On the summability of divergent power series solutions for certain first-order linear PDEs

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§ 1 Introduction

$$(\mathrm{E})\quad \{1+x^2+\beta(x,y)\}y\frac{\partial u}{\partial x}(x,y)+\{x+b(x,y)\}y^2\frac{\partial u}{\partial y}(x,y) + u(x,y) = f(x,y)$$

$x,\ y \in \mathbb{C}$

$\beta,\ b,\ f$: holomorphic at $(x,y)=(0,0)\in\mathbb{C}^2$

$$(1.1)\qquad \beta(x,0)\equiv b(x,0)\equiv 0$$

◎ $^{\exists !}\hat{u}(x,y)=\sum_{n=0}^{\infty}u_n(x)y^n$: formal solution of (E) **divergent**

Problem

When is $\hat{u}(x,y)$ summable?

§ 2 Definition and fundamental result

Definition 2.1

(1) $R > 0, \quad B(R) = \{x \in \mathbb{C};\ |x| \leq R\}$

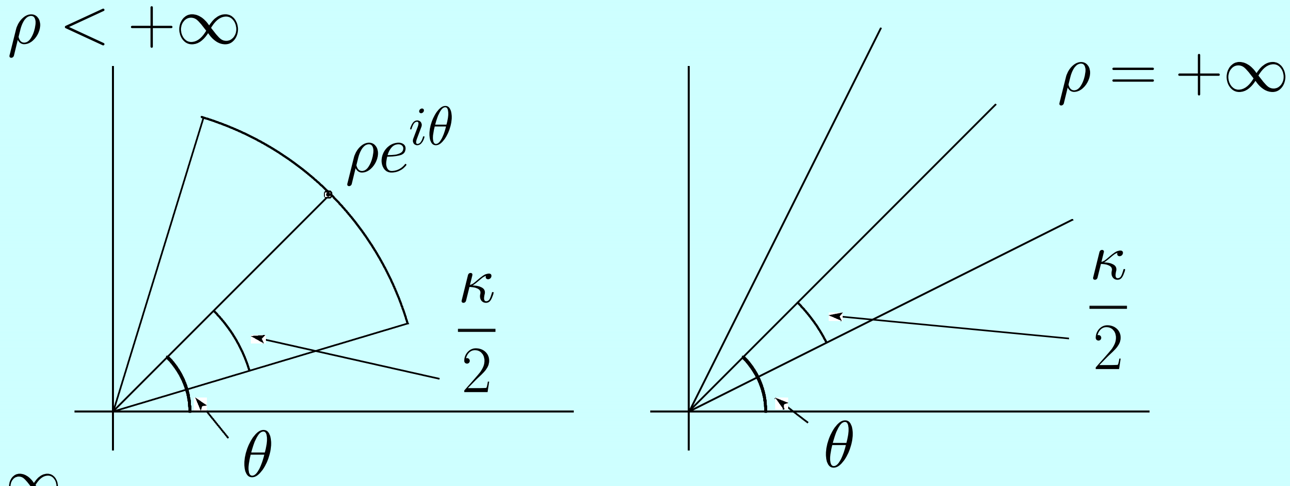
$\mathcal{O}[R]$: ring of holomorphic functions on $B(R)$

(2) $$\mathcal{O}[R][[y]] = \left\{ \hat{u}(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n;\ u_n(x) \in \mathcal{O}[R] \right\}$$

(3) $$\mathcal{O}[R][[y]]_2 = \left\{ \hat{u}(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]];\ {}^{\exists}C, {}^{\exists}K > 0 \text{ s.t. } \max_{|x| \leq R} |u_n(x)| \leq C K^n n! \ \ (n = 0, 1, 2, \dots) \right\}$$

(4) $\theta \in \mathbb{R}, \ \kappa > 0, \ 0 < \rho \leq +\infty$

$$S(\theta, \kappa, \rho) = \left\{ y;\ |\arg(y) - \theta| < \frac{\kappa}{2},\ 0 < |y| < \rho \right\}$$



(5) $\hat{u}(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$ is **Borel summable in $\boldsymbol{\theta}$**

