Linear differential equations on the Riemann sphere and root systems

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Plan of the talk

1. Symmetries of Euler transforms and Weyl groups of Kac-Moody root systems.

2. Classification of Euler transform orbits.

3. Additive Deligne-Simpson problem; Fuchsian and non-Fuchsian.

Symmetries of Euler transforms

The Euler transform, Riemann-Liouville integral

$$I_a^\mu f(x) := \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)\, dt \text{ for } \mu, a \in \mathbb{C}.$$

It is known that for $n \in \mathbb{Z}_{>0}$,

$$I^{-n} f(x) = \frac{d^n}{dx^n} f(x) \quad \text{(by Cauchy's integration theorem).}$$

Also we have the generalized Leibniz rule. Hence we may write

$$\partial^\mu f(x) := I^{-\mu} f(x) \quad \text{(the fractional derivative).}$$

Make new diff eq's from known ones

From a diff op $P(x, \partial)$ with coeff in $\mathbb{C}[x]$, we can make a new diff op $R(x, \partial)$ as follows,

$$P(x, \partial) \overset{\mathrm{Ad}(\partial^{\mu})}{\rightsquigarrow} R(x, \partial) = \partial^{-\mu+m} P(x, \partial)\, \partial^{\mu}.$$

Moreover,

$$P(x, \partial)u = 0 \Rightarrow \begin{array}{rcl} R(x, \partial) I^{\mu} u & = & \partial^{-\mu+m} P \partial^{\mu} I^{\mu} u \\ & = & \partial^{-\mu+m} P u \\ & = & 0 \end{array}.$$

That is,

$$\text{Sol's of } Pu = 0 \overset{I^{\mu}}{\rightsquigarrow} \text{Sol's of } Rv = 0.$$

Natural questions

◦ What is the **orbit** of a differential equation under the action of Euler transforms?

◦ What is the **group** generated by Euler transforms?

Example (Gauss's hypergeometric equation)

$$x(1-x)\partial^2 u + (\gamma - (\alpha+\beta+1)x)\partial u - \alpha\beta u = 0. \tag{1}$$

Then we can see that

$$\begin{aligned}&\partial^{-\beta}(x(1-x)\partial^2 + (\gamma - (\alpha+\beta+1)x)\partial - \alpha\beta)\partial^{\beta-1}\\ &= x(1-x)\partial + (\gamma-\beta) - (\alpha-\beta+1)x. \quad \textbf{First order!}\end{aligned}$$

A solution of the last equation is $x^{\beta-\gamma}(1-x)^{\alpha-\gamma+1}$. Hence solutions of (1) are

$$I_c^{\beta-1}x^{\beta-\gamma}(1-x)^{\alpha-\gamma} = \frac{1}{\Gamma(-\beta)}\int_c^x t^{\beta-\gamma}(1-t)^{\alpha-\gamma}(x-t)^{-\beta}\,dt.$$

Kac-Moody root system (after Crawley-Boevey)

Riemann schemes (tables of exponents of x^μ)

$$\left\{\begin{matrix} x=0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{matrix}\right\} \xrightarrow{E(\beta)} \left\{\begin{matrix} x=0 & 1 & \infty \\ \beta-\gamma & \gamma-\alpha-1 & \alpha-\beta+1 \end{matrix}\right\}$$

Spectral types (tables of multiplicities of x^μ $(\mu \bmod \mathbb{Z})$)

