On Multiple Orthogonal Polynomials

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Orthogonal polynomials

Orthonormal polynomials:

$$\int p_n(x)p_k(x)w(x)dx = \delta_{n,k},$$

where $p_n(x)$ has degree n and integration is over the support of $w(x) \geq 0$ in $\mathbb{R}$.

For a sequence $(p_n)_{n\in\mathbb{N}}$ of orthonormal polynomials one has a three-term recurrence relation:

$$xp_n(x) = \alpha_{n+1}p_{n+1}(x) + \beta_n p_n(x) + \alpha_n p_{n-1}(x)$$

with $p_{-1} = 0$.

Monic polynomials satisfy

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \alpha_n^2 p_{n-1}(x).$$

Multiple orthogonal polynomials

Multiple orthogonal polynomials (MOPs) are generalizations of orthogonal polynomials, which originated from Hermite-Padé approximation in the context of irrationality and transcendence proofs in number theory and further developed in approximation theory.

During the past few years, multiple orthogonal polynomials have also arisen in a natural way in certain models from mathematical physics, including random matrix theory, non-intersecting paths, etc.

MOPs are polynomials of one variable which are defined by orthogonality relations with respect to r different weights $w_1, w_2, \ldots, w_r$, where $r \geq 1$ is a positive integer.

Let $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ be a multi-index of size $|\vec{n}| = n_1 + n_2 + \cdots + n_r$ and suppose $\mu_1, \mu_2, \ldots, \mu_r$ are r measures with supports on certain simple curves in the complex plane.

The type II multiple orthogonal polynomial is the monic polynomial

$$P_{\vec{n}}(x) = x^{|\vec{n}|} + \cdots$$

of degree $|\vec{n}|$ satisfying the conditions

$$\int P_{\vec{n}}(x) x^k w_1(x)\, dx = 0, \qquad k = 0, 1, \ldots, n_1 - 1,$$

$$\vdots$$

$$\int P_{\vec{n}}(x) x^k\, w_r(x)\, dx = 0, \qquad k = 0, 1, \ldots, n_r - 1.$$

Example. Let $r = 2$, $\vec{n} = (n, m)$. The multiple Hermite polynomials are defined by

$$\int_{-\infty}^{\infty} x^k H_{n,m}(x) e^{-x^2 + c_1 x}\, dx = 0, \qquad k = 0, 1, \ldots, n - 1,$$

$$\int_{-\infty}^{\infty} x^k H_{n,m}(x) e^{-x^2 + c_2 x}\, dx = 0, \qquad k = 0, 1, \ldots, m - 1,\ c_1 \neq c_2.$$

Recurrence relations for MOPs

Recall that for monic OPs ($r = 1$) we have a 3-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \alpha_n^2 p_{n-1}(x).$$

What happens for MOPs?

Let $r = 2$. The multi-index $\vec{n}$ is now given by $(n, m) \in \mathbb{N}^2$. The recurrence relations is given by

$$\begin{aligned} xP_{n,m}(x) &= P_{n+1,m}(x) + c_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x), \\ xP_{n,m}(x) &= P_{n,m+1}(x) + d_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x), \end{aligned}$$

with $a_{0,m} = 0$ and $b_{n,0} = 0$ for all $n, m \geq 0$.

For general $r > 1$ we have the following nearest-neighbor recurrence relations:

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_1}(x) + b_{\vec{n},1}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x),$$

$$\vdots$$

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_r}(x) + b_{\vec{n},r}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x),$$

where $\vec{e}_j = (0,\dots,0,1,0,\dots,0)$ is the j-th standard unit vector with 1 on the j-th entry, $(a_{\vec{n},1},\dots,a_{\vec{n},r})$ and $(b_{\vec{n},1},\dots,b_{\vec{n},r})$ are the recurrence coefficients.

For $r = 2$ we use the following notation for the recurrence coefficients:

$$a_{\vec{n},1} = a_{n,m}, \;\; a_{\vec{n},2} = b_{n,m}, \;\; b_{\vec{n},1} = c_{n,m}, \;\; b_{\vec{n},2} = d_{n,m}.$$

Wronskians of multiple orthogonal polynomials

[L. Zhang and G. Filipuk, *On certain Wronskians of multiple orthogonal polynomials*, in preparation.]

We study certain Wronskians with entries given by multiple orthogonal polynomials. We show that depending on the size of the determinant we can either get a strict positivity of the Wronskian (which gives rise to the Turán type inequalities for multiple Hermite and Laguerre orthogonal polynomials (of the first and second kind)) or prove that real zeros of the associated Wronskians strictly interlace. We also present numerical studies of the distribution of zeros of Wronskians in the complex plane.

Orthogonal polynomials associated with an exponential cubic weight

[G. Filipuk, W. Van Assche and L. Zhang, *Multiple orthogonal polynomials associated with an exponential cubic weight*, arXiv:1306.3835, submitted.]

Consider the three rays

$$\Gamma_k = \{z \in \mathbb{C} : \arg z = \omega^k\}, \qquad k = 0, 1, 2,$$

where

$$\omega = e^{2\pi i/3},$$

and the orientations are all taken from left to right.