$\Leftrightarrow\ {}^{\exists}S(\theta,\kappa,\rho)$ (**$\boldsymbol{\kappa > \pi}$**)

${}^{\exists}u(x,y)$: holomorphoic on $B(r) \times S(\theta,\kappa,\rho)$ ($0 < {}^{\exists}r \le R$) s.t.

$$\max_{|x|\le r} \left| u(x,y) - \sum_{n=0}^{N-1} u_n(x) y^n \right| \le {}^{\exists}\mathcal{C}\,{}^{\exists}\mathcal{K}^N N! |y|^N$$

$${}^{\forall}y \in S(\theta,\kappa,\rho),\quad {}^{\forall}N = 1,2,\ldots$$

◎ $\hat{u}$: Borel summable in θ $\Rightarrow$ u: **unique** $\leftarrow$ **Borel sum of $\boldsymbol{\hat{u}}$ in $\boldsymbol{\theta}$**

Definition 2.2 $\hat{u}(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$

$$\hat{\mathcal{B}}[\hat{u}](x,\eta) = \sum_{n=0}^{\infty} u_n(x) \frac{\eta^n}{n!}: \quad \text{formal Borel transform of } \hat{u}$$

$\hat{\mathcal{B}}[\hat{u}](x,\eta)$: holomorphic at $(x,\eta) = (0,0)$

Theorem 2.1 $\hat{u}(x,y) \in \mathcal{O}[R][[y]]_2$

$\hat{u}(x,y)$: Borel summable in θ

$\Updownarrow$

(BS) $\hat{\mathcal{B}}[\hat{u}](x,\eta)$: continued analytically to $B(r_0) \times S(\theta, \kappa_0, +\infty)$ $({}^{\exists}r_0, {}^{\exists}\kappa_0)$

${}^{\exists}C, {}^{\exists}\delta > 0$ s.t. $\max_{|x| \le r_0} |\hat{\mathcal{B}}[\hat{u}](x,\eta)| \le C e^{\delta|\eta|}, \quad \eta \in S(\theta, \kappa_0, +\infty)$

Borel sum of $\hat{u}$ in θ:

$$u(x,\eta) = \frac{1}{y} \int_0^{\infty e^{i\theta}} e^{-\eta/y} \cdot \hat{\mathcal{B}}[\hat{u}](x,\eta)\, d\eta$$

§ 3 Motivation

$$\textbf{(3.1)} \quad A(x,y)\frac{\partial u}{\partial x}(x,y) + B(x,y)\frac{\partial u}{\partial y}(x,y) + u(x,y) = f(x,y)$$

A, B, f: holomorphic at $(x,y) = (0,0) \in \mathbb{C}^2$

$$\textbf{(3.2)} \quad A(x,0) \equiv 0$$

$$\textbf{(3.3)} \quad \frac{\partial A}{\partial y}(0,0) \neq 0$$

$$\textbf{(3.4)} \quad B(x,0) \equiv \frac{\partial B}{\partial y}(x,0) \equiv 0$$

Remark 3.1

$$(3.2),\ (3.3),\ (3.4) \Rightarrow \left.\frac{\partial(A,B)}{\partial(x,y)}\right|_{(x,y)=(0,0)} = \begin{pmatrix} 0 & \dfrac{\partial A}{\partial y}(0,0) \\ 0 & 0 \end{pmatrix}$$

(nilpotent type)

Theorem 3.1 (1999)

(3.2), (3.3), (3.4)

$$\Rightarrow {}^{\exists !}\hat{u}(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2 \ ({}^{\exists}R): \quad \text{formal solution of } (3.1)$$

Problem

θ: given

When is $\hat{u}(x,y)$ Borel summable in θ?

(When does $\hat{\mathcal{B}}[\hat{u}](x,\eta)$ satisfy (BS)?)

