Constructive methods in differential systems & applications

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Global representation of solutions of differential systems

- Combining Borel summability and exponential asymptotic methods with classical techniques in regular regions we can now obtain global, nonperturbative results about solutions of ODEs and PDE, in a perturbative way.
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- The equation for E is obtained by Taylor-Fréchet expansion of $\mathcal{N}$,

$$\mathcal{N}(u_0+E;\epsilon)=0=\mathcal{N}(u_0;\epsilon)+L\,E+\mathcal{N}_1E \Leftrightarrow L\,E=-\delta-\mathcal{N}_1(E)$$

$$E=-L^{-1}\delta-L^{-1}\mathcal{N}_1(E)$$

where $LE=\frac{\partial\mathcal{N}}{\partial u}|_{u=u_0}E=O(\epsilon)$, $\delta=\mathcal{N}\,[u_0]$ and $\mathcal{N}_1(E)=\mathcal{N}(u_0+E)-L\,E$ is expected to be $O(\epsilon^2)$. Then, for rigorous analysis:

- L is inverted in a suitable way, subject to the given initial/boundary conditions, and one makes use of the contractive mapping theorem in an adapted norm.

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- However: many problems are not solvable in closed form and do not come with any obvious small parameter either.
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- (i) One key idea to start with: u_0 in the above analysis does not have to be an *exact* solution. All we need in a perturbative approach is to understand the local and global behavior of u_0 and how to obtain accurate estimates, not a closed form expression... *Approximate* solutions (meaning that $\mathcal{N}[u_0]$ is small in some norm) which are accurate enough should work as well. Then "ϵ" is essentially $\|u - u_0\|$.

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- We used this approach on a number of questions that had been open for some time.
- One of them is the Dubrovin's conjecture: absence of poles of the *tritronqueé* solution of P_I

$$y'' = 6y^2 + z \quad (*)$$

in $\{z : \arg z \in [-3\pi/5, \pi]\}$,

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Transseries and global representations-

- **Local expansions and global information; very simple illustration: erfc**. This solves the equation

$$f' - 2xf = 1$$

The solutions are entire. Take, say, initial condition $f(0) = \sqrt{\pi/2}$ ($\Rightarrow f(x) = \sqrt{\frac{\pi}{2}} e^{x^2} \operatorname{erfc}(x)$).

- At zero we have convergent power series:

$$f(x) = \sqrt{\pi/2} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!!} \quad (*)$$

Though convergent everywhere, this is does not provide in a useful way global information, but only for small enough x: $x = 5$ is not small:For 10^{-3} accuracy about 75 terms need to be kept. f is singular at ∞ and in some sense the influence of the sing. is felt at 5. But the Taylor series can be optimized by replacing Taylor series with, say, Chebyshev-Padé representations.

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- For **large** x, $\arg(x) \in [-\pi/2, \pi/2]$, we have the asymptotic expansion

$$f(x) \sim \sum_{k=0}^{\infty} \frac{(-1)^k (2k-1)!!}{2^{k-1} x^{2k+1}} \ (x \to +\infty) \qquad (*)$$

- This is everywhere *divergent* (but Borel summable). But a three term truncation suffices for 10^{-3} relative accuracy on $[5, +\infty]$.
- Since the series diverges, f *must behave differently* in other *directions* towards ∞. For large **negative** x it is given by

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- The series part is same as (*) $\sqrt{\pi}$ is a *connection constant* (here we have a symmetry, $f(x) + f(-x) = \sqrt{\pi} e^{x^2}$)
- Combining the above, in a sense "everything" in terms of qualitative and quantitative information about erfc follows (i.e., except for explicit connection formulas or values, etc.) *For classical functions all this is of course*