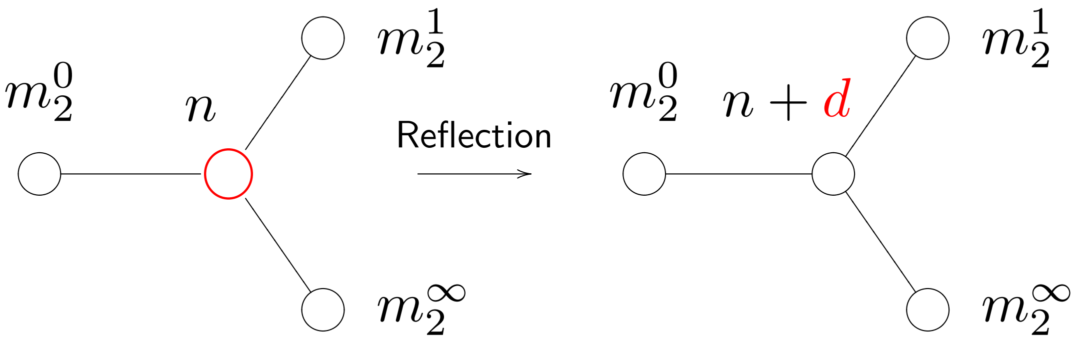
$$\begin{bmatrix} x=0 & 1 & \infty \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} x=0 & 1 & \infty \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Interpretation as reflections

The previous example is a special case of the following computation:

$$\begin{bmatrix} x=0 & 1 & \infty \\ m_1^0 & m_1^1 & m_1^\infty \\ m_2^0 & m_2^1 & m_2^\infty \end{bmatrix} \xrightarrow{\text{Euler trans.}} \begin{bmatrix} x=0 & 1 & \infty \\ m_1^0 + d & m_1^1 + d & m_1^\infty + d \\ m_2^0 & m_2^1 & m_2^\infty \end{bmatrix}$$

for $d = m_2^0 + m_2^1 + m_2^\infty - 2n$. This is the root reflection!!

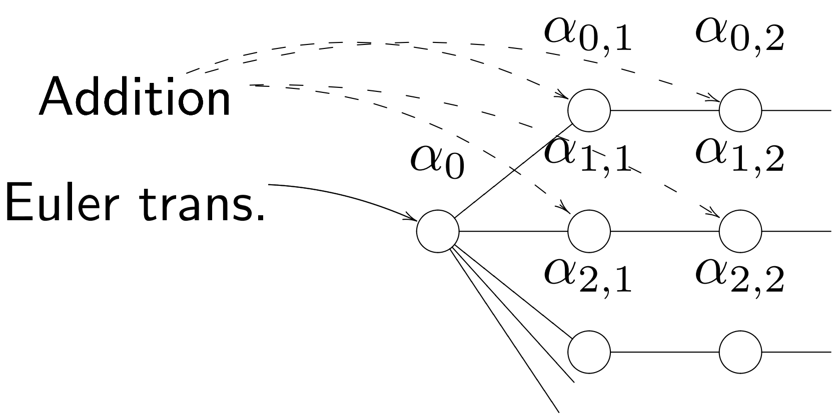


$$n = \begin{cases} m_1^0 + m_2^0 \\ m_1^1 + m_2^1 \\ m_1^\infty + m_2^\infty \end{cases}$$

General case. Let $P(x,\partial)$ be a "good" Fuchsian differential operator with singular points $c_0=\infty, c_1,\dots,c_p$. Then

$$P(x,\partial) \rightsquigarrow \mathbf{m}(P)=\prod_{i=0}^{p}(m_{i,1},\dots,m_{i,l_i}) \quad \text{(multiplicities of } (x-c_i)^{\lambda_{i,j} \pmod{\mathbb{Z}}}\text{)}.$$

We can regard $\mathbf{m}(P)$ as the element $\alpha(\mathbf{m}(P))$ of the root lattice $Q(P)$ with the diagram.



Weyl group symmetry of Euler transforms

Theorem 1 (Crawley-Boevey 2003 (Fuchsian systems), Oshima 2010 (scaler equations)). *Let* $P(x, \partial) \in \mathbb{C}(x)[\partial]$ *be an irreducible Fuchsian differential operator with the spectral type* $\mathbf{m}(P)$*. Then there exists the root lattice* $R(P)$ *of the simply-laced star-shaped diagram* $Q(P)$ *such that the following hold.*

1. $\alpha(\mathbf{m}(P)) \in R(P)^+$ *is a positive root.*

2. *The Weyl group* $W(P)$ *of* $R(P)$ *is generated by Euler transforms.*

3. *# of accessory parameters of* $P(x, \partial)$ *is* $1 - \frac{1}{2}(\alpha(\mathbf{m}(P)), \alpha(\mathbf{m}(P)))$*.*

From Kac to Katz

Theorem 2 (N. Katz '96). *Let* $P(x,\partial)$ *be Fuchsian and irreducible. If* $P(x,\partial)$ *has no accessory parameter, then*

$$P(x,\partial) \rightsquigarrow \partial$$

by a finite iteration of Euler transforms and additions.