Remark 3.2

$\hat{u}(x,y)$: Borel summable

$\Rightarrow$ Borel sum $u(x,y)$: **holomorphic solution**

◎ (3.1) is rewritten

(3.5)

$$\{\alpha(x)+\beta(x,y)\}y\frac{\partial u}{\partial x}(x,y)+\{a(x)+b(x,y)\}y^2\frac{\partial u}{\partial y}(x,y)$$
$$+u(x,y)=f(x,y)$$

α, β, a, b, f: holomorphic at the origin

(3.6) $$\alpha(0)\neq 0$$

(3.7) $$\beta(x,0)\equiv b(x,0)\equiv 0$$

2006 $\alpha(x)$: general, $a(x)\equiv a$ (constant)

2008 $\alpha(x)\equiv\alpha$ (constant), $a(x)$: general

2010 $\alpha(x)=\alpha_0+\alpha_1 x$, $a(x)$: general

In this talk we consider the case

(3.8) $$\alpha(x)=1+x^2, \quad a(x)=x$$

§ 4 Main result

$$\text{(E)} \quad \{1 + x^2 + \beta(x, y)\} y \frac{\partial u}{\partial x}(x, y) + \{x + b(x, y)\} y^2 \frac{\partial u}{\partial y}(x, y) + u(x, y) = f(x, y)$$

Assumptions

We define the region $\Omega_{\theta,\kappa}$ by

$$\Omega_{\theta,\kappa} = \{-\tan[\arcsin(\tau)];\ \tau \in S(\theta, \kappa, +\infty)\}$$

In order to assure the well-definedness of $\Omega_{\theta,\kappa}$ we assume

(A1) $\theta \neq 0,\ \pi$

(A2) $f(x, y)$: continued analytically to $\Omega_{\theta,\kappa} \times \{y \in \mathbb{C};\ |y| \leq c\}$ ($^\exists c > 0$)

$^\exists C$, $^\exists \delta > 0$ s.t. $^\forall x \in \Omega_{\theta,\kappa}$

$$\text{(4.1)} \qquad \max_{|y| \leq c} |f(x, y)| \leq C \exp\left[\delta \left|\sin(\arctan x)\right|\right]$$

(A3) $\beta(x,y)$, $b(x,y)$: continued analytically to $\Omega_{\theta,\kappa} \times \{y \in \mathbb{C};\ |y| \le c\}$
${}^{\exists}K$, ${}^{\exists}L > 0$, ${}^{\exists}p > 1$ s.t. ${}^{\forall}x \in \Omega_{\theta,\kappa}$, ${}^{\forall}m = 1, 2, \ldots$

$$\left|\frac{1}{1+x^2}\frac{\partial^m \beta}{\partial y^m}(x,0)\right| \le KL^m m! |\cos(\arctan x)|^m \tag{4.2}$$

$$\left|\frac{x}{1+x^2}\frac{\partial^m \beta}{\partial y^m}(x,0)\right| \le \frac{KL^m m! |\cos(\arctan x)|^{m+1}}{\{1+|\sin(\arctan x)|\}^p} \tag{4.3}$$

$$\left|\frac{\partial^m b}{\partial y^m}(x,0)\right| \le \frac{KL^m m! |\cos(\arctan x)|^{m+1}}{\{1+|\sin(\arctan x)|\}^p} \tag{4.4}$$

Main theorem

(A1), (A2), (A3)

$\Rightarrow$ The formal solution $\hat{u}(x,y)$ of (E) is Borel summable in θ

§ 5 Proof of Main Theorem

(easy case: $f(x,y) = f(x)$, $\beta(x,y) \equiv b(x,y) \equiv 0$)

$$\textbf{(E)}\quad (y + x^2 y)\frac{\partial u}{\partial x}(x,y) + xy^2\frac{\partial u}{\partial y}(x,y) + u(x,y) = f(x)$$

$\hat{u}(x,y)$: formal solution of (E) $\quad \hat{\mathcal{B}}[\hat{u}](x,\eta)$: formal Borel transform of $\hat{u}$

$$\hat{\mathcal{B}}[\hat{u}](x,\eta)$$
$$= f\Big(-\tan\Big[\arcsin\big\{-\sin(\arctan x) + \cos(\arctan x)\cdot\eta\big\}\Big]\Big)$$