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Nonlinear equations

- Transseries, exponential asymptotics, trans-asymptotic matching and Borel summability are now able to global info for quite general solutions of ODEs, PDEs and difference equations.
- How do solns. of nonlinear equations behave? Consider a simple example,

$$y' + y = \frac{1}{x^2} + y^4 \quad (*)$$

As $x \to \infty$, (*) has a unique power series solution $\tilde{y}_0(x) = \frac{1}{x^2} + \frac{2}{x^3} + \frac{6}{x^4} + \ldots$ which can be shown to be divergent too. The general solution should contain a free parameter. To find possible further terms in a formal expansion we look for

$$\tilde{y} = \tilde{y}_0 + \delta$$

Since $\tilde{y}_0$ formally solves (*), if $\delta \to 0$, then $\delta' + \delta \sim 4\tilde{y}_0\delta$, $\Rightarrow \delta = Ce^{-x}\tilde{y}_1(x) + \delta_1$ where $\tilde{y}_1(x)$ is a (divergent) series.

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- Looking for formal solutions (as $\tilde{y}_0 + E$, E small we get a *transseries*

$$\tilde{y} = \tilde{y}(x; C) = \tilde{y}_0(x) + C\mathrm{e}^{-x}\tilde{y}_1(x) + C^2\mathrm{e}^{-2x}\tilde{y}_2(x) + \dots$$

where $\tilde{y}_k(x)$ are divergent power series, Borel summable, and C is an arbitrary parameter. This is a **transseries** (formal) solution) valid for $x \to \infty$ along directions in the complex plane for which the terms can be well ordered decreasingly, namely for $x \in e^{ia}\mathbb{R}_+$ with $|a| < \frac{\pi}{2}$.

- Beyond $|a| < \frac{\pi}{2}$ nonlinear eq. solutions develop singularities; their expansion is obtained by reordering the terms of the transseries by a general procedure.

- Rewriting of the transseries:

$$\sum_{k=0}^{\infty}(Ce^{-x})^k\tilde{y}_k = \sum_{k,l=0}^{\infty}\frac{a_{kl}}{x^l}(Ce^{-x})^k = \sum_{k,l=0}^{\infty}a_{kl}x^{-l}\xi^k \quad (|x| \to \infty, \, \mathrm{Re}\,(x) > 0)$$

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- For general meromorphic systems, transseries as $x \to +\infty$ are of the form

$$\sum_{\mathbf{k} \geqslant -\mathbf{M}} a_{\mathbf{k}} \boldsymbol{\xi}^{\mathbf{k}}, \ \boldsymbol{\xi}^{\mathbf{k}} =: \xi_1^{k_1} \cdots \xi_l^{k_l}$$

where ξ_i are *transmonomials*, expressions such as $1/x, x^{-\beta} e^{-\lambda x}$, and rarely iterated exps. $\mathrm{Re}\,\lambda_i x > 0$, constructed by induction on level–number of iterated exponentials–possibly applied to an iterated log of x; valid if $\xi_i = o(1)$. To allow ξ to be large we rewrite

$$\sum_{k,l=0}^{\infty} a_{kl} x^{-l} \xi^k = \sum_{k=0}^{\infty} x^{-k} F_k(\xi); \ \xi = Ce^{-x}$$

where now ξ is allowed to be large. Typically, solutions represented by two-scale expansions have arrays of movable singularities (O.C., R. Costin, Invent. Math 2001, IMRN (2012)). For $\mathrm{P_I}$,

$$F_0(\xi) = \frac{144\,\xi}{(\xi-12)^2}, \ F_1(\xi) = \frac{\frac{1}{60}\xi^4 - 3\xi^3 - 210\xi^2 - 216\xi}{(\xi-12)^3}, \ldots, \ F_n(\xi) = \frac{P_n(\xi)}{(\xi-12)^{n+2}} \quad (1)$$

near $\xi = 12$ we have arrays of poles. Transseries & two scale expansions match each-other; these can be shown to match – in compact regular regions– Taylor series, or more generally classical polynomial expansions.

Representations of general functions; transseries

- Transseries were introduced in the pioneering work of Ecalle, $\sim$ 1980. Later made rigorous, extended by Balser, Braaksma, Malgrange, Ramis,...
- We can now provide global information –as matching local expansions covering $\mathbb{C}$ and use the expansions to provide sharp and rigorous estimates.
- Motivated by two specific open problems, we are now applying exponential asymptotics techniques in a **new direction**: that of quantitative global representation of relatively general large classes of functions, for solving existence uniqueness etc. questions when classical method do not apply.