Rough explanation.

$$\begin{aligned} P \text{ is accessory parameter free } &\Leftrightarrow (\alpha(\mathbf{m}(P)), \alpha(\mathbf{m}(P))) = 2 \\ &\Leftrightarrow \alpha(\mathbf{m}(P)) \text{ is a real root} \\ &\Leftrightarrow \alpha(\mathbf{m}(P)) \text{ is in a } W(P) \text{ orbit of a simple root.} \end{aligned}$$

□

Non Fuchsian cases

How about equations with irregular singular points?

Assumption: $P(x, \partial)$ has at most unramified irregular singular points.

We can similarly define the spectral type of $P(x, \partial)$ which counts the multiplicities of formal solutions with leading terms

$$e^{f((x-a)^{-1})}(x-a)^{\mu}, \cdots$$

An explanation with an example.

Consider the **doubly confluent Heun's equation**,

$$[x^2\partial^2 + (cx^2 + (2-b)x + a)\partial + (cdx + \lambda)]u = 0.$$

This has local solutions with the following leading terms,

$$\begin{aligned} &x^0, \qquad e^{\frac{a}{x}}x^b && (x=0),\\ &x^{-d}, \qquad e^{cx}x^{2+b-d} && (x=\infty). \end{aligned}$$

The multiplicities of leading terms are

$x=0$	$x=\infty$
$1,1$	$1,1$

.

More generally let us consider the case with the multiplicities of leading terms,

$x=0$	$x=\infty$
m_1, m_2	n_1, n_2

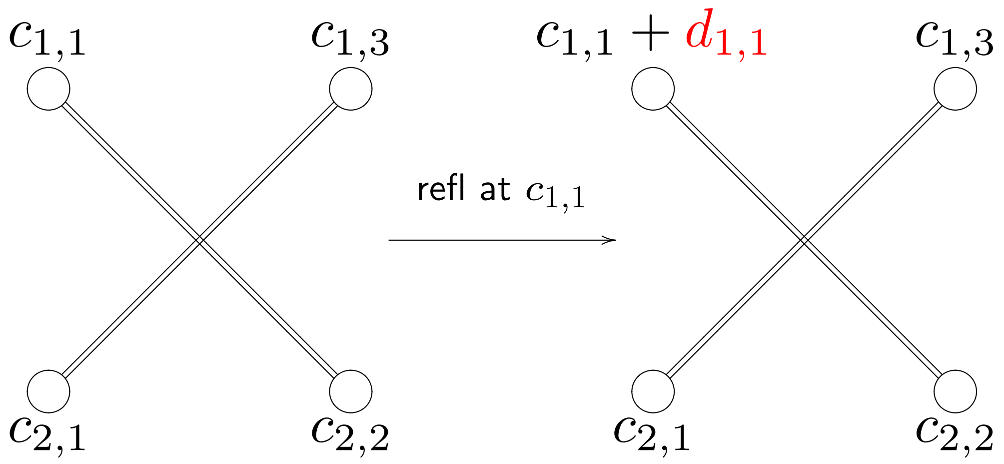
$$(n = m_1 + m_2 = n_1 + n_2).$$

Then middle convolutions change these as follows :

$$(m_1)(m_2),(n_1)(n_2) \quad \xrightarrow{\text{mc}} \quad \begin{array}{c} (m_1+d_{1,1})(m_2),(n_1+d_{1,1})(n_2) \\ (m_1+d_{1,2})(m_2),(n_1)(n_2+d_{1,2}) \\ (m_1)(m_2+d_{2,1}),(n_1+d_{2,1})(n_2) \\ (m_1)(m_2+d_{2,2}),(n_1)(n_2+d_{2,2}). \end{array}$$

Here $d_{i,j} = 2(n - m_i - n_j)$.