(A1) $\theta \neq 0,\ \pi$; **(A2)** $f(x)$ can be continued analytically to $\Omega_{\theta,\kappa} = \{-\tan[\arcsin(\tau)];\ \tau \in S(\theta,\kappa,+\infty)\}$ with the estimate

$$|f(x)| \leq C\exp\Big[\delta|\sin(\arctan x)|\Big],\ x \in \Omega_{\theta,\kappa}$$

$\Rightarrow$ **(BS)** $\hat{\mathcal{B}}[\hat{u}](x,\eta)$ can be continued analytically to $B(r_0)\times S(\theta,\kappa_0,+\infty)$ $({}^{\exists}r_0,\ {}^{\exists}\kappa_0)$ and satisfies

$$\max_{|x|\leq r_0} |\hat{\mathcal{B}}[\hat{u}](x,\eta)| \leq C_0 e^{\delta_0|\eta|},\ \ \eta \in S(\theta,\kappa_0,+\infty)\ \ ({}^{\exists}C_0,\ {}^{\exists}\delta_0)$$

§ 6 Proof of Main Theorem

For simplicity, we consider the case $\beta(x,y) \equiv 0$:

$$\textbf{(E)}\quad \{1+x^2\}yD_x u(x,y) + \{x+b(x,y)\}y^2 D_y u(x,y) + u(x,y) = f(x,y)$$

$$D_x = \frac{\partial}{\partial x},\quad D_y = \frac{\partial}{\partial y}$$

$\hat{u}(x,y)$: formal solution of (E)

$v(x,\eta) = \hat{\mathcal{B}}[\hat{u}](x,\eta)$: formal Borel transform of $\hat{u}$

It is sufficient to prove that $v(x,\eta)$ satisfies (BS):

(BS) $v(x,\eta)$ can be continued analytically to $B(r_0) \times S(\theta,\kappa_0,+\infty)$ ($^{\exists}r_0$, $^{\exists}\kappa_0$) and satisfies

$$\max_{|x|\le r_0} |v(x,\eta)| \le C_0 e^{\delta_0|\eta|},\ \eta \in S(\theta,\kappa_0,+\infty)\ \ (^{\exists}C_0, {}^{\exists}\delta_0)$$

Step 1. Formal Borel transform of equation

$$\text{(E)} \xrightarrow{\text{formal Borel transform}} ?$$

- $A(x,y)yB(x,y) \longrightarrow \int_0^{\eta} \hat{\mathcal{B}}[A](x,\eta-t)\hat{\mathcal{B}}[B](x,t)\,dt$
- $yD_yC(x,y) \longrightarrow \eta D_\eta \hat{\mathcal{B}}[C](x,\eta)$

$v(x,\eta)$ satisfies

$$\begin{aligned} &\{1+x^2\}\int_0^{\eta} v_x(x,t)\,dt + x\int_0^{\eta} t\cdot v_\eta(x,t)\,dt \\ &+\int_0^{\eta} \hat{\mathcal{B}}[b](x,\eta-t)\cdot t\cdot v_\eta(x,t)\,dt + v(x,\eta) = \hat{\mathcal{B}}[f](x,\eta) \end{aligned} \tag{6.1}$$

$$\textbf{(E)}\quad \{1+x^2\}yD_xu(x,y) + \{x+b(x,y)\}y^2D_yu(x,y) + u(x,y) = f(x,y)$$

integration by parts $\longrightarrow$ **"(6.1) $\Leftrightarrow$ (6.2)"**

(6.2)

$$\{1+x^2\}\int_0^\eta v_x(x,t)\,dt + x\int_0^\eta t\cdot v_\eta(x,t)\,dt$$

$$+\int_0^\eta \hat{\mathcal{B}}[b]_\eta(x,\eta-t)\cdot t\cdot v(x,t)\,dt - \int_0^\eta \hat{\mathcal{B}}[b](x,\eta-t)v(x,t)\,dt + v(x,\eta)$$

$$=\hat{\mathcal{B}}[f](x,\eta)$$

$$\begin{aligned}&\{1+x^2\}\int_0^\eta v_x(x,t)\,dt + x\int_0^\eta t\cdot v_\eta(x,t)\,dt \\ &+\int_0^\eta \hat{\mathcal{B}}[b](x,\eta-t)\cdot t\cdot v_\eta(x,t)\,dt + v(x,\eta) = \hat{\mathcal{B}}[f](x,\eta)\end{aligned} \tag{6.1}$$