We shall denote by $p_n^{(1)}$ the monic polynomials satisfying

$$\int_\Gamma p_n(x) x^k e^{-x^3} dx = 0, \qquad k = 0, 1, \ldots, n-1, \tag{1}$$

with $\Gamma = \Gamma_0 \cup \Gamma_1$ and recurrence coefficients $\beta_n^{(1)}$ and $(\alpha_n^{(1)})^2$. In a similar manner, we set $p_n^{(2)}$ to be the polynomials satisfying (1) with $\Gamma = \Gamma_0 \cup \Gamma_2$, and denote by $\beta_n^{(2)}$ and $(\alpha_n^{(2)})^2$ the associated recurrence coefficients.

The three-term recurrence is given by relation

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \alpha_n^2 p_{n-1}(x),$$

where

$$\beta_n = \frac{\int_\Gamma x p_n^2(x) e^{-x^3} dx}{\int_\Gamma p_n^2(x) e^{-x^3} dx}, \qquad \alpha_n^2 = \frac{\int_\Gamma x p_n(x) p_{n-1}(x) e^{-x^3} dx}{\int_\Gamma p_{n-1}^2(x) e^{-x^3} dx},$$

and the initial condition is taken to be $\alpha_0^2 p_{-1} = 0$. It is shown by A. Magnus that the recurrence coefficients β_n and α_n^2 satisfy the string equations

$$\alpha_{n+1}^2 + \beta_n^2 + \alpha_n^2 = 0,$$
$$3\alpha_n^2(\beta_{n-1} + \beta_n) = n.$$

One can determine $(\beta_n^{(1),(2)}, \left(\alpha_n^2\right)^{(1),(2)})$ recursively from the string equations with initial condition $(\frac{\Gamma(2/3)}{\Gamma(1/3)}e^{\pi i/3}, 0)$ and $(\frac{\Gamma(2/3)}{\Gamma(1/3)}e^{-\pi i/3}, 0)$ respectively.

Actually, one can prove that

$$\beta_n^{(1)} = b_n e^{\pi i/3}, \qquad (\alpha_n^{(1)})^2 = a_n e^{-\pi i/3},$$

$$\beta_n^{(2)} = b_n e^{-\pi i/3}, \qquad (\alpha_n^{(2)})^2 = a_n e^{\pi i/3},$$

where

$$a_n + a_{n+1} = b_n^2,$$
$$3a_{n+1}(b_n + b_{n+1}) = n + 1.$$

Multiple orthogonal polynomials associated with exponential cubic weight

For $(k,l) \in \mathbb{N}^2$, we are interested in the polynomials $P_{k,l}$ of degree $k+l$ which satisfy the orthogonality conditions

$$\int_{\Gamma_0 \cup \Gamma_1} x^i P_{k,l}(x) e^{-x^3} dx = 0, \qquad i = 0, 1, \ldots, k-1,$$
$$\int_{\Gamma_0 \cup \Gamma_2} x^i P_{k,l}(x) e^{-x^3} dx = 0, \qquad i = 0, 1, \ldots, l-1.$$

If one of k and l is equal to zero, then $P_{k,l}$ reduce to the usual orthogonal polynomials with respect to the exponential cubic weight e^{-x^3}, i.e.,

$$P_{k,0}(x) = p_k^{(1)}(x), \qquad P_{0,k}(x) = p_k^{(2)}(x).$$

It can be shown that the following Rodrigues formula holds:

$$P_{n,n+m}(x) = \frac{(-1)^n}{3^n} e^{x^3} \frac{d^n}{dx^n}\left(e^{-x^3} P_{0,m}(x)\right),$$
$$P_{n+m,n}(x) = \frac{(-1)^n}{3^n} e^{x^3} \frac{d^n}{dx^n}\left(e^{-x^3} P_{m,0}(x)\right).$$

Result: The recurrence coefficients for MOPs in the nearest neighbor recurrence relations

$$\begin{aligned} xP_{n,n+m}(x) &= P_{n+1,n+m}(x) + c_{n,n+m}P_{n,n+m}(x) \\ &\qquad + a_{n,n+m}P_{n-1,n+m}(x) + b_{n,n+m}P_{n,n+m-1}(x), \\ xP_{n,n+m}(x) &= P_{n,n+m+1}(x) + d_{n,n+m}P_{n,n+m}(x) \\ &\qquad + a_{n,n+m}P_{n-1,n+m}(x) + b_{n,n+m}P_{n,n+m-1}(x), \end{aligned}$$

and

$$\begin{aligned} xP_{n+m,n}(x) &= P_{n+m+1,n}(x) + c_{n+m,n}P_{n+m,n}(x) \\ &\qquad + a_{n+m,n}P_{n+m-1,n}(x) + b_{n+m,n}P_{n+m,n-1}(x), \\ xP_{n+m,n}(x) &= P_{n+m,n+1}(x) + d_{n+m,n}P_{n+m,n}(x) \\ &\qquad + a_{n+m,n}P_{n+m-1,n}(x) + b_{n+m,n}P_{n+m,n-1}(x), \end{aligned}$$

can be expressed explicitly in terms of a_n, b_n satisfying

$$\begin{aligned} a_n + a_{n+1} &= b_n^2, \\ 3a_{n+1}(b_n + b_{n+1}) &= n + 1. \end{aligned}$$

Thank you very much for your attention!