On the Malgrange theorem on convergence of a formal power series solution of an analytic ODE

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Setting the problem

We consider an ordinary differential equation

$$f(z, w, w', \dots, w^{(n)}) = 0, \tag{1}$$

where $f(z, w, w', \dots, w^{(n)})$ is an analytic function (a polynomial) of its variables.

We study some properties of a formal power series solution

$$\varphi = \sum_{k=0}^{\infty} c_k\, z^k \in \mathbb{C}[[z]] \tag{2}$$

of the equation (1).

Mainly, we concern the question of convergence of the formal power series solution (2) of the equation (1).

Brief historical review

The Maillet theorem is one of the well-known theorems that describe properties of formal power series solutions of ODE.

Theorem (E. Maillet, Sur les séries divergentes et les équations différentielles, *Ann. Sci. Ecole Norm. Sup.*, vol. 3, 1903).
If the formal series $\sum_{k=0}^{\infty} c_k z^k$ satisfies the equation

$$f(z, w, w', \ldots, w^{(n)}) = 0,$$

then there exists a real number $s \geqslant 0$ such that the series

$$\sum_{k=0}^{\infty} \frac{c_k}{(k!)^s} z^k \tag{3}$$

converges in some neighborhood of zero.

(Formal power series possessing the property from the Maillet theorem are called the **Gevrey series of order** s.)

In 1978 Jean-Pierre Ramis studied formal power series solutions of a linear differential equation

$$b_n(z)\, w^{(n)} + b_{n-1}(z)\, w^{(n-1)} + \cdots + b_0(z)\, w = 0, \tag{4}$$

where $b_j(z)$ are holomorphic functions in a neighborhood of zero.

We can write the equation (4) in the equivalent form $L\, w = 0$, where L is a linear differential operator

$$L = a_n(z)\, \delta^n + a_{n-1}(z)\, \delta^{n-1} + \cdots + a_0(z), \tag{5}$$

$\delta = z\,\dfrac{d}{dz}$, $a_j(z)$ are holomorphic functions in a neighborhood of zero.

The linear differential operator (5) corresponds to the Newton polygon $N(L)$, that is the bound of the convex hull of the union of sets

$$S_j = \{(x, y) \in \mathbb{R}^2 \mid\ x \leqslant j,\ \ y \geqslant \operatorname{ord}_0 a_j(z)\,\}, \quad j = 0, 1, \ldots, n.$$

The point $z = 0$ is called a **regular** singular point of the operator

$$L = a_n(z)\,\delta^n + a_{n-1}(z)\,\delta^{n-1} + \cdots + a_0(z),$$

if the inequalities $\mathrm{ord}_0\ a_j(z) \geqslant \mathrm{ord}_0\ a_n(z)$ $(j = 0, 1, \ldots, n-1)$ hold, otherwise the point $z = 0$ is called an **irregular** singular point of the operator L.

Note that in the regular case the Newton polygon $N(L)$ consists of the one horizontal and one vertical edge. In the irregular case the Newton polygon $N(L)$ contains edges with positive tangents of the slopes angles.

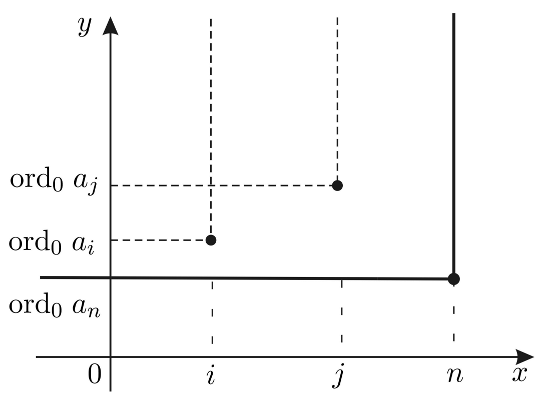

Fig. 1. Regular case.