(6.2) $\xrightarrow{D_\eta}$ (6.3):

$$
(6.3) \quad \begin{cases} \boldsymbol{\{(1+x^2)D_x + (1+x\eta)D_\eta\}v(x,\eta)} \\ \quad \boldsymbol{= -\hat{\mathcal{B}}[b]_\eta(x,0)\cdot\eta\cdot v(x,\eta)} \\ \qquad \boldsymbol{- \int_0^\eta \hat{\mathcal{B}}[b]_{\eta\eta}(x,\eta-t)\cdot t\cdot v(x,t)\,dt} \\ \qquad \boldsymbol{+ \int_0^\eta \hat{\mathcal{B}}[b]_\eta(x,\eta-t)v(x,t)\,dt + \hat{\mathcal{B}}[f]_\eta(x,\eta)} \\ \boldsymbol{v(x,0) = f(x,0)} \quad \textbf{(integro-differential equation)} \end{cases}
$$

(6.2)

$$
\{1+x^2\}\int_0^\eta v_x(x,t)\,dt + x\int_0^\eta t\cdot v_\eta(x,t)\,dt
$$

$$
+ \int_0^\eta \hat{\mathcal{B}}[b]_\eta(x,\eta-t)\cdot t\cdot v(x,t)\,dt - \int_0^\eta \hat{\mathcal{B}}[b](x,\eta-t)v(x,t)\,dt + v(x,\eta)
$$

$$
= \hat{\mathcal{B}}[f](x,\eta)
$$

$$
(6.3) \quad \begin{cases} \mathscr{L} v(x,\eta) = -\hat{\mathcal{B}}[b]_\eta(x,0)\cdot\eta\cdot v(x,\eta) \\ \qquad\qquad - \int_0^\eta \hat{\mathcal{B}}[b]_{\eta\eta}(x,\eta-t)\cdot t\cdot v(x,t)\,dt \\ \qquad\qquad + \int_0^\eta \hat{\mathcal{B}}[b]_\eta(x,\eta-t)v(x,t)\,dt + g(x,\eta) \\ v(x,0) = f(x,0) \quad \text{(integro-differential equation)} \end{cases}
$$

$$
(6.4) \qquad \mathscr{L} = (1+x^2)D_x + (1+x\eta)D_\eta
$$

$$
g(x,\eta) = \hat{\mathcal{B}}[f]_\eta(x,\eta)
$$

Does $v(x,\eta)$ satisfy (BS)?

Step 2. Integro-differential equation $\rightarrow$ Integral equation

$$\begin{cases} \mathscr{L} w(x,\eta) = k(x,\eta) \\ w(x,0) = l(x) \end{cases}$$

(6.5)

$$w(x,\eta) = l\Big(-\tan\big(\mathscr{A}(x,\eta)\big)\Big) + \int_0^{\eta} k\Big(-\tan\big(\mathscr{A}(x,\eta-z)\big),\ \mathscr{E}(x,\eta-z)\cdot z\Big)\cdot \mathscr{E}(x,\eta-z)\, dz$$

$$\mathscr{A}(x,\eta) = \arcsin\Big\{-\sin(\arctan x) + \cos(\arctan x)\cdot\eta\Big\}$$

$$\mathscr{E}(x,\eta) = \frac{\cos(\arctan x)}{\cos(\mathscr{A}(x,\eta))}$$

$\left(\begin{array}{c}\text{(6.5)} \\ \text{integration by substitution}\end{array}\right) \longrightarrow$ **“(6.3) $\Leftrightarrow$ (6.6)”**

