, \qquad \nu > -1$$

- **P A Clarkson & K Jordaan**, "The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation", *Constr. Approx.*, to appear [arXiv:1301.4134]

Semi-classical Laguerre weight

Consider monic orthogonal polynomials with respect to the **semi-classical Laguerre weight**

$$\omega(x;t) = x^\nu \exp(-x^2 + tx), \qquad x \in \mathbb{R}^+, \qquad \nu > -1 \tag{1}$$

which satisfy the three-term recurrence relation

$$xP_n(x;t) = P_{n+1}(x;t) + \alpha_n(t)P_n(x;t) + \beta_n(t)P_{n-1}(x;t) \tag{2}$$

Theorem

(**Boelen & van Assche [2011]**)

The coefficients $\alpha_n(t)$ and $\beta_n(t)$ in the three-term recurrence relation (2) associated with the semi-classical Laguerre weight (1)

$$(2\alpha_n - t)(2\alpha_{n-1} - t) = \frac{(2\beta_n - n)(2\beta_n - n - \nu)}{\beta_n}$$

$$2\beta_n + 2\beta_{n+1} + \alpha_n(2\alpha_n - t) = 2n + 1 + \nu$$

Theorem

The coefficients $\alpha_n(t)$ and $\beta_n(t)$ in the three-term recurrence relation (2) associated with the semi-classical Laguerre weight (1) satisfy the Toda system

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = \beta_{n+1} - \beta_n, \qquad \frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n(\alpha_n - \alpha_{n-1})$$

$$\omega(x;t) = x^{\nu}\exp(-x^2 + tx), \qquad x \in \mathbb{R}^+, \qquad \nu > -1 \tag{1}$$

$$xP_n(x;t) = P_{n+1}(x;t) + \alpha_n(t)P_n(x;t) + \beta_n(t)P_{n-1}(x;t) \tag{2}$$

Theorem (**Filipuk, van Assche & Zhang [2012]**)

The coefficient $\alpha_n(t)$ in the recurrence relation (2) associated with the semi-classical Laguerre weight (1) is given by

$$\alpha_n(t) = \tfrac{1}{2}q_n(\tfrac{1}{2}t) + \tfrac{1}{2}t$$

where $q_n(z)$ satisfies

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}z^2} = \frac{1}{2q_n}\left(\frac{\mathrm{d}q_n}{\mathrm{d}z}\right)^2 + \tfrac{3}{2}q_n^3 + 4zq_n^2 + 2(z^2 - 2n - 1 - \nu)q_n - \frac{2\nu^2}{q_n} \tag{3}$$

which is $\mathrm{P_{IV}}$ *with parameters*

$$(A, B) = \left(2n + 1 + \nu, -2\nu^2\right) \tag{4}$$

Remarks:

- **Filipuk, van Assche & Zhang [2012]** did **not** specify the specific solution of (3).
- The parameters (4) satisfy the condition for $\mathrm{P_{IV}}$ to have solutions expressible in terms of **parabolic cylinder functions**.

Theorem (**PAC & Jordaan [2013]**)

For the semi-classical Laguerre weight

$$\omega(x;t) = x^{\nu}\exp(-x^2+tx)$$

the moment $\mu_0(t;\nu)$ is given by

$$\mu_0(t;\nu) = \begin{cases} \dfrac{\Gamma(\nu+1)\exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}}\, D_{-\nu-1}\left(-\frac{1}{2}\sqrt{2}\,t\right), & \textit{if} \quad \nu \neq n \in \mathbb{N} \\ \frac{1}{2}\sqrt{\pi}\,\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}\left\{\exp\left(\frac{1}{4}t^2\right)\left[1+\operatorname{erf}(\frac{1}{2}t)\right]\right\}, & \textit{if} \quad \nu = n \in \mathbb{N} \end{cases}$$

with $D_{\nu}(\zeta)$ the **parabolic cylinder function** *and* $\operatorname{erf}(z)$ *the* **error function**.

Proof. The **parabolic cylinder function** $D_{\nu}(\zeta)$ has the integral representation

$$D_{\nu}(\zeta) = \frac{\exp(-\frac{1}{4}\zeta^2)}{\Gamma(-\nu)}\int_0^{\infty} s^{-\nu-1}\exp(-\tfrac{1}{2}s^2-\zeta s)\,\mathrm{d}s$$

Hence for the semi-classical Laguerre weight the moment $\mu_0(t;\nu)$ is given by

$$\begin{aligned} \mu_0(t;\nu) &= \int_0^{\infty} x^{\nu}\exp(-x^2+tx)\,\mathrm{d}x \\ &= 2^{-(\nu+1)/2}\int_0^{\infty} s^{\nu}\exp\left(-\tfrac{1}{2}s^2+\tfrac{1}{2}\sqrt{2}\,t\,s\right)\,\mathrm{d}s \\ &= \frac{\Gamma(\nu+1)\exp\left(\frac{1}{8}t^2\right)}{2^{(\nu+1)/2}}\, D_{-\nu-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right) \end{aligned}$$

If $\nu = n \in \mathbb{N}$, then

$$D_{-n-1}(\zeta) = \sqrt{\frac{\pi}{2}} \frac{(-1)^n}{n!} \exp(-\tfrac{1}{4}\zeta^2) \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \left\{ \exp(\tfrac{1}{2}\zeta^2) \operatorname{erfc}\left(\tfrac{1}{2}\sqrt{2}\,\zeta\right) \right\},$$

with $\operatorname{erfc}(z)$ the complementary error function. Since $\operatorname{erfc}(-z) = 1 + \operatorname{erf}(z)$, then

$$\mu_0(t;n) = \tfrac{1}{2}\sqrt{\pi}\, \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left\{ \exp\left(\tfrac{1}{4}t^2\right) \left[1 + \operatorname{erf}(\tfrac{1}{2}t)\right] \right\}$$

Corollary

The moment $\mu_0(t;\nu)$ satisfies the ordinary differential equation

$$\frac{\mathrm{d}^2\mu_0}{\mathrm{d}t^2} - \tfrac{1}{2}t\frac{\mathrm{d}\mu_0}{\mathrm{d}t} - \tfrac{1}{2}(\nu+1)\mu_0 = 0 \tag{1}$$

Proof. The parabolic cylinder function $D_\nu(\zeta)$ satisfies

$$\frac{\mathrm{d}^2 D_\nu}{\mathrm{d}\zeta^2} + \left(\nu + \tfrac{1}{2} - \tfrac{1}{4}\zeta^2\right) D_\nu = 0$$

and so it follows from its definition that the moment $\mu_0(t;\nu)$ satisfies equation (1).

Theorem (PAC & Jordaan [2013])

The coefficients $\alpha_n(t)$ and $\beta_n(t)$ in the three-term recurrence relation

$$xP_n(x;t) = P_{n+1}(x;t) + \alpha_n(t)P_n(x;t) + \beta_n(t)P_{n-1}(x;t)$$

associated with the semi-classical Laguerre weight

$$\omega(x;t) = x^\nu \exp(-x^2 + tx), \qquad x \in \mathbb{R}^+, \qquad \nu > -1$$

are given by

$$\alpha_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)}, \qquad \beta_n(t) = \frac{\Delta_{n+1}(t)\Delta_{n-1}(t)}{\Delta_n^2(t)}, \qquad n \geq 0$$

where $\Delta_n(t)$ is the Hankel determinant given by

$$\Delta_n(t) = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right), \qquad n \geq 1$$

$\Delta_0(t) = 1$ *and* $\Delta_{-1}(t) = 0$, *with*

$$\mu_0(t;\nu) = \begin{cases} \dfrac{\Gamma(\nu+1)\exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}} D_{-\nu-1}\left(-\frac{1}{2}\sqrt{2}\,t\right), & \textit{if} \quad \nu \neq n \in \mathbb{N} \\ \frac{1}{2}\sqrt{\pi}\,\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}\left\{\exp\left(\frac{1}{4}t^2\right)\left[1 + \operatorname{erf}(\frac{1}{2}t)\right]\right\}, & \textit{if} \quad \nu = n \in \mathbb{N} \end{cases}$$

$D_\nu(\zeta)$ *the parabolic cylinder function and* $\operatorname{erf}(z)$ *the error function.*

Remarks:

- The Hankel determinant $\Delta_n(t)$ satisfies the **Toda equation**

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t) = \frac{\Delta_{n-1}(t)\Delta_{n+1}(t)}{\Delta_n^2(t)}$$

and the fourth-order, bi-linear equation

$$\Delta_n\frac{\mathrm{d}^4\Delta_n}{\mathrm{d}t^4} - 4\frac{\mathrm{d}^3\Delta_n}{\mathrm{d}t^3}\frac{\mathrm{d}\Delta_n}{\mathrm{d}t} + 3\left(\frac{\mathrm{d}^2\Delta_n}{\mathrm{d}t^2}\right)^2 - (\tfrac{1}{4}t^2+4n+2\nu)\left\{\Delta_n\frac{\mathrm{d}^2\Delta_n}{\mathrm{d}t^2} - \left(\frac{\mathrm{d}\Delta_n}{\mathrm{d}t}\right)^2\right\} + \tfrac{1}{4}t\Delta_n\frac{\mathrm{d}\Delta_n}{\mathrm{d}t} + \tfrac{1}{2}n(n+\nu)\Delta_n^2 = 0$$

