On solvability of linear differential systems by quadratures

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Linear differential systems

Consider a linear diff. system of p equations:

$$\frac{dy}{dz} = B(z)\, y, \qquad y(z) \in \mathbb{C}^p;$$

$B(z)$ is a merom. $(p \times p)$-matrix, that is, its entries are from the field $\mathbb{C}(z)$ of rational functions,

$a_1 \ldots, a_n \in \overline{\mathbb{C}}$ are singular points of $B(z)$.

Two types of singularities: **regular** and **irregular**.

The Picard–Vessiot extension $\mathbb{C}(z) \subset F$ is a field obtained by adjoining to $\mathbb{C}(z)$ all elements of a fundamental matrix of the system.

A linear diff. system is **solvable by quadratures** if there is a tower of fields

$$\mathbb{C}(z) \subset F_1 \subset \ldots \subset F_k, \qquad F \subset F_k,$$

such that each F_{i+1} is obtained by adjoining to F_i an exponential or integral of some element from F_i.

Galois group

The Galois group G of a linear diff. system is a group of diff. automorphisms of the Picard–Vessiot extension $\mathbb{C}(z) \subset F$

$$G = \{\sigma : F \to F \,|\, \sigma(f) = f \quad \forall f \in \mathbb{C}(z)\}.$$

For a fundamental matrix $Y(z)$ of the system one has

$$\sigma : Y(z) \mapsto Y(z)C, \qquad C \in \mathsf{GL}(p, \mathbb{C}),$$

and $G \subset \mathsf{GL}(p, \mathbb{C})$ is a matrix algebraic subgroup.

The monodromy group M of a linear diff. system is a subgroup of its Galois group: $M \subset G$. And for a system with regular singularities $G = \overline{M}$ (the closure in the Zariski topology).

Example. The following system of two equations has one irregular singularity $z = \infty$:

$$\frac{dy}{dz} = \begin{pmatrix} 0 & 1 \\ 0 & 2z \end{pmatrix} y, \qquad Y(z) = \begin{pmatrix} 1 & \int e^{z^2} dz \\ 0 & e^{z^2} \end{pmatrix}.$$

The system is solvable, $M = \{\mathrm{Id}\}$, $G \neq \overline{M}$ is upper triangular.

Theorem (Picard–Vessiot). A linear diff. system is solvable by quadratures $\iff$ the Galois group G is solvable, that is, there is a tower of normal subgroups

$$e \subset G_1 \subset \ldots \subset G_k = G$$

such that all the factors G_{i+1}/G_i are commutative.

How can one understand if a system is solvable by quadratures looking at its coeff. matrix?

First consider a **Fuchsian** system

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i}{z - a_i} \right) y;$$

$B_1, \ldots, B_n \in \mathrm{Mat}(p, \mathbb{C})$ are residue matrices.

The eigenvalues $\beta_i^1, \ldots, \beta_i^p$ of the matrix B_i are called the **exponents** of the system at the point a_i.

Solvability of Fuchsian systems with bounded exponents

Theorem 1. Let the eigenvalues β_i^j of the residue matrices B_i satisfy inequalities

$$\operatorname{Re}\beta_i^j > -1/(n(p-1)),$$

and for each pair $\beta_i^j \neq \beta_i^l$, $i = 1,\ldots,n$, one of the following conditions holds:

$$1)\, \operatorname{Re}\beta_i^j - \operatorname{Re}\beta_i^l \notin \mathbb{Q}; \qquad 2)\, \operatorname{Im}\beta_i^j \neq \operatorname{Im}\beta_i^l.$$

Then the Fuchsian system is solvable by quadratures iff all the matrices B_i are (upper-)triangular (in some basis).

Remark. One of the conditions

$$1)\,\mathrm{Re}\,\beta_i^j-\mathrm{Re}\,\beta_i^l\notin\mathbb{Q},\qquad 2)\,\mathrm{Im}\,\beta_i^j\neq\mathrm{Im}\,\beta_i^l$$

of the previous theorem holds iff

$$(\mu_i^j/\mu_i^l)^N\neq 1\qquad\forall N\in\mathbb{N}$$

for each pair $\mu_i^j\neq\mu_i^l$ of the eigenvalues of the monodromy matrix M_i, $i=1,\ldots,n$ (since $\mu_i^j=e^{2\pi\mathbf{i}\beta_i^j}$).

Theorem 2. If the Jordan form of each monodromy matrix of the Fuchsian system consists of one block, then this system is solvable by quadratures iff all its residue matrices B_i are triangular (in some basis).

Example (A. Bolibrukh). There are 4 upper-triangular matrices $M_i \in \mathrm{GL}(7, \mathbb{C})$ such that there exists a Fuchsian system having them as the monodromy matrices. But the coeff. matrix of this system can not be transformed to the upper-triangular form.

Thus, this Fuchsian system is solvable by quadratures (the monodromy group $M = \langle M_1, \ldots, M_4 \rangle$ is solvable) but its residue matrices are not triangular.