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The problem and the strategy, in a nutshell-

- Assume we are seeking to show existence and uniqueness and find the properties in a large region of the solution of a linear or nonlinear problem, $\mathcal{N}(y(x), x) = 0$.
- *First we find $\tilde{y}$, an approximate global solution* (i.e. $\mathcal{N}(\tilde{y}, x)$ small), made of matched transseries, double-scale expansions and classical polynomials.
- Then, by definition/construction, we have a $\tilde{y}$ s.t. $\mathcal{N}(\tilde{y}(x), x) = \varepsilon(x)$ with ε small enough in suitable norms. Then seek y in the form $y = \tilde{y} + E$.
- Exactly as in perturbation theory, E, expected to be small, satisfies

$$LE = \varepsilon(x) + \mathcal{N}_1(E); \quad L = \frac{\partial \mathcal{N}}{\partial y}(*)$$

where $\mathcal{N}_1(E)$ is a "small" nonlinearity. We solve the quasi-linear equation (*) by, writing it in **contractive integral form**, typically

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- *Next, contractive mapping arguments (for $L^{-1}\mathcal{N}_1$) show that the solution E* exists and is small. For the estimates, rough bounds $L^{-1}\mathcal{N}_1$ suffice.

The problem and the strategy, in a nutshell-

- Assume we are seeking to show existence and uniqueness and find the properties in a large region of the solution of a linear or nonlinear problem, $\mathcal{N}(y(x),x)=0$.
- *First we find $\tilde{y}$, an approximate global solution* (i.e. $\mathcal{N}(\tilde{y},x)$ small), made of matched transseries, double-scale expansions and classical polynomials.
- Then, by definition/construction, we have a $\tilde{y}$ s.t. $\mathcal{N}(\tilde{y}(x),x)=\varepsilon(x)$ with ε small enough in suitable norms. Then seek y in the form $y=\tilde{y}+E$.
- Exactly as in perturbation theory, E, expected to be small, satisfies

$$LE=\varepsilon(x)+\mathcal{N}_1(E); \quad L=\frac{\partial\mathcal{N}}{\partial y}(*)$$

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- This is a quite general approach, applicable roughly in all low dimensional problems for which it is possible to "check the result numerically" (numerical checking may be a hard task!). All this is of course natural; the new element is set of techniques to represent solutions (i) accurately, (ii) "economically" and (iii) with rigorously controlled errors, for many problems for quite general ODEs, difference equations and some classes of PDEs.
- I will illustrate this on the solution of two problems which were open.

  –Conditional stability of solitons in focusing NLS;

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NLS (OC, M Huang, W Schlag, Nonlinearity 2011)-

- $$i\psi_t + \Delta\psi + |\psi|^{2p}\psi = 0; \quad x \in \mathbb{R}^d \tag{2}$$
- Here, $p \in (0, \frac{2}{d-2})$ if $d \geqslant 3$. Standing wave solutions ("solitons") $\psi = e^{it\alpha^2}Q(x)$ where we restrict to the case where Q is a ground state, that is
$$\alpha^2 Q - \Delta Q = Q^{2p+1}; \quad Q > 0 \tag{3}$$
- Known: such Q exist, they are radial $Q = Q(|x|)$, smooth, expo decaying, (Strauss, Berestycki, Lions; for uniqueness, Coffman, McLeod, Serrin, and Kwong). In one dimension $d = 1$, these ground states are explicitly given as
$$Q(x) = (p+1)^{1/2p}\cosh^{-1/p}(px) \tag{4}$$
- In 3d, Q is likely non-explicit (equation is non-integrable).
- An important question: is the scattering property of the (unstable) soliton: whether with I.C. on the center stable manifold, solutions decompose into soliton + free wave $+o(1)$ in t (Buslaev, Strauss, Grillakis, Weinstein and for the latter, Soffer-Weinstein, Perelman, Cuccagna, Schlag.)