- The function $S_n(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}\ln\Delta_n(t)$ satisfies

$$4\left(\frac{\mathrm{d}^2S_n}{\mathrm{d}t^2}\right)^2 - \left(t\frac{\mathrm{d}S_n}{\mathrm{d}t} - S_n\right)^2 + 4\frac{\mathrm{d}S_n}{\mathrm{d}t}\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t} - n\right)\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t} - n - \nu\right) = 0$$

which is equivalent to S_{IV}, the P_{IV} σ-equation (let $S_n(t) = \frac{1}{2}\sigma(z)$, with $z = 2t$), so

$$\alpha_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_{n+1}(t)}{\Delta_n(t)} = S_{n+1}(t) - S_n(t), \qquad \beta_n(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t) = \frac{\mathrm{d}S_n}{\mathrm{d}t}$$

Theorem

(PAC & Jordaan [2013])

If $\alpha_n(t)$ and $\beta_n(t)$ satisfy the system

$$\begin{aligned}\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} &= -\alpha_n(\alpha_n - \tfrac{1}{2}t) - 2\beta_n + \tfrac{1}{2}(2n+1+\nu)\\ \frac{\mathrm{d}\beta_n}{\mathrm{d}t} &= (\alpha_n - \tfrac{1}{2}t)\beta_n - \frac{(2\beta_n - n)(2\beta_n - n - \nu)}{2(2\alpha_n - t)}\end{aligned} \tag{1}$$

then

$$S_n(t) = 2\alpha_n(t)\beta_n(t) + \frac{[2\beta_n(t) - n][2\beta_n(t) - n - \nu]}{2\alpha_n(t) - t}$$

satisfies

$$4\left(\frac{\mathrm{d}^2 S_n}{\mathrm{d}t^2}\right)^2 - \left(t\frac{\mathrm{d}S_n}{\mathrm{d}t} - S_n\right)^2 + 4\frac{\mathrm{d}S_n}{\mathrm{d}t}\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t} - n\right)\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t} - n - \nu\right) = 0 \tag{2}$$

which is equivalent to $\mathrm{S_{IV}}$, *the* $\mathrm{P_{IV}}$ σ*-equation.*

Conversely if $S_n(t)$ satisfies equation (2) then

$$\alpha_n(t) = \frac{2\dfrac{\mathrm{d}^2 S_n}{\mathrm{d}t^2} + t\dfrac{\mathrm{d}S_n}{\mathrm{d}t} + S_n}{4\dfrac{\mathrm{d}S_n}{\mathrm{d}t}}, \qquad \beta_n(t) = \frac{\mathrm{d}S_n}{\mathrm{d}t}$$

are solutions of the system (1).

Theorem

The system

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = -\alpha_n(\alpha_n - \tfrac{1}{2}t) - 2\beta_n + \tfrac{1}{2}(2n+1+\nu) \tag{1a}$$

$$\frac{\mathrm{d}\beta_n}{\mathrm{d}t} = (\alpha_n - \tfrac{1}{2}t)\beta_n - \frac{(2\beta_n - n)(2\beta_n - n - \nu)}{2(2\alpha_n - t)} \tag{1b}$$

is equivalent to the system

$$\frac{\mathrm{d}q_n}{\mathrm{d}z} = 4q_n p_n - q_n^2 - 2zq_n - 2\nu \tag{2a}$$

$$\frac{\mathrm{d}p_n}{\mathrm{d}z} = -2p_n^2 + 2p_n q_n + 2zp_n - n - \nu \tag{2b}$$

which is the Hamiltonian system associated with $\mathrm{P_{IV}}$, *with Hamiltonian*

$$\mathcal{H}_{\mathrm{IV}}(q_n, p_n, z; n, \nu) = 2q_n p_n^2 - (q_n^2 + 2zq_n + 2\nu)p_n + (n+\nu)q_n$$

Proof. Making the transformation

$$\alpha_n(t) = \tfrac{1}{2}q_n(z) + \tfrac{1}{2}t, \qquad \beta_n(t) = -\tfrac{1}{2}q_n(z)p_n(z) + \tfrac{1}{2}(n+\nu), \qquad z = \tfrac{1}{2}t$$

in the system (1) yields the system (2). The inverse transformation is

$$q_n(z) = 2\alpha_n(t) - t, \qquad p_n(z) = -\frac{2\beta_n(t) - n - \nu}{2\alpha_n(t) - t}, \qquad t = 2z$$

The first few coefficients in the recurrence relation are given by

$$\alpha_0(t) = \tfrac{1}{2}t - \frac{D_{-\nu}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)}{D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)} \equiv \Psi_\nu(t)$$

$$\alpha_1(t) = \tfrac{1}{2}t - \Psi_\nu(t) - \frac{\Psi_\nu(t)}{2\Psi_\nu^2(t) - t\Psi_\nu(t) - \nu - 1}$$

$$\begin{aligned}\alpha_2(t) = \tfrac{1}{2}t + \frac{2\nu+4}{t} &+ \frac{\Psi_\nu(t)}{2\Psi_\nu^2(t) - t\Psi_\nu(t) - \nu - 1}\\
&- \frac{2[(\nu+1)t^2 + 4(\nu+2)(2\nu+3)]\Psi_\nu^2(t) - (\nu+1)t[t^2+2(4\nu+9)]\Psi_\nu(t)}{2t\big[2t\Psi_\nu^3(t) - (t^2-4\nu-6)\Psi_\nu^2(t) - 3(\nu+1)t\Psi_\nu(t) - 2(\nu+1)^2\big]}\\
&+ \frac{(\nu+1)^2[t^2+8(\nu+2)]}{2t\big[2t\Psi_\nu^3(t) - (t^2-4\nu-6)\Psi_\nu^2(t) - 3(\nu+1)t\Psi_\nu(t) - 2(\nu+1)^2\big]}\end{aligned}$$

$$\beta_1(t) = -\Psi_\nu^2(t) + \tfrac{1}{2}t\Psi_\nu(t) + \tfrac{1}{2}(\nu+1),$$

$$\beta_2(t) = -\frac{2t\Psi_\nu^3(t) - (t^2-4\nu-6)\Psi_\nu^2(t) - 3(\nu+1)t\Psi_\nu(t) - 2(\nu+1)^2}{2\Big[\Psi_\nu^2(t) - \tfrac{1}{2}t\Psi_\nu(t) - \tfrac{1}{2}(\nu+1)\Big]^2}$$

Hence, using the three-term recurrence relation

$$P_{n+1}(x;t) = [x - \alpha_n(t)]P_n(x;t) - \beta_n(t)P_{n-1}(x;t), \qquad n = 0, 1, 2, \ldots$$

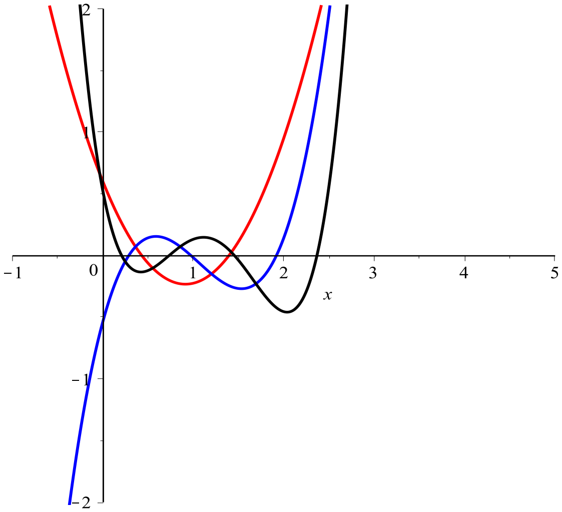
with $P_0(x;t) = 1$ and $P_{-1}(x;t) = 0$, then the first few polynomials are given by

$$P_1(x;t) = x - \Psi_\nu$$

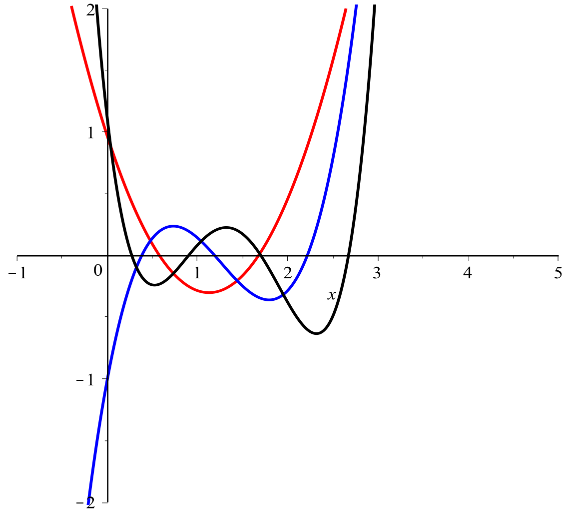
$$P_2(x;t) = x^2 - \frac{2t\Psi_\nu^2 - (t^2+2)\Psi_\nu - (\nu+1)t}{2\left[\Psi_\nu^2 - \frac{1}{2}t\Psi_\nu - \frac{1}{2}(\nu+1)\right]}\,x - \frac{2(\nu+2)\Psi_\nu^2 - (\lambda+1)\Psi_\nu - (\lambda+1)^2}{2\left[\Psi_\nu^2 - \frac{1}{2}t\Psi_\nu - \frac{1}{2}(\nu+1)\right]}$$