This can be seen as the Weyl group action as follows.



Here $c_{i,j}$are defined by

$$c_{i,1} + c_{i,2} = m_i \quad c_{1,j} + c_{2,j} = n_j.$$

Thus the doubly confluent Heun lives in $(A_1 \oplus A_1)^{(1)}$ root lattice with Weyl group symmetry.

Theorem 3 (H 2013, (cf. H-Oshima 2013)). *If $P(x,\partial) \in \mathbb{C}(x)[\partial]$ is irreducible and has at most unramified irregular singular points. There exists symmetric Kac-Moody root lattice $R(P)$ such that the following hold.*

1. *The spectral type $\mathbf{m}(P)$ corresponds to a positive root $\alpha(P)$ of $R(P)$.*

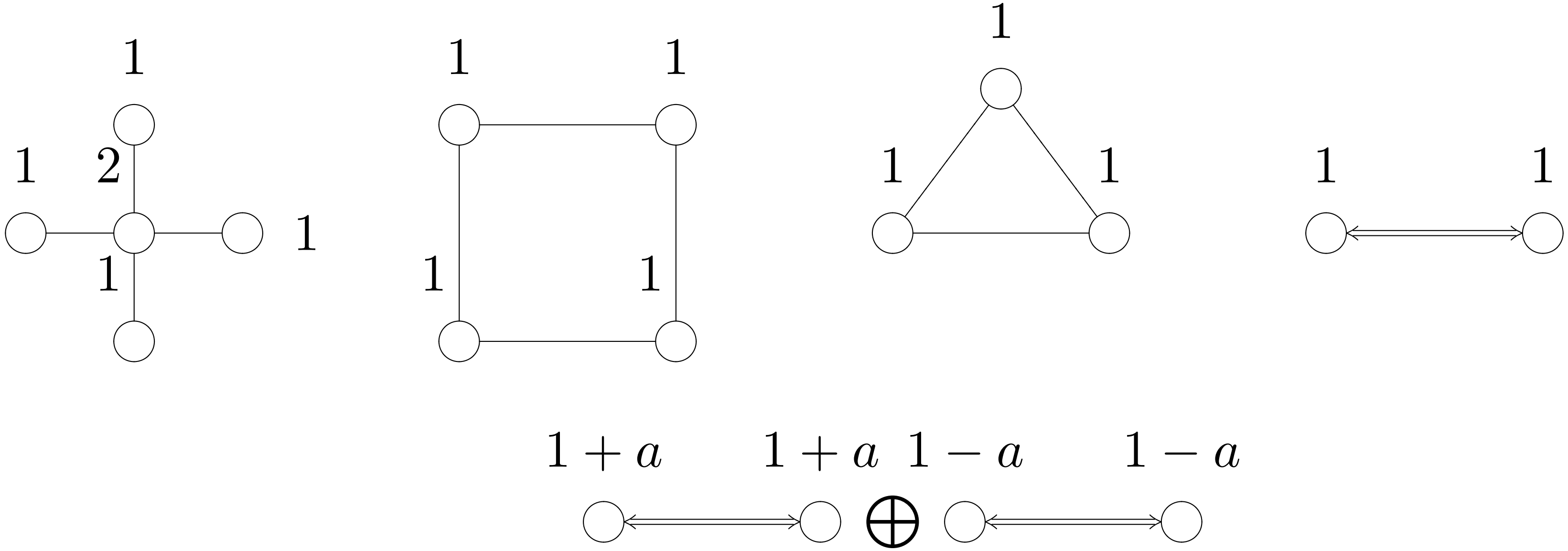
2. *The Weyl group $W(P)$ is generated by Euler transforms.*