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- In order to study stability, one generally linearizes around the standing wave.
- This leads to matrix Schrödinger operators of the form

$$\begin{pmatrix} -\Delta+\alpha^2 & 0 \\ 0 & \Delta-\alpha^2 \end{pmatrix} + \begin{pmatrix} -(p+1)Q^{2p} & -pQ^{2p} \\ pQ^{2p} & (p+1)Q^{2p} \end{pmatrix} \tag{5}$$

- and upon conjugation with $\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ we get the operator

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- Schlag (Ann. of Math (2009), showed conditional scattering for the unstable case $p=1$ (focusing, supercritical). The results are conditional on the fact that neither L_+ nor L_- have any eigenvalues in the gap $(0;\alpha^2]$ ($\alpha=1$) and L_- has no resonance at 1 (reduction uses ideas of Perelman).
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- One difficulty with standard approaches is that there is no clear intrinsic reason for the gap condition to hold; "on the contrary" the problem is dangerously close (± 0.01 in the power p of the nonlinearity), to one in which the answer is different.
- Our new method was introduced to solve this problem, but has evolved and simplified much since then.
- The soliton "Q" is spherically symmetric, and is the unique regular, decaying, positive solution of

$$-Q''(r) - \frac{2}{r}Q'(r) + Q(r) - Q^3(r) = 0 \tag{7}$$

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Representation as $r \to \infty$ (here $|r| > 2.5$): First the "Jost" solution:

Lemma

There exists a unique positive solution $y(r;\beta)$ to (7) with the property that $y(r;\beta) \sim \beta r^{-1} e^{-r}$ as $r \to \infty$. It satisfies

$$\left| \frac{y(r;\beta)}{y_3(r;\beta)} - 1 \right| < 4.6 \cdot 10^{-6} \qquad \forall\, r \geqslant \frac{5}{2} \tag{8}$$

where [a]

$$y_3(r;\beta) = r^{-1}\beta e^{-r} + \beta^3 - r^{-1}\left(2e^r \mathrm{Ei}(-4r) - e^{-r}\mathrm{Ei}(-2r)\right)$$

[a] $\beta \in (1,3)$; a specific β is chosen later by matching.

- For a global rep., we match with Legendre poly. expansion on $[0, 2.5]$.
- Set

$$\tilde{Q}(r) := \begin{cases} p_1(r) & \text{for} \quad 0 \leqslant r < 5/2 \\ y_3(r;\beta) & \text{for} \quad r \geqslant 5/2 \end{cases} \tag{9}$$

where p_1 is given in terms of a piecewise-polynomial (2 pieces) of degree 11, with explicit rational coefficients.

Lemma (O C, M. Huang, W. Schlag, Nonlinearity 2011)

Let Q be the exact ground state of (7) and $\tilde{Q}$ be the approximate one given above. Then one has the error bound

$$\left| \frac{\tilde{Q}(r)}{Q(r)} - 1 \right| \leqslant 7 \cdot 10^{-5} \quad \forall\, r \geqslant 0 \tag{10}$$

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- Is this "extravagant" accuracy really needed? Surprisingly perhaps, it is.
- *Idea of proof.* It is first checked that $\tilde{Q}$ satisfies the equation within an error $\varepsilon, \|\varepsilon\| \sim 10^{-6}$ relative error in L^∞. By contractve mapping arguments we show that there is an actual solution within the error bound listed above.
- Contraction mapping estimates, requiring taking absolute values where cancellations might occur, etc., account for the loss of accuracy from 10^{-6} to $7 \cdot 10^{-5}$
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Theorem (Gap property)

In $[0,1]$, $L_+ = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + 1 - 3Q^2$ has no eigenvalue or resonance and L_- has no eigenvalue.