$$\begin{aligned}
P_3(x;t) = x^3 &- \left\{\frac{4(t^2+2\nu+4)\Psi_\nu^3 - 2t(t^2-\nu-1)\Psi_\nu^2}{2\left[2t\Psi_\nu^3 - (t^2-4\nu-6)\Psi_\nu^2 - 3(\nu+1)t\Psi_\nu - 2(\nu+1)^2\right]}\right.\\
&\qquad \left. - \frac{(\nu+1)(5t^2+4\lambda+6)\Psi_\nu + 3(\nu+1)^2 t}{2\left[2t\Psi_\nu^3 - (t^2-4\nu-6)\Psi_\nu^2 - 3(\nu+1)t\Psi_\nu - 2(\nu+1)^2\right]}\right\} x^2\\
&+ \left\{\frac{2t(t^2+2\nu+4)\Psi_\nu^3 - \left[t^4 + 4(2\nu+5)(\nu+2)\right]\Psi_\nu^2}{4\left[2t\Psi_\nu^3 - (t^2-4\nu-6)\Psi_\nu^2 - 3(\nu+1)t\Psi_\nu - 2(\nu+1)^2\right]}\right.\\
&\qquad \left. - \frac{2(\nu+1)t(t^2-\nu-5)\Psi_\nu + (\nu+1)^2(t^2-4\nu-12)}{4\left[2t\Psi_\nu^3 - (t^2-4\nu-6)\Psi_\nu^2 - 3(\nu+1)t\Psi_\nu - 2(\nu+1)^2\right]}\right\} x\\
&+ \frac{2\left[(\nu+1)t^2 + 4(\nu+2)^2\right]\Psi_\nu^3 - (\nu+1)t(t^2+2\nu+8)\Psi_\nu^2}{4\left[2t\Psi_\nu^3 - (t^2-4\nu-6)\Psi_\nu^2 - 3(\nu+1)t\Psi_\nu - 2(\nu+1)^2\right]}\\
&\qquad - \frac{2(\nu+1)^2(t^2+2\nu+5)\Psi_\nu + (\nu+1)^3 t}{4\left[2t\Psi_\nu^3 - (t^2-4\nu-6)\Psi_\nu^2 - 3(\nu+1)t\Psi_\nu - 2(\nu+1)^2\right]}
\end{aligned}$$

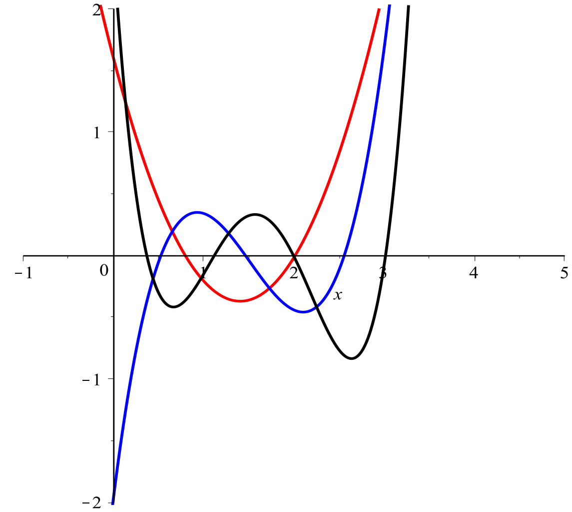
$$\color{red}{P_2(x;t)} \quad \color{blue}{P_3(x;t)} \quad P_4(x;t)$$



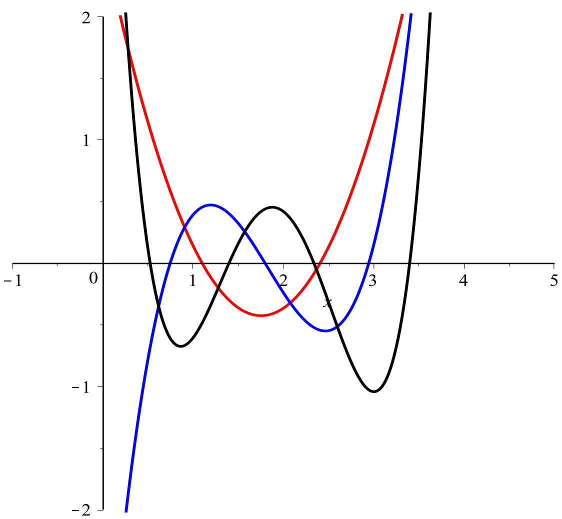
$t = 0$



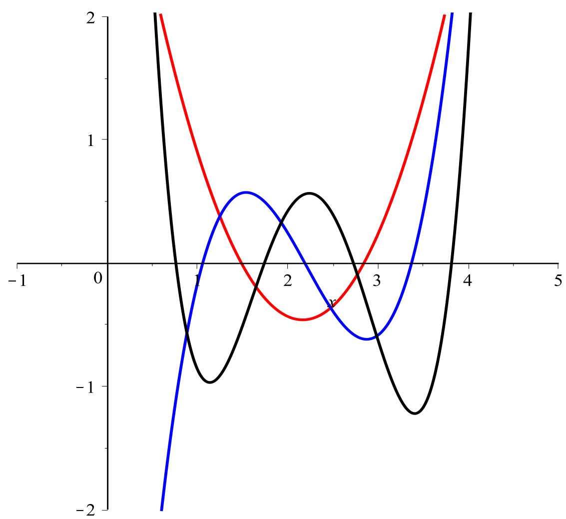
$t = 1$



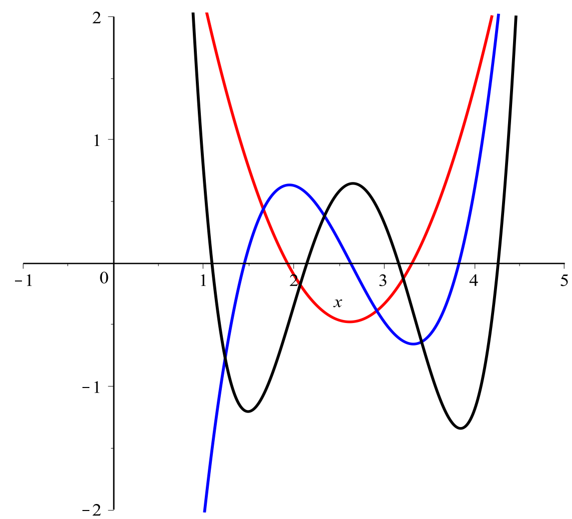
$t = 2$



$t = 3$



$t = 4$



$t = 5$

Semi-classical Hermite Weight

$$\omega(x;t) = |x|^{\nu} \exp(-x^2 + tx), \qquad x \in \mathbb{R}, \qquad \nu > -1$$

- **P A Clarkson & K Jordaan**, "The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation", *Constr. Approx.*, to appear [arXiv:1301.4134]

Semi-classical Hermite weight

Consider the **semi-classical Hermite weight**

$$\omega(x;t) = |x|^\nu \exp(-x^2 + tx), \qquad x \in \mathbb{R}, \quad \nu > -1$$

- The moment $\mu_k(t;\nu)$ is given by

$$\begin{aligned}\mu_k(t;\nu) &= \int_{-\infty}^{\infty} x^k |x|^\nu \exp(-x^2 + tx)\,\mathrm{d}x \\ &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\int_{-\infty}^{\infty} |x|^\nu \exp(-x^2 + tx)\,\mathrm{d}x\right) = \frac{\mathrm{d}^k \mu_0}{\mathrm{d}t^k}\end{aligned}$$

- The Hankel determinant $\Delta_n(t)$ is given by

$$\Delta_n(t) = \det\Big[\mu_{j+k}(t)\Big]_{j,k=0}^{n-1} \equiv \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right)$$

where

$$\mu_0(t;\nu) = \begin{cases} \dfrac{\Gamma(\nu+1)\exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}}\Big\{D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) + D_{-\nu-1}\big(\tfrac{1}{2}\sqrt{2}\,t\big)\Big\}, & \nu \notin \mathbb{N} \\ \sqrt{\pi}\,\big(-\tfrac{1}{2}\mathrm{i}\big)^{2N} H_{2N}\big(\tfrac{1}{2}\mathrm{i}t\big)\exp\big(\tfrac{1}{4}t^2\big), & \nu = 2N \\ \sqrt{\pi}\,\dfrac{\mathrm{d}^{2N+1}}{\mathrm{d}t^{2N+1}}\Big\{\mathrm{erf}\big(\tfrac{1}{2}t\big)\,\exp\big(\tfrac{1}{4}t^2\big)\Big\}, & \nu = 2N+1 \end{cases}$$

Theorem (PAC & Jordaan [2013])

The recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ in the three-term recurrence relation

$$xP_n(x;t) = P_{n+1}(x;t) + \alpha_n(t)P_n(x;t) + \beta_n(t)P_{n-1}(x;t),$$

for monic polynomials orthogonal with respect to the semi-classical Hermite weight

$$\omega(x;t) = |x|^\nu \exp(-x^2 + tx), \qquad x \in \mathbb{R}, \quad \nu > -1$$

are given by

$$\alpha_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)}, \qquad \beta_n(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln \Delta_n(t)$$

where $\Delta_n(t)$ is the Hankel determinant

$$\Delta_n(t) = \det \Big[\mu_{j+k}(t)\Big]_{j,k=0}^{n-1} \equiv \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right)$$

with

$$\mu_0(t;\nu) = \begin{cases} \dfrac{\Gamma(\nu+1)\exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}}\Big\{D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) + D_{-\nu-1}\big(\tfrac{1}{2}\sqrt{2}\,t\big)\Big\}, & \nu \notin \mathbb{N} \\ \sqrt{\pi}\,\big(-\tfrac{1}{2}\mathrm{i}\big)^{2N} H_{2N}\big(\tfrac{1}{2}\mathrm{i}t\big) \exp\big(\tfrac{1}{4}t^2\big), & \nu = 2N \\ \sqrt{\pi}\,\dfrac{\mathrm{d}^{2N+1}}{\mathrm{d}t^{2N+1}}\left\{\mathrm{erf}(\tfrac{1}{2}t)\,\exp\big(\tfrac{1}{4}t^2\big)\right\}, & \nu = 2N+1 \end{cases}$$