3. *# of accessory parameters is $1 - \frac{1}{2}(\alpha(P), \alpha(P))$.*

4. *The Weyl group acts on the space of characteristic exponents of the formal solutions.*

Confluent family of Heun's equations

Heun	Confluent Heun	Biconf Heun	Triconf Heun	Doubleconf Heun
$1+1+1+1$	$1+1+2$	$1+3$	4	$2+2$
$D_4^{(1)}$	$A_3^{(1)}$	$A_2^{(1)}$	$A_1^{(1)}$	$A_1^{(1)} \oplus A_1^{(1)}$



- All roots are null imaginary roots of affine root systems.

- All of them are fixed by Weyl group. However, characteristic exponents are changed by Euler transforms.

- Weyl group symmetries coincide with them of Bäcklund transform of corresponding Painlevé equations!

Classification of spectral types (joint work with T. Oshima)

Let us classify Euler transform orbits of diff. eq's.

Assumption: $P(x,\partial) \in \mathbb{C}(x)[\partial]$ has at most unramified irregular singular points.

[0-acc. param.]

Theorem 4 (Katz, Arinkin). *If $P(x,\partial)$ is irreducible and has no accessory parameter, then P is in the Euler transform orbit of ∂.*

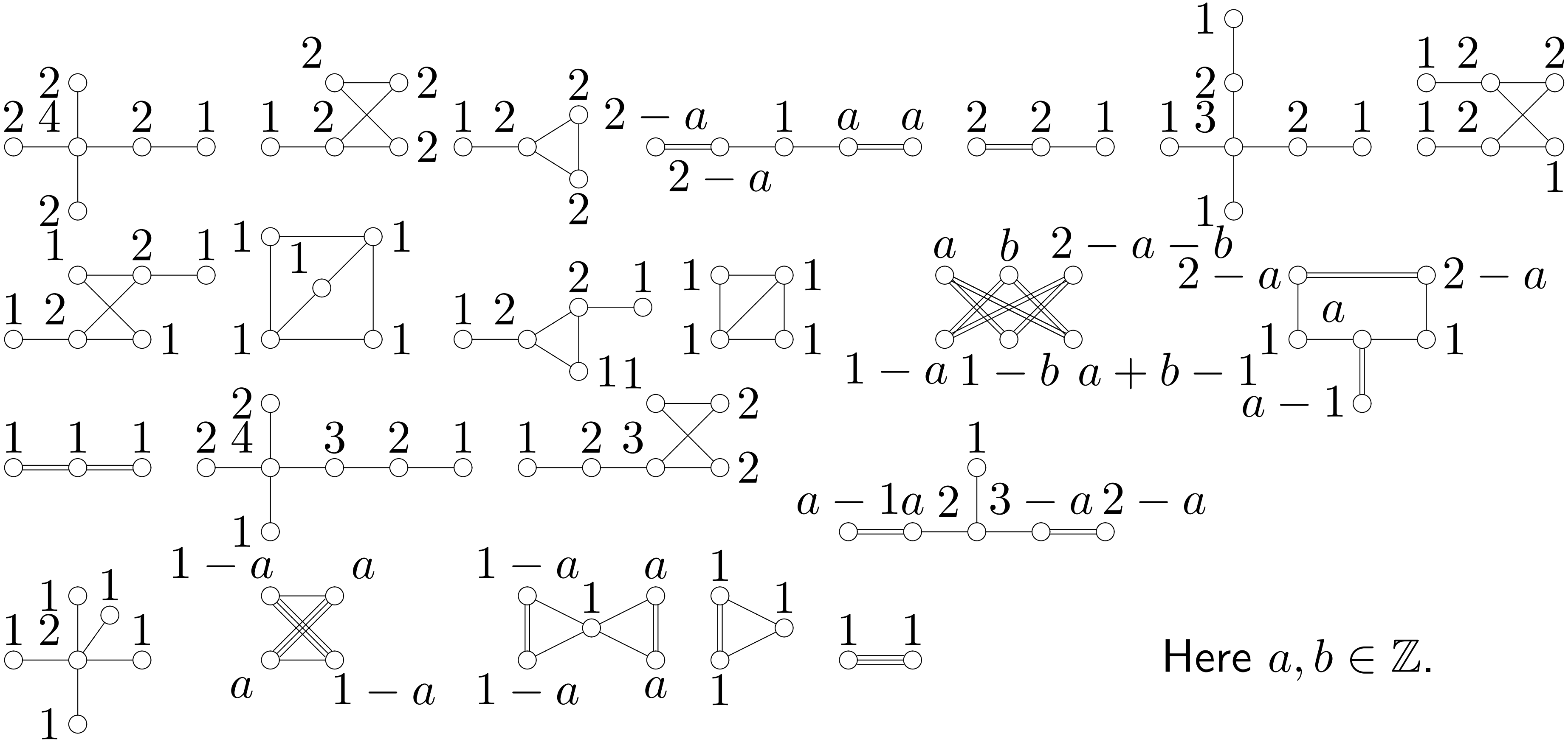
[1-acc. param.]