Essentially the same strategy is now applied to stability (spectral) problem

$$L_+ u = \lambda u; \; L_- u = \lambda u \tag{11}$$

where (as before) $L_+ = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + 1 - 3Q^2$, $L_- = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + 1 - Q^2$.
We have
Standard ODE analysis shows that there are two solutions $u_1(r;\lambda)$ and $u_2(r;\lambda)$ of the equation (11) with the properties $u_1(0;\lambda) = 1$ and $u_2(r;\lambda) = r^{-1} e^{-r\sqrt{1-\lambda}}(1 + o(1)), \; r \to \infty$.
Let $W = u_1 u_2' - u_2 u_1'$ be the Wronskian of these two special solutions. Clearly, the existence of an eigenvalue of L_+ is equivalent to $W = 0$ for some λ. The lemma thus follows from the result below.

Lemma

We have the following estimate

$$\inf_{\lambda \in [0,1]} |W(\tfrac{5}{2}; \lambda)| \geqslant \frac{1}{250} \tag{12}$$

from which the gap condition follows. The low value of the Wronskian is what requires working with such accuracy. (Note: the numerically calculated inf is 40% larger –still very small–this is why we need such accuracy throughout.)

solving Dubrovin's conjecture (OC, M Huang, S Tanveer, Duke M. J. 2013

- The Painlevé equation P1 is

$$y'' = 6y^2 + z$$

  Painlevé equations occur in numerous applications, random matrices, combinatorics, number theory, KdV etc.

- The Dubrovin conjecture relates to a special solution of P_I, the *tri-tronquée*, y_t, unique mod. symmetries (the eq. has a 5-fold symmetry) given by the property that it has no poles for large z, $\arg z \in [-3\pi/5, \pi]$.

  Dubrovin's conjecture($\sim$ 1993, crucial for instance in understanding blow-up in NLS) states y_t is analytic for *all* z $\arg z \in [-3\pi/5, \pi]$.

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The Dubrovin conjecture follows from the following

Theorem (OC, M. Huang, S. Tanveer (DMJ, 2013))

The tritronquée y_t is analytic in the region

$$\left\{z \neq 0 : \arg z \in [-3\pi/5, \pi]\right\} \cup \left\{z : |z| < \frac{37}{20}\right\} \tag{13}$$

- **Proof.** We first use the symmetry of the solution w.r.t. $i\mathbb{R}^+$ to reduce the question to showing existence in a bisected sector, $\arg x \in [-\pi/2, \pi/2]$, where $x = \frac{e^{i\pi/4}}{30}(24z)^{5/4}$.
- The solution is given, in the four of the five[1] sectors of symmetry the tritronquée, by four matched expressions, (14), (15), (16) below, + a Taylor series near zero.

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The Dubrovin conjecture follows from the following

Theorem (OC, M. Huang, S. Tanveer (DMJ, 2013))

The tritronquée y_t is analytic in the region

$$\left\{z \neq 0 : \arg z \in [-3\pi/5, \pi]\right\} \cup \left\{z : |z| < \frac{37}{20}\right\} \tag{13}$$

- **Proof.** We first use the symmetry of the solution w.r.t. $i\mathbb{R}^+$ to reduce the question to showing existence in a bisected sector, $\arg x \in [-\pi/2, \pi/2]$, where $x = \frac{e^{i\pi/4}}{30}(24z)^{5/4}$.
- The solution is given, in the four of the five[1] sectors of symmetry the tritronquée, by four matched expressions, (14), (15), (16) below, + a Taylor series near zero.

[1] The fifth is a sector with poles.

Main steps

- The quasisolution we use in the region $|z| \geqslant 1.7$, $\arg x \in [-\pi/4, \pi/2]$ is simply the asymptotic series with one nontrivial order,

$$y_0(z) = i\sqrt{\frac{z}{6}}\left(1 - \frac{4}{25x^2}\right) \text{ with } x = \frac{e^{i\pi/4}}{30}(24z)^{5/4} \tag{14}$$