Recurrence coefficients for $\omega(x;t) = x^2 \exp(-x^2 + tx)$

$$\alpha_0(t) = \tfrac{1}{2}t + \frac{2t}{t^2+2}$$

$$\alpha_1(t) = \tfrac{1}{2}t + \frac{4t^3}{t^4+12} - \frac{2t}{t^2+2}$$

$$\alpha_2(t) = \tfrac{1}{2}t + \frac{6t(t^4+12-4t^2)}{t^6-6t^4+36t^2+72} - \frac{4t^3}{t^4+12}$$

$$\alpha_3(t) = \tfrac{1}{2}t + \frac{8t^3(t^4+60-12t^2)}{t^8-16t^6+120t^4+720} - \frac{6t(t^4+12-4t^2)}{t^6-6t^4+36t^2+72}$$

$$\alpha_4(t) = \tfrac{1}{2}t + \frac{10t(t^8+216t^4+720-24t^6-480t^2)}{t^{10}-30t^8+360t^6-1200t^4+3600t^2+7200} - \frac{8t^3(t^4+60-12t^2)}{t^8-16t^6+120t^4+720}$$

$$\beta_1(t) = \tfrac{1}{2} - \frac{2(t^2-2)}{(t^2+2)^2}$$

$$\beta_2(t) = 1 - \frac{4t^2(t^2-6)(t^2+6)}{(t^4+12)^2}$$

$$\beta_3(t) = \tfrac{3}{2} - \frac{6(t^4-12t^2+12)(t^6+6t^4+36t^2-72)}{(t^6-6t^4+36t^2+72)^2}$$

$$\beta_4(t) = 2 - \frac{8t^2(t^4-20t^2+60)(t^8+72t^4-2160)}{(t^8-16t^6+120t^4+720)^2}$$

Hence, using the three-term recurrence relation

$$P_{n+1}(x;t) = [x - \alpha_n(t)]P_n(x;t) - \beta_n(t)P_{n-1}(x;t), \qquad n = 0, 1, 2, \ldots$$

with $P_0(x;t) = 1$ and $P_{-1}(x;t) = 0$, then the first few polynomials are given by

$$P_1(x;t) = x - \frac{t(t^2+6)}{2(t^2+2)}$$

$$P_2(x;t) = x^2 - \frac{t(t^4+4t^2+12)}{t^4+12}x + \frac{t^6+6t^4+36t^2-72}{4(t^4+12)}$$

$$\begin{aligned}P_3(x;t) = x^3 &- \frac{3t(t^6-2t^4+20t^2+120)}{2(t^6-6t^4+36t^2+72)}x^2 + \frac{3(t^8+40t^4-240)}{4(t^6-6t^4+36t^2+72)}x \\ &- \frac{t(t^8+72t^4-2160)}{8(t^6-6t^4+36t^2+72)}\end{aligned}$$

$$\begin{aligned}P_4(x;t) = x^4 &- \frac{2t(t^8-12t^6+72t^4+240t^2+720)}{t^8-16t^6+120t^4+720}x^3 \\ &+ \frac{3(t^{10}-10t^8+80t^6+1200t^2-2400)}{2(t^8-16t^6+120t^4+720)}x^2 \\ &- \frac{t(t^{10}-10t^8+120t^6-240t^4-1200t^2-7200)}{2(t^8-16t^6+120t^4+720)}x \\ &+ \frac{t^{12}-12t^{10}+180t^8-480t^6-3600t^4-43200t^2+43200}{16(t^8-16t^6+120t^4+720)}\end{aligned}$$

Freud Weight

$$\omega(x;t) = \exp\left(-\tfrac{1}{4}x^4 - tx^2\right), \qquad x \in \mathbb{R}$$

Freud weight (Magnus [1995])

Consider the monic orthogonal polynomials with respect to the **Freud weight**

$$\omega(x;t) = \exp\left(-\tfrac{1}{4}x^4 - tx^2\right), \qquad x,t \in \mathbb{R}$$

which satisfy the three-term recurrence relation

$$xP_n(x;t) = P_{n+1}(x;t) + \beta_n(t)P_{n-1}(x;t)$$

It is well known that $\beta_n(t)$ satisfies

$$\begin{aligned}
&\beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}) + 2t\beta_n = n \\
&\frac{\mathrm{d}^2\beta_n}{\mathrm{d}t^2} = \frac{1}{2\beta_n}\left(\frac{\mathrm{d}\beta_n}{\mathrm{d}t}\right)^2 + \tfrac{3}{2}\beta_n^3 + 4t\beta_n^2 + 2(t^2 + \tfrac{1}{2}n)\beta_n - \frac{n^2}{2\beta_n}
\end{aligned}$$

which are $\mathrm{dP_I}$ and $\mathrm{P_{IV}}$ with $A = -\frac{1}{2}n$ and $B = -\frac{1}{2}n^2$, respectively.

Remark. The link between these equations is given by

$$\begin{aligned}
\beta_{n+1} &= \frac{1}{2\beta_n}\left(n - \frac{\mathrm{d}\beta_n}{\mathrm{d}t} - 2t\beta_n - \beta_n^2\right) \\
\beta_{n-1} &= \frac{1}{2\beta_n}\left(n + \frac{\mathrm{d}\beta_n}{\mathrm{d}t} - 2t\beta_n - \beta_n^2\right)
\end{aligned}$$

which are the $\mathrm{P_{IV}}$ Bäcklund transformations $\mathcal{T}_2^+$ and $\mathcal{T}_1^-$, respectively.

For the Freud weight

$$\omega(x;t) = \exp\left(-\tfrac{1}{4}x^4 - tx^2\right), \qquad x \in \mathbb{R}$$

the moments are

$$\begin{aligned} \mu_0(t) &= \int_{-\infty}^{\infty} \exp\left(-\tfrac{1}{4}x^4 - tx^2\right) \mathrm{d}x = \sqrt{2}\int_0^{\infty} y^{-1/2} \exp\left(-y^2 - 2ty\right) \mathrm{d}y \\ &= 2^{1/4}\sqrt{\pi}\, \exp(\tfrac{1}{2}t^2) D_{-1/2}\big(\sqrt{2}\,t\big) \end{aligned}$$

$$\mu_{2n}(t) = (-1)^n \frac{\mathrm{d}^n \mu_0}{\mathrm{d}t^n}, \qquad \mu_{2n+1}(t) = 0, \qquad n = 1, 2, \ldots$$

Remark. Solutions of $\mathrm{P_{IV}}$ with $A = -\frac{1}{2}n$ and $B = -\frac{1}{2}n^2$, i.e.

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}t^2} = \frac{1}{2q_n}\left(\frac{\mathrm{d}q_n}{\mathrm{d}t}\right)^2 + \tfrac{3}{2}q_n^3 + 4tq_n^2 + 2(t^2 + \tfrac{1}{2}n)q_n - \frac{n^2}{2q_n}, \qquad n = 1, 2, \ldots$$

are known as the "**half-integer hierarchy**", which arise in quantum gravity (**Fokas, Its & Kitaev [1991, 1992]**) and were studied by **Bassom, PAC & Hicks [1995]**. The first solution in this hierarchy is given by

$$q(t; -\tfrac{1}{2}, -\tfrac{1}{2}) = -2t + \sqrt{2}\,\frac{C_1 D_{1/2}\big(\sqrt{2}\,t\big) - C_2 D_{1/2}\big(-\sqrt{2}\,t\big)}{C_1 D_{-1/2}\big(\sqrt{2}\,t\big) + C_2 D_{-1/2}\big(-\sqrt{2}\,t\big)}$$

with C_1 and C_2 arbitrary constants and $D_\nu(\zeta)$ the **parabolic cylinder function**.

Generalized Freud Weight

$$\omega(x;t) = |x|^{2\nu-1} \exp\left(-\tfrac{1}{4}x^4 - tx^2\right), \qquad x \in \mathbb{R}, \quad \nu > 0$$

Generalized Freud weight

For the **generalized Freud weight**

$$\omega(x;t) = |x|^{2\nu-1}\exp\left(-\tfrac{1}{4}x^4 - tx^2\right), \qquad x \in \mathbb{R}$$

the moments are

$$\begin{aligned}
\mu_0(t) &= \int_{-\infty}^{\infty} |x|^{2\nu-1}\exp\left(-\tfrac{1}{4}x^4 - tx^2\right)\mathrm{d}x \\
&= 2^{\nu}\int_0^{\infty} y^{\nu-1}\exp\left(-y^2 - 2ty\right)\mathrm{d}y \\
&= 2^{\nu/2}\Gamma(\nu)\,\exp(\tfrac{1}{2}t^2)D_{-\nu}\big(\sqrt{2}\,t\big)
\end{aligned}$$

$$\begin{aligned}
\mu_{2n}(t) &= \int_{-\infty}^{\infty} x^{2n}|x|^{2\nu-1}\exp\left(-\tfrac{1}{4}x^4 - tx^2\right)\mathrm{d}x \\
&= (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\int_{-\infty}^{\infty} |x|^{2\nu-1}\exp\left(-\tfrac{1}{4}x^4 - tx^2\right)\mathrm{d}x\right) \\
&= (-1)^n\frac{\mathrm{d}^n\mu_0}{\mathrm{d}t^n}, \qquad n = 1, 2, \ldots
\end{aligned}$$

$$\begin{aligned}
\mu_{2n+1}(t) &= \int_{-\infty}^{\infty} x^{2n+1}|x|^{2\nu-1}\exp\left(-\tfrac{1}{4}x^4 - tx^2\right)\mathrm{d}x \\
&= 0, \qquad n = 1, 2, \ldots
\end{aligned}$$

Theorem (**PAC [2013]**)