Theorem 5 (Kostov, Takemura, H-Oshima). *If $P(x,\partial)$ is irreducible and has one accessory parameters, then the corresponding root is in the Weyl group orbit of the null imaginary root of one of the following Euclidean root systems.*

$$E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, D_4^{(1)}, A_3^{(1)}, A_2^{(1)}, A_1^{(1)}, (A_1 \oplus A_1)^{(1)}.$$

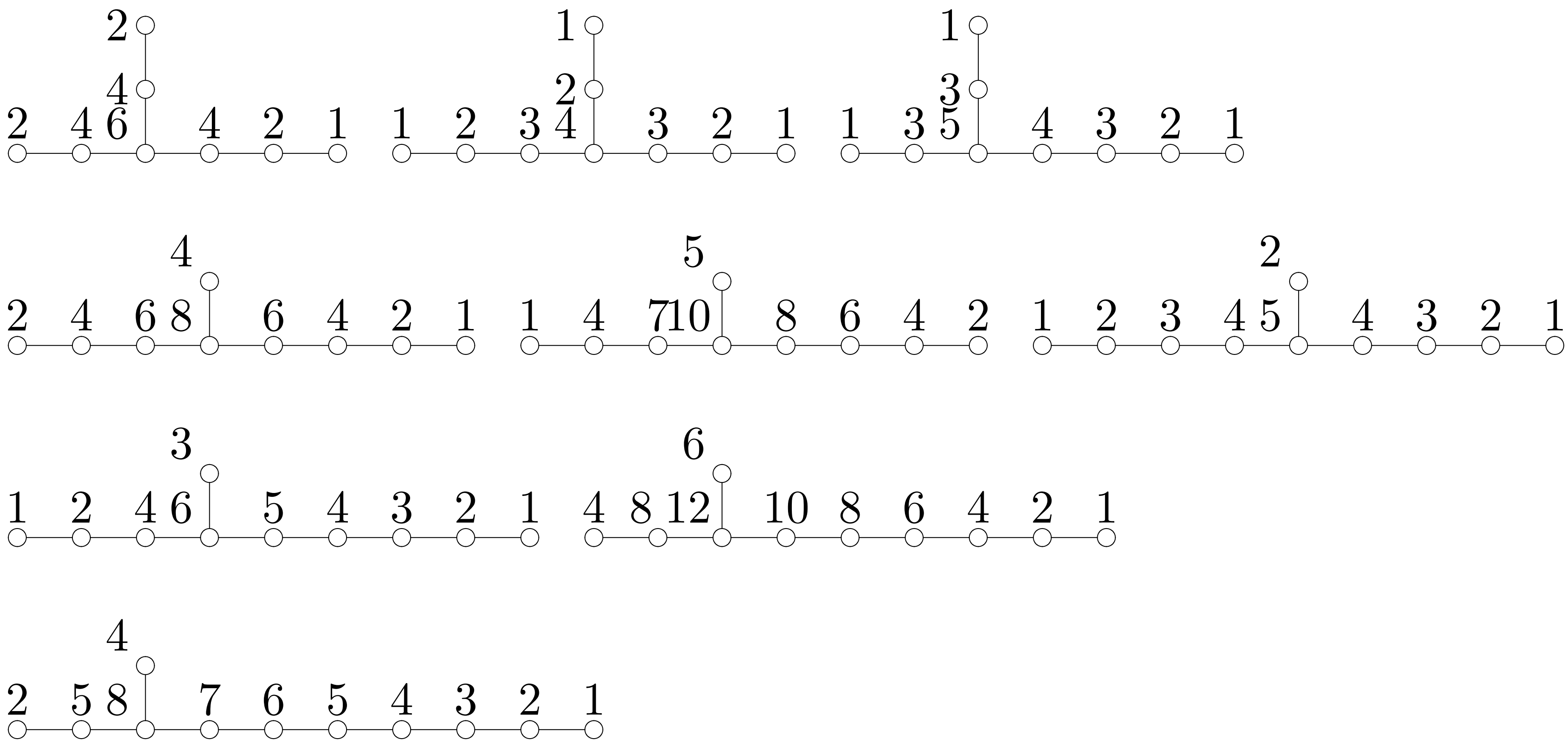
[2-acc. param.]