(6.6)

$$\boldsymbol{v(x,\eta) = f\big(-\tan\big(\mathscr{A}(x,\eta)\big),\, 0\big) + \int_0^\eta g\Big(-\tan\big(\mathscr{A}(x,\eta-z)\big),\, \mathscr{E}(x,\eta-z)\cdot z\Big)\cdot \mathscr{E}(x,\eta-z)\,dz + \sum_{i=1}^{3} I_i v(x,\eta)} \quad \textbf{(integral equation)}$$

$$I_1 v(x,\eta) = -\int_0^\eta \mathscr{E}(x,\eta-z)^2 \cdot z \cdot \hat{\mathcal{B}}[b]_\eta\big(-\tan\big(\mathscr{A}(x,\eta-z)\big),\, 0\big) \times v\Big(-\tan\big(\mathscr{A}(x,\eta-z)\big),\, \mathscr{E}(x,\eta-z)\cdot z\Big)\,dz$$

$$I_2 v(x,\eta) = -\int_0^\eta \int_0^z \mathscr{E}(x,\eta-z)^3 \cdot \hat{\mathcal{B}}[b]_{\eta\eta}\Big(-\tan\big(\mathscr{A}(x,\eta-z)\big),\, \mathscr{E}(x,\eta-z)\cdot(z-s)\Big) \times s\cdot v\Big(-\tan\big(\mathscr{A}(x,\eta-z)\big),\, \mathscr{E}(x,\eta-z)\cdot s\Big)\,ds\,dz$$

$$I_3 v(x,\eta) = \int_0^\eta \int_0^z \mathscr{E}(x,\eta-z)^2 \cdot \hat{\mathcal{B}}[b]_\eta\Big(-\tan\big(\mathscr{A}(x,\eta-z)\big),\, \mathscr{E}(x,\eta-z)\cdot(z-s)\Big) \times v\Big(-\tan\big(\mathscr{A}(x,\eta-z)\big),\, \mathscr{E}(x,\eta-z)\cdot s\Big)\,ds\,dz$$

Does a solution $v(x,\eta)$ of (6.6) satisfy (BS)?

Step 3. Iteration

- $v_0(x,\eta)$
$$= f\Big(-\tan\big(\mathscr{A}(x,\eta)\big),\ 0\Big)$$
$$+ \int_0^\eta g\Big(-\tan\big(\mathscr{A}(x,\eta-z)\big),\ \mathscr{E}(x,\eta-z)\cdot z\Big)\cdot \mathscr{E}(x,\eta-z)\,dz$$
- $v_{n+1}(x,\eta) = v_0(x,\eta) + \sum_{i=1}^{3} I_i v_n(x,\eta) \quad (n \geq 0)$
- $w_0(x,\eta) = v_0(x,\eta)$
- $w_n(x,\eta) = v_n(x,\eta) - v_{n-1}(x,\eta) \quad (n \geq 1)$

Lemma 6.1

$w_n(x,\eta)$: continued analytically to $B(r_0) \times S(\theta,\kappa_0,+\infty) \quad ({}^{\exists}r_0,\ {}^{\exists}\kappa_0 > 0)$

${}^{\exists}C_0$, ${}^{\exists}\delta_0$, ${}^{\exists}M > 0$ s.t. ${}^{\forall}\eta \in S(\theta,\kappa_0,+\infty)$

$$\max_{|x|\leq r_0} |w_n(x,\eta)| \leq C_0 e^{\delta_0} M^n \Big(1 + \frac{1}{p-1}\Big)^n \frac{|\eta|^n}{n!} \tag{6.7}$$

(A3) $\beta(x,y)$, $b(x,y)$: continued analytically to $\Omega_{\theta,\kappa} \times \{y \in \mathbb{C};\ |y| \le c\}$
$^\exists K$, $^\exists L > 0$, $^\exists p > 1$ s.t. $^\forall x \in \Omega_{\theta,\kappa}$, $^\forall m = 1, 2, \dots$

$$(4.2) \qquad \left|\frac{1}{1+x^2}\frac{\partial^m \beta}{\partial y^m}(x,0)\right| \le K L^m m! |\cos(\arctan x)|^m$$

$$(4.3) \qquad \left|\frac{x}{1+x^2}\frac{\partial^m \beta}{\partial y^m}(x,0)\right| \le \frac{K L^m m! |\cos(\arctan x)|^{m+1}}{\{1 + |\sin(\arctan x)|\}^p}$$

$$(4.4) \qquad \left|\frac{\partial^m b}{\partial y^m}(x,0)\right| \le \frac{K L^m m! |\cos(\arctan x)|^{m+1}}{\{1 + |\sin(\arctan x)|\}^p}$$