The recurrence coefficient $\beta_n(t)$ in the three-term recurrence relation

$$xP_n(x;t) = P_{n+1}(x;t) + \beta_n(t)P_{n-1}(x;t),$$

is given by

$$\beta_{2n}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_n^{[0]}(t)}{\Delta_n^{[2]}(t)}, \qquad \beta_{2n+1}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_n^{[2]}(t)}{\Delta_{n+1}^{[0]}(t)}$$

where $\Delta_n^{[0]}(t)$ and $\Delta_n^{[2]}(t)$ are the Hankel determinants, respectively given by

$$\Delta_n^{[0]}(t) = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right)$$

$$\Delta_n^{[2]}(t) = \mathcal{W}\left(\mu_2, \frac{\mathrm{d}\mu_2}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_2}{\mathrm{d}t^{n-1}}\right) = (-1)^n\mathcal{W}\left(\frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \frac{\mathrm{d}^2\mu_0}{\mathrm{d}t^2}, \ldots, \frac{\mathrm{d}^n\mu_0}{\mathrm{d}t^n}\right)$$

Remark: Note that

$$\beta_{2n}(t) = q\big(t; 1-n-2\nu, -2n^2\big)$$
$$\beta_{2n+1}(t) = q\big(t; \nu-n-1, -2(\nu+n)^2\big)$$

where $q(t;A,B)$ satisfies $\mathrm{P_{IV}}$

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 + \frac{3}{2}q^3 + 4tq^2 + 2(t^2-A)q + \frac{B}{q}$$

Discrete Orthogonal Polynomials

- **P A Clarkson**, "Recurrence coefficients for discrete orthonormal polynomials and the Painlevé equations", *J. Phys. A* **46** (2013) 185205

Discrete Orthonormal Polynomials

Discrete orthonormal polynomials $\{p_n(x)\}$, $n = 0, 1, 2, \ldots$, with respect to a discrete weight $\omega(k)$,

$$\sum_{k=0}^{\infty} p_m(k)p_n(k)\omega(k) = \delta_{m,n} = \begin{cases} 1, & \text{if} \quad m = n \\ 0, & \text{if} \quad m \neq n \end{cases}$$

satisfy the three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x;t) + a_n p_{n-1}(x)$$

The moments are

$$\mu_n = \sum_{k=0}^{\infty} k^n \omega(k), \qquad n = 0, 1, 2, \ldots$$

and the recurrence coefficients

$$a_n^2 = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \qquad b_n = \frac{\widetilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\widetilde{\Delta}_n}{\Delta_n}$$

where

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-1} \\ \mu_1 & \mu_2 & \ldots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-2} \end{vmatrix}, \qquad \widetilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \ldots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}$$

In the case when the discrete weight has the form

$$\omega(k) = c(k)t^k, \qquad t > 0$$

which is the case for the **Charlier polynomials** and **Meixner polynomials**, then

$$\mu_0 = \sum_{k=0}^{\infty} c(k)t^k \qquad \Rightarrow \qquad \mu_n = \sum_{k=0}^{\infty} k^n c(k)t^k = \delta^n(\mu_0), \quad \delta(\phi) = t\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

Hence the determinants Δ_n and $\widetilde{\Delta}_n$ are given by

$$\Delta_n(t) = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-2} \end{vmatrix} = \begin{vmatrix} \mu_0 & \delta(\mu_0) & \dots & \delta^{n-1}(\mu_0) \\ \delta(\mu_0) & \delta^2(\mu_0) & \dots & \delta^n(\mu_0) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{n-1}(\mu_0) & \delta^n(\mu_0) & \dots & \delta^{2n-2}(\mu_0) \end{vmatrix} =: \widetilde{\mathcal{W}}_n(\mu_0)$$

$$\widetilde{\Delta}_n(t) = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix} = \delta(\Delta_n) = \delta\big(\widetilde{\mathcal{W}}_n(\mu_0)\big)$$

and the recurrence coefficients are given by

$$a_n^2(t) = \delta^2(\ln \Delta_n) = \delta^2\big(\ln \widetilde{\mathcal{W}}_n(\mu_0)\big), \quad b_n(t) = \delta\left(\ln \frac{\Delta_{n+1}}{\Delta_n}\right) = \delta\left(\ln \frac{\widetilde{\mathcal{W}}_{n+1}(\mu_0)}{\widetilde{\mathcal{W}}_n(\mu_0)}\right)$$

Charlier Polynomials

The **Charlier polynomials** given by

$$C_n(k;t) = (-1)^n n! L_n^{(-1-k)}\left(-\frac{1}{t}\right)$$

where $L_n^{(\alpha)}(z)$ is the **Laguerre polynomial**, are orthogonal on $\mathbb{N}$ with respect to the discrete weight

$$\omega(k) = \frac{t^k}{k!}, \qquad t > 0$$

Here

$$\mu_0(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} = \mathrm{e}^t$$

$$\Delta_n(t) = \widetilde{\mathcal{W}}_n(\mu_0) = t^{n(n-1)/2} \exp(nt) \prod_{k=1}^{n-1}(k!)$$

$$\widetilde{\Delta}_n(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\Delta_n = \left[\tfrac{1}{2}n(n-1) + nt\right]\Delta_n$$

and the recurrence coefficients are given by

$$a_n^2(t) = \delta^2(\ln \Delta_n) = nt, \qquad b_n(t) = \delta\left(\ln \frac{\Delta_{n+1}}{\Delta_n}\right) = n + t$$

Meixner Polynomials

The **Meixner polynomials** given by

$$M_n(k;\alpha,t) = {}_2F_1\left(-n,-k;-\alpha;1-\frac{1}{t}\right), \qquad \alpha>0, \quad 0<t<1$$

where ${}_2F_1(a,b;c;z)$ is the **hypergeometric function**, are orthogonal on $\mathbb{N}$ with respect to the discrete weight

$$\omega(k) = \frac{(\alpha)_k\, t^k}{k!}, \qquad \alpha>0, \quad t>0$$

with $(\alpha)_k = \Gamma(\alpha+k)/\Gamma(\alpha)$ the Pochhammer symbol. Here

$$\mu_0(t) = \sum_{k=0}^{\infty} \frac{(\alpha)_k\, t^k}{k!} = (1-t)^{-\alpha}$$

$$\Delta_n(t) = \widetilde{\mathcal{W}}_n(\mu_0) = \frac{t^{n(n-1)/2}}{(1-t)^{n(n+\alpha-1)}} \prod_{k=1}^{n-1} k!(\alpha+k)^{n-k-1}$$

$$\widetilde{\Delta}_n(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\Delta_n = \frac{n(n-1)+n(n+2\alpha-1)t}{2(1-t)}\Delta_n$$

and the recurrence coefficients are given by

$$a_n^2(t) = \delta^2(\ln\Delta_n) = \frac{n(n+\alpha-1)t}{(1-t)^2}, \qquad b_n(t) = \delta\left(\ln\frac{\Delta_{n+1}}{\Delta_n}\right) = \frac{n+(n+\alpha)t}{1-t}$$

Discrete Pearson Equation

The **discrete Pearson equation** has the form

$$\Delta\Big[\sigma(k)\omega(k)\Big] = \tau(k)\omega(k)$$

where Δ is the forward difference operator

$$\Delta f(k) = f(k+1) - f(k)$$

- **Classical discrete orthogonal polynomials**: $\sigma(k)$ and $\tau(k)$ are polynomials with $\deg(\sigma) \le 2$ and $\deg(\tau) = 1$

	$\omega(k)$	$\sigma(k)$	$\tau(k)$
Charlier	$\dfrac{t^k}{k!}$	k	$t-k$
Meixner	$\dfrac{(\alpha)_k\, t^k}{k!}$	k	$(t-1)k + t\alpha$

- **Semi-classical discrete orthogonal polynomials**: $\sigma(k)$ and $\tau(k)$ are polynomials either $\deg(\sigma) > 2$ or $\deg(\tau) > 1$

	$\omega(k)$	$\sigma(k)$	$\tau(k)$
Generalized Charlier	$\dfrac{t^k}{(\beta)_k\, k!}$	$k(k+\beta-1)$	$-k^2 + (1-\beta)k + t$
Generalized Meixner	$\dfrac{(\alpha)_k\, t^k}{(\beta)_k\, k!}$	$k(k+\beta-1)$	$-k^2 + (1+t-\beta)k + t\alpha$

Generalized Charlier Polynomials

The **generalized Charlier polynomials** are orthogonal on $\mathbb{N}$ with respect to the discrete weight

$$\omega(k) = \frac{t^k}{(\beta)_k \, k!}, \qquad \beta > 0$$

with $(\beta)_k = \Gamma(\beta + k)/\Gamma(\beta)$ the Pochhammer symbol.

Theorem (Smet & van Assche [2012])

The recurrence coefficients $a_n(t)$ and $b_n(t)$ for orthonormal polynomials associated with the generalized Charlier weight

$$\omega(k) = \frac{t^k}{(\beta)_k \, k!}, \qquad \beta > 0$$

on the lattice $\mathbb{N}$ satisfy the discrete system

$$\begin{aligned} &(a_{n+1}^2 - t)(a_n^2 - t) = t(b_n - n)(b_n - n + \beta - 1), \\ &b_n + b_{n-1} - n + \beta = nt/a_n^2, \end{aligned} \tag{2}$$

with initial conditions

$$a_0^2 = 0, \qquad b_0 = \frac{\sqrt{t}\, I_\beta(2\sqrt{t})}{I_{\beta-1}(2\sqrt{t})} = t\frac{\mathrm{d}}{\mathrm{d}t} \ln\left(t^{(1-\beta)/2} I_{\beta-1}(2\sqrt{t})\right), \tag{3}$$

with $I_\nu(x)$ the modified Bessel function.