Theorem 6 (Oshima, H-Oshima)**.** *If* $P(x, \partial)$ *is irreducible and has two accessory parameters, then the corresponding root is in the Weyl group orbit of the null imaginary root of one of the following Euclidean root systems.*



Here $a, b \in \mathbb{Z}$.

And MORE!!


2
4
2 4 6 4 2 1
1
2
1 2 3 4 3 2 1
1
3
1 3 5 4 3 2 1
4
2 4 6 8 6 4 2 1
5
1 4 7 10 8 6 4 2
2
1 2 3 4 5 4 3 2 1
3
1 2 4 6 5 4 3 2 1
6
4 8 12 10 8 6 4 2 1
4
2 5 8 7 6 5 4 3 2 1

Remark 7. *Our classification coincides with the degeneration scheme of 4-dimensional Painlevé equations given by Kawakami-Nakamura-Sakai.*

[General cases]

Theorem 8 (Oshima, H-Oshima)**.** *# of Weyl group orbits of differential equations with the fixed number of accessory parameters is finite.*

Additive Deligne-Simpson problem

Definition 9. *An additive Deligne-Simpson problem consists of a collection of points* $a_1, \ldots, a_p \in \mathbb{C}$ *and of conjugacy classes* $C_0, C_1, \ldots, C_p$ *of* $M(n, \mathbb{C})$*. The solution of the problem is irreducible* Fuchsian system

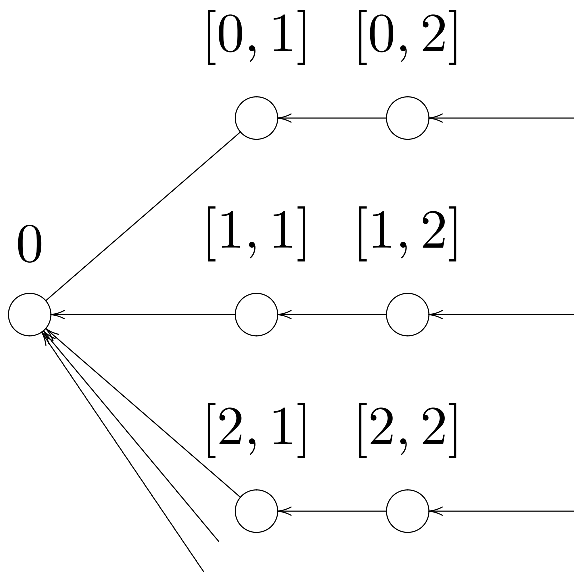
$$\frac{d}{dx}Y = \sum_{i=1}^{p} \frac{A_i}{x - a_i} Y$$

satisfying that $A_i \in C_i$*. Here* $A_0 := -\sum_{i=1}^{p} A_i$*.*

W. Crawley-Bovevey gave a necessary and sufficient condition for the existence of a solution by using the representation theory of quivers.