Now consider a linear diff. system

$$\frac{dy}{dz} = B(z)\, y, \qquad y(z) \in \mathbb{C}^p,$$

with **non-resonant** irregular singularities $a_1, \ldots, a_n$. This means that the leading term B_{i,r_i} of an expansion

$$B(z) = \frac{B_{i,r_i}}{(z-a_i)^{r_i}} + \ldots + \frac{B_{i,1}}{z-a_i} + B_{i,0} + \ldots$$

of the coeff. matrix $B(z)$ near each singular point has p pairwise distinct eigenvalues.

A formal fundamental matrix near each singular point:

$$\widehat{Y}_i(z) = \widehat{F}_i(z)(z - a_i)^{\Lambda_i}\, e^{Q_i(z)},$$

$\widehat{F}_i(z)$ is an invertible matrix formal Taylor series in $(z - a_i)$;
$\Lambda_i = \mathrm{diag}(\lambda_i^1, \ldots, \lambda_i^p)$ is a diagonal matrix of **formal** exponents;
$Q_i(z)$ is a diagonal matrix with polynomials in $1/(z - a_i)$ on the diagonal.

Solvability of irregular systems with bounded formal exponents

Theorem 3. Let for each irregular singularity a_i all formal exponents λ_i^j are **distinct** and satisfy inequalities

$$\mathrm{Re}\,\lambda_i^j > -1/(n(p-1)),$$

moreover one of the following conditions holds:

$$1)\,\mathrm{Re}\,\lambda_i^j - \mathrm{Re}\,\lambda_i^l \notin \mathbb{Q}; \qquad 2)\,\mathrm{Im}\,\lambda_i^j \neq \mathrm{Im}\,\lambda_i^l.$$

Then the system is solvable by quadratures iff there exists a constant matrix $C \in \mathrm{GL}(p, \mathbb{C})$ such that the matrix $C^{-1}B(z)C$ is triangular.

Linear independence of some elements of the Picard–Vessiot extension

Now we consider a scalar linear diff. equation

$$u^{(p)} + b_1(z)u^{(p-1)} + \ldots + b_p(z)u = 0, \qquad b_j \in \mathbb{C}(z),$$

of order p with **regular** singular points $a_1, \ldots, a_n \in \overline{\mathbb{C}} \setminus \{0\}$.

a_i is regular $\Longleftrightarrow b_j(z) = \frac{\alpha_{ij}}{(z-a_i)^j} + o(1/(z-a_i)^j), \qquad z \to a_i.$

The **exponents** $\beta_i^1, \ldots, \beta_i^p$ at the point a_i are roots of the equation

$$\lambda(\lambda-1)\ldots(\lambda-p+1) + \alpha_{i1}\lambda(\lambda-1)\ldots(\lambda-p+2) + \ldots + \alpha_{ip} = 0.$$

Consider the Picard–Vessiot extension $\mathbb{C}(z) \subset F$ obtained by adjoining to $\mathbb{C}(z)$ fundamental solutions $u_1(z), \ldots, u_p(z)$ of the equation and their derivatives.

An assumption that the elements of the finite subset

$$\mathcal{A}_M = \{u_1^{k_1}(z) \ldots u_p^{k_p}(z) \mid (k_1, \ldots, k_p) \in \mathbb{Z}_+^p, \;\; k_1 + \ldots + k_p = M\} \subset F$$

($\#\mathcal{A}_M = C_{p+M-1}^M$) are linear independent over $\mathbb{C}$ implies the following statement.

Theorem 4. Let the exponents of the linear diff. equation satisfy the conditions $\mathrm{Re}\,\beta_i^j \geqslant t_i \in \mathbb{Z}$ and

$$N \geqslant \frac{1}{2}(n-2)(C_{p+M-1}^{M}-1) - M\sum_{i=1}^{n} t_i$$

is an integer. Then the elements of the family

$$\mathcal{A}_{M,N} = \{z^{kN}u_1^{k_1}(z)\ldots u_p^{k_p}(z) \mid k \in \mathbb{Z}_+, \quad k_1+\ldots+k_p = M\} \subset F$$

are also linear independent over $\mathbb{C}$.

Using ideas of the proof of this theorem one can also obtain the following estimate.

Proposition. Let $P(x_1,\ldots,x_p)$ be a homogeneous polynomial of degree M and $u_1(z),\ldots,u_p(z)$ fundamental solutions of the linear diff. equation with exponents satisfying the conditions $\operatorname{Re}\beta_i^j \geqslant t_i \in \mathbb{Z}$. If $P(u_1(z),\ldots,u_p(z)) \not\equiv 0$, then

$$\begin{aligned}\mathrm{ord}_0 P(u_1(z),\ldots,u_p(z)) \; &< \; \frac{1}{2}(n-2)C^M_{p+M-1}(C^M_{p+M-1}-1)- \\ &-C^M_{p+M-1}M\sum_{i=1}^{n} t_i.\end{aligned}$$

(Estimates of Yu. Nesterenko, D. Bertrand, A. Bolibrukh for $\mathrm{ord}_0 P(y^1(z),\ldots,y^p(z))$, where $y=(y^1,\ldots,y^p)^\top$ is a solution of a linear differential system.)