Lemma (PAC [2013])

For the **generalized Charlier polynomials**

$$\mu_0(t) = \sum_{k=0}^{\infty} \frac{t^k}{(\beta)_k\, k!} = \Gamma(\beta) t^{(1-\beta)/2} I_{\beta-1}\big(2\sqrt{t}\big)$$

with $I_\nu(z)$ *the* **modified Bessel function**.

Proof. Since the modified Bessel function $I_\nu(x)$ has the series expansion

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}x\right)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)},$$

then the expression for the moment $\mu_0(t)$ follows immediately.

Corollary (PAC [2013])

The Hankel determinant $\Delta_n(t)$ *is given by*

$$\Delta_n(t) = \widetilde{\mathcal{W}}_n(\mu_0) = \left[\Gamma(\beta)\right]^n \widetilde{\mathcal{W}}_n\Big(t^{(1-\beta)/2} I_{\beta-1}\big(2\sqrt{t}\big)\Big)$$

and the recurrence coefficients $a_n(t)$ *and* $b_n(t)$ *have the form*

$$a_n^2(t) = \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \big(\ln \Delta_n(t)\big), \qquad b_n(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)}\right)$$

Theorem (PAC [2013])

The function $S_n(t) = t\dfrac{\mathrm{d}}{\mathrm{d}t}\ln\Delta_n(t)$ *where*

$$\Delta_n(t) = \left[\Gamma(\beta)\right]^n \widetilde{\mathcal{W}}_n\left(t^{(1-\beta)/2} I_{\beta-1}(2\sqrt{t})\right)$$

satisfies

$$\left[t\frac{\mathrm{d}^2 S_n}{\mathrm{d}t^2}\right]^2 = \left[n - (n+\beta-1)\frac{\mathrm{d}S_n}{\mathrm{d}t}\right]^2 - 4\frac{\mathrm{d}S_n}{\mathrm{d}t}\left[\frac{\mathrm{d}S_n}{\mathrm{d}t} - 1\right]\left[t\frac{\mathrm{d}S_n}{\mathrm{d}t} - S_n + \tfrac{1}{2}n(n-1)\right]$$

which is equivalent to $\mathrm{S}_{\mathrm{III}'}$*, the* $\mathrm{P}_{\mathrm{III}'}$ σ*-equation.*

Proof. Making the transformation

$$S_n(t;\beta) = \sigma(t) + \tfrac{1}{2}t + \tfrac{1}{4}n^2 - \tfrac{1}{2}n(\beta+1) - \tfrac{1}{4}\beta^2$$

yields

$$\left(t\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}\right)^2 + \left\{4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - 1\right\}\left(t\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \sigma\right) + (n^2-\beta^2)\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \tfrac{1}{2}(n^2+\beta^2)$$

which is the $\mathrm{P}_{\mathrm{III}'}$ σ-equation with parameters $(\theta_0,\theta_\infty) = (n+\beta, n-\beta)$.
Then we set $\nu=\beta$, $\varepsilon_1=-1$ and $\varepsilon_2=1$, in the following Theorem.

Theorem (**Okamoto [1987]; Forrester & Witte [2002]**)

Special function solutions of the $\mathrm{P}_{\mathrm{III}'}$ σ*-equation*

$$\left(t\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}\right)^2 + \left\{4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - 1\right\}\left(t\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \sigma\right) + \vartheta_0\vartheta_\infty\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \tfrac{1}{4}(\vartheta_0^2 + \vartheta_\infty^2) \qquad \mathrm{S}_{\mathrm{III}'}$$

are given by

$$\sigma(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\ln\tau_{n,\nu}(t) + \tfrac{1}{2}\varepsilon_1\varepsilon_2 t + \tfrac{1}{4}\nu^2 + \tfrac{1}{2}n(1-\varepsilon_1\nu) - \tfrac{1}{4}n^2$$

for the parameters $(\vartheta_0, \vartheta_\infty) = (\nu + n, \varepsilon_1\varepsilon_2(\nu - n))$*, where* $\tau_{n,\nu}(t)$ *is the determinant*

$$\tau_{n,\nu}(t) = \det\left[\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j+k}\psi_\nu(t)\right]_{j,k=0}^{n-1}$$

with $\psi_\nu(t)$ *given by*

$$\psi_\nu(t) = \begin{cases} t^{\nu/2}\left\{C_1 J_\nu\left(2\sqrt{t}\right) + C_2 Y_\nu\left(2\sqrt{t}\right)\right\}, & \text{if} \quad \varepsilon_1 = 1, \quad \varepsilon_2 = 1 \\ t^{-\nu/2}\left\{C_1 J_\nu\left(2\sqrt{t}\right) + C_2 Y_\nu\left(2\sqrt{t}\right)\right\}, & \text{if} \quad \varepsilon_1 = -1, \quad \varepsilon_2 = -1 \\ t^{\nu/2}\left\{C_1 I_\nu\left(2\sqrt{t}\right) + C_2 K_\nu\left(2\sqrt{t}\right)\right\}, & \text{if} \quad \varepsilon_1 = 1, \quad \varepsilon_2 = -1 \\ t^{-\nu/2}\left\{C_1 I_\nu\left(2\sqrt{t}\right) + C_2 K_\nu\left(2\sqrt{t}\right)\right\}, & \text{if} \quad \varepsilon_1 = -1, \quad \varepsilon_2 = 1 \end{cases}$$

C_1 *and* C_2 *arbitrary constants,* $J_\nu(z)$, $Y_\nu(z)$, $I_\nu(z)$ *and* $K_\nu(z)$ **Bessel functions**.

Generalized Meixner polynomials

The **generalized Meixner polynomials** are orthogonal on $\mathbb{N}$ with respect to the discrete weight

$$\omega(k) = \frac{(\alpha)_k\, t^k}{(\beta)_k\, k!}, \qquad \alpha > 0, \quad \beta > 0$$

with $(\alpha)_k = \Gamma(\alpha + k)/\Gamma(\alpha)$ the Pochhammer symbol.

Theorem

(**Smet & van Assche [2012]**)

The recurrence coefficients $a_n(z)$ and $b_n(z)$ for orthonormal polynomials associated with the generalized Meixner weight on the lattice $\mathbb{N}$ satisfy

$$a_n^2 = nt - (\alpha - 1)x_n, \qquad b_n = n + \alpha - \beta + t - (\alpha - 1)y_n/t$$

where x_n and y_n satisfy the discrete system

$$(x_n + y_n)(x_{n+1} + y_n) = \frac{\alpha - 1}{t^2} y_n(y_n - t)\left(y_n - t\,\frac{\alpha - \beta}{\alpha - 1}\right),$$

$$(x_n + y_n)(x_n + y_{n-1}) = \frac{(\alpha - 1)x_n(x_n + t)}{(\alpha - 1)x_n - nt}\left(x_n + t\,\frac{\alpha - \beta}{\alpha - 1}\right),$$

with initial conditions

$$a_0^2 = 0, \qquad b_0 = \frac{\alpha t}{\beta}\,\frac{M(\alpha + 1, \beta + 1, t)}{M(\alpha, \beta, t)} = t\frac{\mathrm{d}}{\mathrm{d}t} \ln M(\alpha, \beta, t)$$

and $M(\alpha, \beta, t)$ is the Kummer function.

Lemma (PAC [2013])

For the **generalized Meixner polynomials**

$$\mu_0(t) = \sum_{k=0}^{\infty} \frac{(\alpha)_k\, t^k}{(\beta)_k\, k!} = M(\alpha,\beta,t)$$

with $M(\alpha,\beta,t)$ *the* **Kummer function**.

Corollary (PAC [2013])

The Hankel determinant $\Delta_n(t)$ *is given by*

$$\Delta_n(t) = \widetilde{\mathcal{W}}_n(\mu_0) = \widetilde{\mathcal{W}}_n\Big(M(\alpha,\beta,t)\Big)$$

and the recurrence coefficients $a_n(t)$ *and* $b_n(t)$ *have the form*

$$a_n^2(t) = \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \big(\ln \Delta_n(t)\big), \qquad b_n(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)}\right)$$

Theorem (PAC [2013])

The function

$$S_n(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\ln \widetilde{\mathcal{W}}_n\Big(M(\alpha,\beta,t)\Big)$$

satisfies

$$\left(t\frac{\mathrm{d}^2S_n}{\mathrm{d}t^2}\right)^2 = \left[(t+n+\beta-1)\frac{\mathrm{d}S_n}{\mathrm{d}t} - \tfrac{1}{2}n(n-1+2\alpha)\right]^2 - 4\frac{\mathrm{d}S_n}{\mathrm{d}t}\left(\frac{\mathrm{d}S_n}{\mathrm{d}t} - n - \alpha + \beta\right)\left[t\frac{\mathrm{d}S_n}{\mathrm{d}t} - S_n + \tfrac{1}{2}n(n-1)\right]$$

which is equivalent to $\mathrm{S_V}$, *the* $\mathrm{P_V}$ σ*-equation.*

Proof. Making the transformation

$$S_n(t) = \sigma(z) + \tfrac{1}{4}(2\alpha-\beta+3n-1)z + \tfrac{5}{8}n^2 + \tfrac{1}{4}(2\alpha-3\beta-1)n + \tfrac{1}{8}(2\alpha-\beta-1)^2$$

with $z=t$, yields the $\mathrm{P_V}$ σ-equation

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - \left\{2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - z\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right\}^2 + 4\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \kappa_j\right) = 0 \qquad \mathrm{S_V}$$

with parameters

$$\kappa_1 = \tfrac{1}{4}(2\alpha-\beta-n-1) \qquad \kappa_3 = \tfrac{1}{4}(2\alpha-\beta+3n-1)$$
$$\kappa_2 = -\tfrac{1}{4}(2\alpha+\beta+n-3) \qquad \kappa_4 = -\tfrac{1}{4}(2\alpha-3\beta+n+1)$$