Main theorem

(A1), (A2), (A3)

$\Rightarrow$ The formal solution $\hat{u}(x,y)$ of (E) is Borel summable in θ

Lemma 6.1

$$\Downarrow$$

$$\begin{aligned}\max_{|x|\le r_0}\sum_{n=0}^{\infty}|w_n(x,\eta)| &\le C_0 e^{\delta_0|\eta|}\sum_{n=0}^{\infty}M^n\Big(1+\frac{1}{p-1}\Big)^n\frac{|\eta|^n}{n!}\\ &= C_0 e^{\delta_1|\eta|},\qquad \eta\in S(\theta,\kappa_0,+\infty)\\ &\quad\Big[\delta_1=\delta_0+M\Big(1+\frac{1}{p-1}\Big)\Big]\end{aligned}$$

$$\Rightarrow v(x,\eta)=\lim_{n\to\infty}v_n(x,\eta)=\sum_{n=0}^{\infty}w_n(x,\eta):$$

holomorphic solution of (6.6) on $B(r_0)\times S(\theta,\kappa_0,+\infty)$

$$\max_{|x|\le r_0}|v(x,\eta)|\le C_0 e^{\delta_1|\eta|},\quad \eta\in S(\theta,\kappa_0,+\infty)$$

□

Proof of Lemma 6.1

(A1) and (A2) $\Longrightarrow$ Lemma 6.1 for $n = 0$

(A3) and "analytic continuation property for $w_n(x, \eta)$"

$\Longrightarrow$ analytic continuation property for $w_{n+1}(x, \eta)$

$$w_{n+1}(x, \eta) = I_1 w_n(x, \eta) + I_2 w_n(x, \eta) + I_3 w_n(x, \eta)$$

$$\begin{aligned}
I_1 w_n(x, \eta) \;&=\; -\int_0^{\eta} \mathscr{E}(x, \eta - z)^2 \cdot z \cdot \hat{\mathcal{B}}[b]_{\eta}\Big(-\tan\big(\mathscr{A}(x, \eta - z)\big),\, 0\Big) \\
&\quad \times w_n\Big(-\tan\big(\mathscr{A}(x, \eta - z)\big),\, \mathscr{E}(x, \eta - z) \cdot z\Big)\, dz \\
I_2 w_n(x, \eta) \;&=\; -\int_0^{\eta}\int_0^{z} \mathscr{E}(x, \eta - z)^3 \cdot \hat{\mathcal{B}}[b]_{\eta\eta}\Big(-\tan\big(\mathscr{A}(x, \eta - z)\big),\, \mathscr{E}(x, \eta - z) \cdot (z - s)\Big) \\
&\quad \times s \cdot w_n\Big(-\tan\big(\mathscr{A}(x, \eta - z)\big),\, \mathscr{E}(x, \eta - z) \cdot s\Big)\, ds\, dz \\
I_3 w_n(x, \eta) \;&=\; \int_0^{\eta}\int_0^{z} \mathscr{E}(x, \eta - z)^2 \cdot \hat{\mathcal{B}}[b]_{\eta}\Big(-\tan\big(\mathscr{A}(x, \eta - z)\big),\, \mathscr{E}(x, \eta - z) \cdot (z - s)\Big) \\
&\quad \times w_n\Big(-\tan\big(\mathscr{A}(x, \eta - z)\big),\, \mathscr{E}(x, \eta - z) \cdot s\Big)\, ds\, dz
\end{aligned}$$

On the estimate (6.7)?