- The region $|z| \geqslant 1.7$ $\arg x \in [-\pi/2, -\pi/4]$ is close to the antistokes line $-i\mathbb{R}^-$ along which the solution starts oscillating and beyond which poles will develop. Here we use a truncated two-scale expansion $(1/x, \xi)$, $\xi = Cx^{-1/2}e^{-x}$ explained before, as $y_0(z) = i\sqrt{\frac{z}{6}}\left(1 - \frac{4}{25x^2} + h_0(x)\right)$ with

$$h_0(x) = \left(\xi + \frac{\xi^2}{6} + \frac{\xi^3}{48} + \frac{\xi^4}{432} + \frac{5\xi^5}{20736}\right) + \frac{1}{x}\left(-\frac{\xi}{8} - \frac{11}{72}\xi^2 - \frac{43}{1152}\xi^3\right) + \frac{9\xi}{128x^2}, \tag{15}$$

- We show that there is an actual solution close to the quasisolution. This takes care of the nbd of ∞, meaning here $|z| > 1.7$.

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- To cover now the disk, $|z| \leqslant 1.85$ we first solve the eq. from $|z| = 1.7$ down to zero.
- We first use the substitution $g(t) = e^{2\pi i/5} y(-te^{i\pi/5})$ which makes the equation and solution real valued, $g'' = 6g^2 + t$. The region of interest is $0 \geqslant t \geqslant t_0 = -1.7$. Here we do a more careful calculation to find g, g' within $5 \cdot 10^{-3}$, and use these values at t_0 as an initial condition for solving the ODE up to $t = 0$ and match to a power series.
- On the segment $t \in [t_0, 0]$, the quasisolution is obtained through Chebyshev projection (of the power series at zero obtained by matching at t_0).

$$g_0(t) = -\frac{280}{519} + \frac{150s}{1013} + \frac{239s^2}{10331} + \frac{110s^3}{14779} - \frac{32s^4}{9853} + \frac{9s^5}{4397} - \frac{16s^6}{39505} + \frac{8s^7}{49105}, \text{ where } s = t - t_0. \tag{16}$$

- The relative difference between g_0 and the actual solution is around 1% in $L^\infty([-1.7, 0])$.
- We thus get g, g' at zero within 1%. The last step is to show that there is a convergent power series solution in a ball of radius at least 1.85 if the first two coefficients c_0, c_1 are within 1% of $g(0), g'(0)$. This is done by relatively straightforward estimated on the recurrence of the Taylor coefficients; this ends the proof.

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Comments; directions of future research

- This new approach, computational but rigorous, applies to many concrete problems. It is straightforward and "robust", as the precision can be increased as needed.
- Whenever qualitative methods exist, they would usually still be preferable since they are more likely to provide "a simple reason" for which a result holds. Insofar as $\mathrm{P_I}$ goes, for us the Dubrovin conjecture holds because the Stokes multiplier μ is numerically quite small while first pole position is increasing as a function of μ.
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- We are working on the extension of the method to PDEs, where there are many open questions amenable to this method.

  Many solutions of the thin film equation

  $$h_t + (h_{xxx})_x = 0$$

  break down (pinch) in finite time. There is very convincing numerical evidence, but no proof and the problem has been open for a long time.

  However, when trying to prove the expected result we obtained inconsistencies instead... It turned out that the standard pinching shape ansatz assumed in the literature and based on the (relatively vast) numerical literature, missed some logarithmic corrections, virtually invisible unless explicitly searched for, and the first step for us was to redo the numerical analysis carefully...

Thank you.

Borel summability in a nutshell

- A formal solution of $y' + y = 1/x$ as $x \to +\infty$ is

$$\tilde{y} = \sum_{k=0}^{\infty} \frac{k!}{(-x)^{k+1}}$$

nowhere convergent. At the core of Borel summability: note that $k!/x^{k+1} = \int_0^\infty e^{-px} p^k dp$ and thus:

$$\sum_{k=0}^{\infty} \frac{k!}{(-x)^{k+1}} \text{"} = \text{"} \sum_{k=0}^{\infty} \int_0^\infty e^{-px}(-p)^k dp \text{"} = \text{"} \int_0^\infty \frac{e^{-px}}{1+p} dp$$

It turns out that for generic linear or nonlinear systems of ODEs, many PDEs etc at irregular singular points, modulo normalizing changes of variables, factorial divergence is the only one encountered. A rigorous version of the above (plus some generalizations) allows for dealing with large classes of systems.