Theorem (**Okamoto [1987]; Forrester & Witte [2002]**)

Special function solutions of the $\mathrm{P_V}$ *σ-equation*

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - \left\{2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - z\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right\}^2 + 4\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \kappa_j\right) = 0 \qquad \mathrm{S_V}$$

are given by

$$\begin{aligned}\sigma(z) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln\mathcal{W}_n(\varphi_{\alpha,\beta}) &- \tfrac{1}{4}(3n+2\alpha-\beta-1)z \\ &- \tfrac{5}{8}n^2 - \tfrac{1}{4}(2\alpha-3\beta-1)n - \tfrac{1}{8}(2\alpha-\beta-1)^2\end{aligned}$$

for the parameters

$$\begin{aligned}\kappa_1 &= \tfrac{1}{4}(2\alpha-\beta-n-1), & \kappa_3 &= \tfrac{1}{4}(2\alpha-\beta+3n-1) \\ \kappa_2 &= -\tfrac{1}{4}(2\alpha+\beta+n-3), & \kappa_4 &= -\tfrac{1}{4}(2\alpha-3\beta+n+1)\end{aligned}$$

where $\mathcal{W}_n(\varphi_{\alpha,\beta})$ *is the determinant given by*

$$\mathcal{W}_n(\varphi_{\alpha,\beta}) = \det\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j+k}\varphi_{\alpha,\beta}(z)\right]_{j,k=0}^{n-1}$$

with

$$\varphi_{\alpha,\beta}(z) = C_1 M(\alpha,\beta,z) + C_2 U(\alpha,\beta,z)$$

C_1 *and* C_2 *arbitrary constants,* $M(\alpha,\beta,z)$ *and* $U(\alpha,\beta,z)$ **Kummer functions**.

Special function solutions of Painlevé equations

	Number of (essential) parameters	Special function	Number of parameters	Associated orthogonal polynomial	Number of parameters
$\mathrm{P_I}$	0	—			
$\mathrm{P_{II}}$	1	**Airy** $\mathrm{Ai}(z), \mathrm{Bi}(z)$	0	—	
$\mathrm{P_{III}}$	2	**Bessel** $J_\nu(z), Y_\nu(z), I_\nu(z), K_\nu(z)$	1	—	
$\mathrm{P_{IV}}$	2	**Parabolic cylinder** $D_\nu(z)$	1	**Hermite** $H_n(z)$	0
$\mathrm{P_V}$	3	**Kummer** $M(a,b,z), U(a,b,z)$ **Whittaker** $M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$	2	**Associated Laguerre** $L_n^{(k)}(z)$	1
$\mathrm{P_{VI}}$	4	**hypergeometric** ${}_2F_1(a,b;c;z)$	3	**Jacobi** $P_n^{(\alpha,\beta)}(z)$	2

Further Examples

$$\omega(x;z) = x^{\nu} \exp\left(-x - t/x\right), \qquad \nu > 0, \qquad x \in \mathbb{R}^{+}$$

$$\omega(x;z) = x^{\nu}(x+z)^{\lambda} \exp(-x), \qquad \nu, \lambda > 0, \qquad x \in \mathbb{R}^{+}$$

Perturbed Laguerre weight

Consider orthogonal polynomials with respect to the **perturbed Laguerre weight**

$$\omega(x;z) = x^{\nu}\exp\left(-x - t/x\right), \qquad x \in \mathbb{R}^+, \qquad \nu > 0$$

Define the Hankel determinant

$$\Delta_n(t) = \det\left[\mu_{j+k}(t)\right]_{j,k=0}^{n-1}, \qquad \mu_k(t) = \int_0^{\infty} x^{\nu+k}\exp\left(-x - t/x\right)\mathrm{d}x$$

Then **Chen & Its [2010]** show that

$$H_n(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\ln\Delta_n(t)$$

satisfies

$$\left(t\frac{\mathrm{d}^2 H_n}{\mathrm{d}t^2}\right)^2 = \left[(2n+\nu)\frac{\mathrm{d}H_n}{\mathrm{d}t} - n\right]^2 - 4\frac{\mathrm{d}H_n}{\mathrm{d}t}\left(\frac{\mathrm{d}H_n}{\mathrm{d}t} - 1\right)\left[t\frac{\mathrm{d}H_n}{\mathrm{d}t} - H_n + n(n+\nu)\right]$$

which is equivalent to a special case of $\mathrm{S}_{\mathrm{III}'}$, the $\mathrm{P}_{\mathrm{III}'}$ σ-equation. Specifically, letting

$$H_n(t) = \sigma + \tfrac{1}{2}t + \tfrac{1}{4}n^2 - \tfrac{1}{2}n(\nu+1) - \tfrac{1}{4}\nu^2$$

yields

$$\left(t\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}\right)^2 + \left\{4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - 1\right\}\left(t\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \sigma\right) + (n^2-\nu^2)\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \tfrac{1}{2}(n^2+\nu^2)$$

which is $\mathrm{S}_{\mathrm{III}'}$ with $(\vartheta_0, \vartheta_\infty) = (n+\nu, n-\nu)$.

For the **perturbed Laguerre weight**

$$\omega(x;t) = x^\nu \exp\left(-x - t/x\right), \qquad x \in \mathbb{R}^+, \quad \nu > 0, \quad t > 0$$

the associated moments are

$$\mu_k(t) = \int_0^\infty x^{\nu+k} \exp\left(-x - t/x\right) \mathrm{d}x = 2t^{(\nu+k+1)/2} K_{\nu+k+1}\left(2\sqrt{t}\right)$$

with $K_\nu(z)$ the **modified Bessel function**. Hence the Hankel determinant is given by

$$\begin{aligned}\Delta_n(t) = \det\left[\mu_{j+k}(t)\right]_{j,k=0}^{n-1} &= 2^n t^{n(n+\nu)/2} \det\left[K_{\nu+j+k+1}\left(2\sqrt{t}\right)\right]_{j,k=0}^{n-1} \\ &= 2^n t^{n(\nu+1)/2} \det\left[\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j+k} K_{\nu+n}\left(2\sqrt{t}\right)\right]_{j,k=0}^{n-1}\end{aligned}$$

using properties of Bessel functions. Then

$$H_n(t) = t\frac{\mathrm{d}}{\mathrm{d}t} \ln \Delta_n(t)$$

satisfies

$$\left(t\frac{\mathrm{d}^2 H_n}{\mathrm{d}t^2}\right)^2 = \left[n - (2n+\nu)\frac{\mathrm{d}H_n}{\mathrm{d}t}\right]^2 - 4\frac{\mathrm{d}H_n}{\mathrm{d}t}\left(\frac{\mathrm{d}H_n}{\mathrm{d}t} - 1\right)\left[t\frac{\mathrm{d}H_n}{\mathrm{d}t} - H_n + n(n+\nu)\right]$$

which is equivalent to a special case of $\mathrm{S}_{\mathrm{III'}}$, the $\mathrm{P}_{\mathrm{III'}}$ σ-equation.

Deformed Laguerre weight

Consider orthogonal polynomials with the respect to the **deformed Laguerre weight**

$$\omega(x;z) = x^{\nu}(x+z)^{\lambda}\mathrm{e}^{-x}, \qquad x \in \mathbb{R}^{+}, \qquad \nu > 0, \quad \lambda > 0$$

Define the Hankel determinant

$$\Delta_n(z;\nu,\lambda) = \det\left[\mu_{j+k}(z;\nu,\lambda)\right]_{j,k=0}^{n-1}$$

where

$$\mu_k(z;\nu,\lambda) = \int_0^{\infty} x^{\nu+k}(x+z)^{\lambda}\mathrm{e}^{-x}\,\mathrm{d}x$$

Chen & McKay [2012] (also **Basor, Chen & McKay [2013]**) show that

$$H_n(z;\nu,\lambda) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln\Delta_n(z;\nu,\lambda)$$

satisfies

$$\left(z\frac{\mathrm{d}^2H_n}{\mathrm{d}z^2}\right)^2 = \left[(z+2n+\nu+\lambda)\frac{\mathrm{d}H_n}{\mathrm{d}z} - H_n + n\lambda\right]^2 - 4\frac{\mathrm{d}H_n}{\mathrm{d}z}\left(\frac{\mathrm{d}H_n}{\mathrm{d}z}+\lambda\right)\left[z\frac{\mathrm{d}H_n}{\mathrm{d}z} - H_n + n(n+\nu+\lambda)\right]$$

which is equivalent to a special case of $\mathrm{S_V}$, the $\mathrm{P_V}$ σ-equation.

Remarks

- For the **deformed Laguerre weight**

$$\omega(x;z) = x^{\nu}(x+z)^{\lambda}\mathrm{e}^{-x}, \qquad x \in \mathbb{R}^+, \qquad \nu > 0, \quad \lambda > 0$$

the kth moment is

$$\begin{aligned}\mu_k(z;\nu,\lambda) &= \int_0^\infty x^{\nu+k}(x+z)^{\lambda}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= z^{\nu+\lambda+k+1}\int_0^\infty s^{\nu+k}(1+s)^{\lambda}\mathrm{e}^{-sz}\,\mathrm{d}x \\ &= \Gamma(\nu+k+1)z^{\nu+\lambda+k+1}U(\nu+k+1,\nu+\lambda+k+2,t)\end{aligned}$$

with $U(a,b,z)$ the **Kummer function** of the second kind.