$$\max_{|x|\le r_0} |w_n(x,\eta)| \le C_0 e^{\delta_0} M^n \Big(1+\frac{1}{p-1}\Big)^n \frac{|\eta|^n}{n!} \tag{6.7}$$

$$A(x,y) = \sum_{m=0}^{\infty} A_m(x) y^m$$

$$\Rightarrow \int_0^\eta \mathcal{B}[A](x,\eta-t)V(x,t)\,dt = \sum_{m=0}^{\infty} \frac{A_m(x)}{m!} \int_0^\eta (\eta-t)^m \cdot V(x,t)\,dt$$

$$= \sum_{m=0}^{\infty} A_m(x) \int_0^\eta \int_0^{\eta_1} \cdots \int_0^{\eta_m} V(x,\eta_{m+1})\,d\eta_{m+1} \cdots d\eta_2\,d\eta_1$$

$$w_{n+1}(x,\eta) = \mathscr{I}_1 w_n(x,\eta) + \mathscr{I}_2 w_n(x,\eta) \tag{6.8}$$

$$
\begin{aligned}
&\mathscr{I}_1 w_n(x,\eta) = I_1 w_n(x,\eta) + I_2 w_n(x,\eta) \\
&= -\sum_{m=1}^{\infty} \int_0^{\eta} b_m \left(-\tan\left(\mathscr{A}(x,\eta-\eta_1)\right)\right) \cdot \mathscr{E}(x,\eta-\eta_1)^{m+1} \\
&\qquad \times \int_0^{\eta_1} \cdots \int_0^{\eta_{m-1}} \eta_m \cdot w_n\Big(-\tan\left(\mathscr{A}(x,\eta-\eta_1)\right),\, \mathscr{E}(x,\eta-\eta_1)\cdot\eta_m\Big)\, d\eta_m \cdots d\eta_2\, d\eta_1 \\
&\mathscr{I}_2 w_n(x,\eta) = I_3 w_n(x,\eta) \\
&= \sum_{m=1}^{\infty} \int_0^{\eta} b_m \Big(-\tan\left(\mathscr{A}(x,\eta-\eta_1)\right)\Big) \cdot \mathscr{E}(x,\eta-\eta_1)^{m+1} \\
&\qquad \times \int_0^{\eta_1} \cdots \int_0^{\eta_m} w_n\Big(-\tan\left(\mathscr{A}(x,\eta-\eta_1)\right),\, \mathscr{E}(x,\eta-\eta_1)\cdot\eta_{m+1}\Big)\, d\eta_{m+1} \cdots d\eta_2\, d\eta_1
\end{aligned}
$$

$$
\boldsymbol{b_m(x) = \frac{1}{m!}\frac{\partial^m b}{\partial y^m}(x,0)}
$$

Estimate (6.7) for $w_n(x,\eta)$, (4.4), (6.8)

$\Longrightarrow$ Estimate (6.7) for $w_{n+1}(x,\eta)$ $\square$

(A3) $\beta(x,y)$, $b(x,y)$: continued analytically to $\Omega_{\theta,\kappa} \times \{y \in \mathbb{C};\ |y| \le c\}$
$^{\exists}K$, $^{\exists}L > 0$, $^{\exists}p > 1$ s.t. $^{\forall}x \in \Omega_{\theta,\kappa}$, $^{\forall}m = 1, 2, \ldots$

$$\left|\frac{1}{1+x^2}\frac{\partial^m \beta}{\partial y^m}(x,0)\right| \le KL^m m! |\cos(\arctan x)|^m \tag{4.2}$$

$$\left|\frac{x}{1+x^2}\frac{\partial^m \beta}{\partial y^m}(x,0)\right| \le \frac{KL^m m! |\cos(\arctan x)|^{m+1}}{\{1+|\sin(\arctan x)|\}^p} \tag{4.3}$$

$$\left|\frac{\partial^m b}{\partial y^m}(x,0)\right| \le \frac{KL^m m! |\cos(\arctan x)|^{m+1}}{\{1+|\sin(\arctan x)|\}^p} \tag{4.4}$$

Main theorem

(A1), (A2), (A3)

$\Rightarrow$ The formal solution $\hat{u}(x,y)$ of (E) is Borel summable in θ