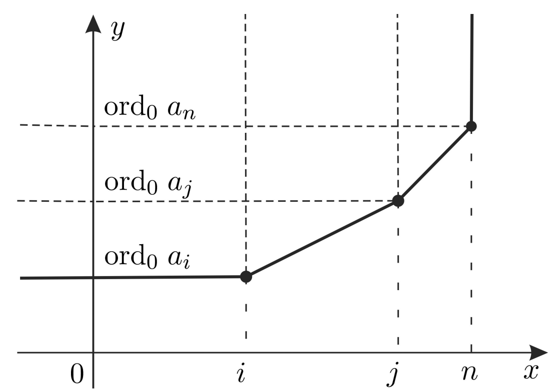

Fig. 2. Irregular case.

Let $0 < r_1 < \ldots < r_m < \infty$ be all the positive tangents of the slopes angles of the edges of the Newton polygon $N(L)$ $(m \leqslant n)$.

Theorem (J.-P. Ramis, Dévissage Gevrey, *Astérisque*, vol. 59/60, 1978, pp. 173–204).

Any formal power series solution $\sum_{k=0}^{\infty} c_k\, z^k$ of the equation

$$L\, w = 0$$

has an exact Gevrey order

$$s \in \{0, 1/r_1, \ldots, 1/r_m\}.$$

Further we consider a non-linear ODE written in the form

$$F(z, w, \delta w, \ldots, \delta^n w) = 0, \tag{6}$$

where $\delta = z\dfrac{d}{dz}$ and $F(z, w_0, w_1, \ldots, w_n)$ is a holomorphic function in a neighborhood of the point $0 \in \mathbb{C}^{n+2}$.

Theorem (B. Malgrange, Sur le théorème de Maillet, *Asympt. Anal.*, vol. 2 1989, pp. 1–4).
Let the formal power series $\varphi = \sum_{k=0}^{\infty} c_k z^k$ satisfy the equation (6), $\varphi(0) = 0$, and the condition

$$\frac{\partial F(z, \Phi)}{\partial w_n} \not\equiv 0, \qquad \Phi = (\varphi, \delta\varphi, \ldots, \delta^n \varphi),$$

holds. Then

a) if the point $z=0$ is a regular singular point of the operator

$$L_\varphi = \sum_{i=0}^{n} \frac{\partial F}{\partial w_i}(z,\Phi)\,\delta^i, \tag{7}$$

then the formal power series φ converges in some neighborhood of $z=0$;

b) if the point $z=0$ is an irregular singular point of the operator L_φ and r is the smallest positive tangent of the slopes angles of the Newton polygon $N(L_\varphi)$, then the formal power series φ has the Gevrey order $s=1/r$.

One year later the item b) of the Malgrange theorem have been specified by Yasutaka Sibuya.

Theorem (Y. Sibuya, Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation, Transl. Math. Monographs, vol. 82, A.M.S., 1990).
The formal power series solution φ has the exact Gevrey order

$$s \in \{0, 1/r_1, \ldots, 1/r_m\},$$

where $0 < r_1 < \ldots < r_m < \infty$ are all the positive tangents of the slopes angles of the Newton polygon $N(L_\varphi)$ of the linear differential operator

$$L_\varphi = \sum_{i=0}^{n} \frac{\partial F}{\partial w_i}(z, \Phi)\, \delta^i.$$

Concerning proofs of the Malgrange theorem

Malgrange and Sibuya proved the item a) of the Malgrange theorem (the regular case) in different ways, however, in both proofs the initial equation is reduced to a special form.

Lemma (B. Malgrange). If the equation

$$F(z, w, \delta w, \ldots, \delta^n w) = 0 \tag{6}$$

has the formal power series solution $\varphi = \sum\limits_{k=0}^{\infty} c_k z^k$, $\varphi(0) = 0$, and the condition $\dfrac{\partial F(z, \Phi)}{\partial w_n} \not\equiv 0$ holds, then there exists the integer $N \geqslant 0$ such that for all $m \geqslant N$ by means of the transformation

$$w = \sum_{k=0}^{m} c_k z^k + z^m v$$

the equation (6) is transformed into the special form

$$\overline{L}(\delta + m)v = z\,\mu(z, v, \delta v, \ldots, \delta^n v), \tag{8}$$

where $\overline{L}$ is a polynomial of degree less than or equal to n and the function $\mu(z, v_0, v_1, \ldots, v_n)$ is holomorphic in a neighborhood of the point $0 \in \mathbb{C}^{n+2}$.