- In the special case of the deformed Laguerre weight when $\lambda = m \in \mathbb{Z}^+$ then

$$\begin{aligned}\mu_k(z;\nu,m) &= \int_0^\infty x^{\nu+k}(x+z)^{m}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \Gamma(\nu+k+1)z^{\nu+m+k+1}U(\nu+k+1,\nu+m+k+2,t) \\ &= \Gamma(\nu+k+1)\,(-1)^m\,m!\,L_m^{(-\nu-m-k-1)}(z)\end{aligned}$$

with $L_n^{(\alpha)}(z)$ the **associated Laguerre polynomial**, since

$$z^{\alpha+m}U(\alpha,\alpha+m+1,z) = (-1)^m\,m!\,L_m^{(-\alpha-m)}(z), \qquad m \in \mathbb{Z}^+$$

The **Kummer functions** $M(a,b,z)$ and $U(a,b,z)$ have the integral representations

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 \mathrm{e}^{zs}\, s^{a-1}(1-s)^{b-a-1}\, \mathrm{d}s$$

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-zs}\, s^{a-1}(1+s)^{b-a-1}\, \mathrm{d}s$$

- For the **perturbed Jacobi weight** (**Basor, Chen & Ehrhardt [2010]**)

$$\omega(x;z) = (1-x)^{\alpha-1}(1+x)^{\beta-1}\mathrm{e}^{-zx}, \qquad x \in [-1,1], \qquad \alpha > 0, \quad \beta > 0$$

  the moments are given by

$$\mu_0(z;\alpha,\beta) = 2^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,\mathrm{e}^{-z}M(\alpha,\alpha+\beta,2z)$$

$$\mu_k(z;\alpha,\beta) = (-1)^k\frac{\mathrm{d}^k}{\mathrm{d}z^k}\mu_0(z;\alpha,\beta)$$

- For the **Pollaczek-Jacobi weight** (**Chen & Dai [2010]**)

$$\omega(x;z) = x^{\alpha-1}(1-x)^{\beta-1}\mathrm{e}^{-z/x}, \qquad x \in [0,1], \qquad \alpha > 0, \quad \beta > 0$$

  the kth moment is

$$\mu_k(z;\alpha,\beta) = \Gamma(\beta)\,\mathrm{e}^{-z}U(\beta, 1-\alpha-k, z)$$

For both these weights $H_n(z) = z\dfrac{\mathrm{d}}{\mathrm{d}z}\ln\Delta_n(z)$, with $\Delta_n(z) = \det\Big[\mu_{j+k}(z)\Big]_{j,k=0}^{n-1}$, satisfies an equation which is equivalent to a special case of $\mathrm{S_V}$, the $\mathrm{P_V}$ σ-equation.

Discontinuous Weights

$$\omega(x;z) = \{1 + \xi - 2\xi\mathcal{H}(x-z)\} \exp\left(-x^2\right), \qquad 0 < \xi < 1, \quad x, z \in \mathbb{R}$$

$$\omega(x;z) = \{1 - \xi\mathcal{H}(x-z)\} |x-z|^{\lambda} x^{\alpha} \exp(-x), \qquad \nu, \lambda > 0, \qquad x, z \in \mathbb{R}^+$$

where $\mathcal{H}(x)$ is the Heaviside step function.

Discontinuous Hermite weight

Consider the **discontinuous Hermite weight**

$$\omega(x;z) = \{1+\xi-2\xi\mathcal{H}(x-z)\}\exp\left(-x^2\right), \qquad 0<\xi<1, \quad x,z\in\mathbb{R}$$

with $\mathcal{H}(x)$ the Heaviside step function. In this case

$$\mu_0(z) = \int_{-\infty}^{\infty}\omega(x;z)\,\mathrm{d}x = \sqrt{\pi}\left[1+\xi\,\mathrm{erf}(z)\right]$$

and define the Hankel determinant

$$\Delta_n(z) = \det\left[\mu_{j+k}(z)\right]_{j,k=0}^{n-1}$$

Then it can be shown that

$$S_n(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\Delta_n(z)$$

satisfies

$$\left(\frac{\mathrm{d}^2S_n}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}S_n}{\mathrm{d}z}-S_n\right)^2 + 4\left(\frac{\mathrm{d}S_n}{\mathrm{d}z}\right)^2\left(\frac{\mathrm{d}S_n}{\mathrm{d}z}+2n\right) = 0$$

which is $\mathrm{S_{IV}}$, the $\mathrm{P_{IV}}$ σ-equation, with $(\vartheta_0,\vartheta_\infty)=(n,0)$, and

$$w_n(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\Delta_{n+1}(z)}{\Delta_n(z)} = S_{n+1}(z)-S_n(z)$$

satisfies $\mathrm{P_{IV}}$ with $(A,B)=(2n+1,0)$.

Discontinuous Laguerre weight

Consider the **discontinuous Laguerre weight**

$$\omega(x;z) = \{1 - \xi\mathcal{H}(x-z)\}\,|x-z|^{\lambda}\,x^{\nu}\,\exp(-x), \qquad \nu,\lambda > 0, \quad x,z \in \mathbb{R}^{+}$$

with $\mathcal{H}(x)$ the Heaviside step function.

Since

$$\int_0^z x^{\nu}(z-x)^{\lambda}\mathrm{e}^{-x}\,\mathrm{d}x = B(\lambda+1,\lambda+k+1)\,z^{\nu+\lambda+1}\,\mathrm{e}^{-z}\,M(\lambda+1,\nu+\lambda+2,z)$$

$$\int_z^{\infty} x^{\nu}(x-z)^{\lambda}\mathrm{e}^{-x}\,\mathrm{d}x = \Gamma(\lambda+1)\,z^{\nu+\lambda+1}\,\mathrm{e}^{-z}\,U(\lambda+1,\nu+\lambda+2,z)$$

with $B(\lambda+1,\lambda+k+1) = \Gamma(\nu+1)\Gamma(\lambda+1)/\Gamma(\nu+\lambda+2)$ the **Beta function**, and $M(a,b,z)$ and $U(a,b,z)$ the **Kummer functions**, then the kth moment is given by

$$\begin{aligned}
\mu_k(z;\nu,\lambda) &= \int_0^{\infty} [1-\xi\mathcal{H}(x-z)]\,x^{\nu+k}\,|x-z|^{\lambda}\,\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_0^{z} x^{\nu+k}(z-x)^{\lambda}\,\mathrm{e}^{-x}\,\mathrm{d}x + (1-\xi)\int_z^{\infty} x^{\nu+k}(x-z)^{\lambda}\,\mathrm{e}^{-x}\,\mathrm{d}x \\
&= z^{\nu+\lambda+k+1}\,\mathrm{e}^{-z}\,\{B(\lambda+1,\lambda+k+1)\,M(\lambda+1,\nu+\lambda+k+2,z) \\
&\qquad + (1-\xi)\Gamma(\lambda+1)U(\lambda+1,\nu+\lambda+k+2,z)\}
\end{aligned}$$

Define the Hankel determinant

$$\Delta_n(z;\nu,\lambda) = \det\left[\mu_{j+k}(z;\nu,\lambda)\right]_{j,k=0}^{n-1}$$

then

$$H_n(z;\nu,\lambda) = z\frac{\mathrm{d}}{\mathrm{d}z}\ln\Delta_n(z;\nu,\lambda)$$

satisfies

$$\begin{aligned} z^2\left(\frac{\mathrm{d}^2H_n}{\mathrm{d}z^2}\right)^2 &= \left[(z+2n+\nu+\lambda)\frac{\mathrm{d}H_n}{\mathrm{d}z} - H_n + (2n+2\nu+\lambda)\,n\right]^2 \\ &\quad - 4\left(\frac{\mathrm{d}H_n}{\mathrm{d}z}+n\right)\left(\frac{\mathrm{d}H_n}{\mathrm{d}z}+n+\nu\right)\left[z\frac{\mathrm{d}H_n}{\mathrm{d}z} - H_n + (n+\nu+\lambda)n\right] \end{aligned}$$

which is equivalent to a special case of $\mathrm{S_V}$, the $\mathrm{P_V}$ σ-equation. Specifically, letting

$$H_n(z;\nu,\lambda) = \sigma - \tfrac{1}{4}(2n+\nu-\lambda)z + \tfrac{1}{2}n^2 + \tfrac{1}{2}n(\nu+\lambda) + \tfrac{1}{8}(\nu-\lambda)^2$$

yields

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - \left\{2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - z\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right\}^2 + 4\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}+\kappa_j\right) = 0 \qquad \mathrm{S_V}$$

with

$$\begin{aligned} \kappa_1 &= \tfrac{1}{2}n + \tfrac{3}{4}\nu + \tfrac{1}{4}\lambda, & \kappa_3 &= -\tfrac{1}{2}n - \tfrac{1}{4}\nu - \tfrac{3}{4}\lambda, \\ \kappa_2 &= \tfrac{1}{2}n - \tfrac{1}{4}\nu + \tfrac{1}{4}\lambda & \kappa_4 &= -\tfrac{1}{2}n - \tfrac{1}{4}\nu + \tfrac{1}{4}\lambda \end{aligned}$$

Conclusions

- The coefficients in the three-term recurrence relations associated with semi-classical generalizations of orthogonal polynomials and discrete orthogonal polynomials can often be expressed in terms of solutions of the Painlevé equations.
- These coefficients can be expressed as Hankel determinants which arise in the solution of the Painlevé equations and particularly the Painlevé σ-equations, the second-order, second-degree equations associated with the Hamiltonian representation of the Painlevé equations.
- These Hankel determinants arise in the special cases of the Painlevé equations when they have solutions in terms of the classical special functions, the "classical solutions" of the Painlevé equations.
- The moments of the semi-classical weight and the discrete weight provide the link between the orthogonal polynomials and the associated Painlevé equation.
- These results illustrate the increasing significance of the Painlevé equations in the field of orthogonal polynomials and special functions.