(A regular linear differential operator L_φ corresponds to the case when the polynomial $\overline{L}$ in the equation (8) is of degree n exactly.)

On Malgrange's and Sibuya's proofs

Further convergence of the corresponding formal power series solution ψ of the equation

$$\overline{L}(\delta+m)v = z\,\mu(z,v,\delta v,\dots,\delta^n v) \tag{8}$$

(in the regular case) was proved by Malgrange with the help of the implicit mapping theorem for Banach spaces, while Sibuya used the fundamental Ramis–Sibuya theorem on asymptotic expansions.

Note that Malgrange's and Sibuya's proofs do not contain an estimate of the radius of convergence of the formal power series solution of the ODE $F(z,w,\,\delta w,\,\dots,\delta^n w)=0$.

We propose an analytic proof of the Malgrange theorem in its regular case. Our proof is based on the majorant method and allows to get an estimate of the radius of convergence of the formal power series solution ψ of the equation (8).

Analytic proof of the Malgrange theorem in the regular case

Let the series expansion of the function $\mu(z, v_0, v_1, \ldots, v_n)$ has the form

$$\mu(z, v_0, v_1, \ldots, v_n) = \sum_{p=0}^{\infty} \sum_{q \in \mathbb{Z}_+^{n+1}} \alpha_{p,q}\; z^p\, v_0^{q_0} v_1^{q_1} \ldots v_n^{q_n}.$$

Consider a function $M(z, v_n)$ holomorphic in a neighborhood of the point $0 \in \mathbb{C}^2$ which is constructed from the function $\mu(z, v_0, v_1, \ldots, v_n)$ as follows:

$$M(z, v_n) = \sum_{p=0}^{\infty} \sum_{q \in \mathbb{Z}_+^{n+1}} |\alpha_{p,q}|\;\; z^p\, v_n^{q_0} v_n^{q_1} \ldots v_n^{q_n}.$$

The key idea of our proof is that the equation

$$\overline{L}(\delta+m)v = z\,\mu(z,v,\delta v,\ldots,\delta^n v) \tag{8}$$

is majorated by the equation

$$\sigma\,\delta^n v = z\,M(z,\,\delta^n v), \qquad \sigma = \inf_{k\in\mathbb{N}} \frac{|\overline{L}(k+m)|}{k^n} > 0, \tag{9}$$

in the following sense.

Lemma. The equation (9) has the unique analytic solution

$$\Psi = \sum_{k=1}^{\infty} C_k\,z^k, \quad C_k \in \mathbb{R}, \quad C_k \geqslant 0,$$

in a neighborhood of zero ($\Psi(0)=0$), which is majorant for the formal solution $\psi = \sum\limits_{k=1}^{\infty} c_k\,z^k$ of the equation (8), that is,

$$|c_k| \leqslant C_k, \qquad k = 1,2,\ldots$$

As a consequence of the previous lemma we have the following statement.

Proposition. Let the function $\mu(z, v_0, v_1, \ldots, v_n)$ be holomorphic in a neighborhood of a closed polydisk

$$\overline{\Delta} = \{|z| \leqslant r, |w_0| \leqslant \rho, \ldots, |w_n| \leqslant \rho\}, \qquad \mathcal{M} = \max_{\overline{\Delta}} |\mu|.$$

Then the formal solution $\psi = \sum_{k=1}^{\infty} c_k z^k$ of the equation

$$\overline{L}(\delta + m)v = z\, \mu(z, v, \delta v, \ldots, \delta^n v)$$

converges in a disk

$$\left\{|z| < r\,\frac{\rho}{\rho + \mathcal{M}r/\sigma N}\right\}, \qquad N = (n+1)^{n+1}/(n+2)^{n+2}.$$