Let us choose $\xi_{[i,1]}, \ldots, \xi_{[i,d_i]} \in \mathbb{C}$ so that $\prod_{j=1}^{d_i}(A_i - \xi_{[i,j]} I_n) = 0$ for all $A_i \in C_i$ ($\approx$ eigenvalues).

Then consider the following quiver.



Define a collection of positive integers $\alpha = (\alpha_0, \alpha_{[i,j]})$ and of complex numbers $\lambda = (\lambda_0, \lambda_{[i,j]})$ by

$$\alpha_0 = 0, \qquad \alpha_{[i,j]} = \operatorname{rank} \prod_{k=1}^{j} (A_i - \xi_{[i,k]} I),$$

$$\lambda_0 = -\sum_{i=0}^{p} \xi_1, \qquad \lambda_{[i,j]} = \xi_{[i,j]} - \xi_{[i,j+1]}.$$

Theorem 10 (Crawley-Boevey). *The additive Deligne-Simpson problem has a solution if and only if the following are satisfied.*

1. *α is a positive root.*

2. *For any decomposition $\alpha = \beta_1 + \beta_2 + \cdots$ where β_t is a positive root with $\lambda \cdot \beta_t = 0$, we have the equality $p(\alpha) > p(\beta_1) + p(\beta_2) + \cdots$.*

Here $p(\beta) = 1 - \frac{1}{2}(\beta, \beta)$.

Remark 11. *A corresponding result for scalar equations is obtained by T. Oshima.*

Additive Deligne-Simpson problems for non-Fuchsian equations

$$
\begin{aligned}
G_k &= \mathrm{GL}(n, \mathbb{C}[[x]]/x^k\mathbb{C}[[x]]) \\
&= \left\{ \sum_{i=0}^{k-1} A_i x^i \,\middle|\, A_0 \in \mathrm{GL}(n,\mathbb{C}),\ A_i \in M(n,\mathbb{C}) \right\} \\
\mathfrak{g}_k^* &= M(n, x^{-k}\mathbb{C}[[x]]/\mathbb{C}[[x]]) \\
&= \left\{ \sum_{i=1}^{k} \frac{A_i}{x^i} \,\middle|\, A_i \in M(n,\mathbb{C}) \right\}
\end{aligned}
$$

Definition 12 (Hukuhara-Turrittin-Levelt normal forms).

$$B = \mathrm{diag}(q_1(x^{-1})I_{n_1} + R_1 x^{-1}, \ldots, q_m(x^{-1})I_{n_m} + R_m x^{-1}) \in \mathfrak{g}_k^*$$

with $q_i(x) \in x^2\mathbb{C}[x]$ *satisfying* $q_i \neq q_j$ *if* $i \neq j$ *and* $R_i \in M(n_i, \mathbb{C})$.

Fuchsian		non-Fuchisan
Jordan normal forms	$\leftrightarrow$	HTL normal forms
$\mathrm{GL}(n,\mathbb{C})$-orbits	$\leftrightarrow$	G_k-orbits

Definition 13. *An generalized additive Deligne-Simpson problem consists of a collection of points* $a_1, \ldots, a_p \in \mathbb{C}$*, of nonzero positive integers* $k_0, \ldots, k_p$ *and of HTL normal forms* $B_i \in \mathfrak{g}^*_{k_i}$ *for* $i = 0, \ldots, p$*. A solution of the problem is an irreducible equation*

$$\frac{d}{dx}Y = \left(\sum_{i=1}^{p} \sum_{j=1}^{k_i} \frac{A_{i,j}}{(x-a_i)^j} + \sum_{j=1}^{k_0-1} A_{0,j+1} x^j \right) Y$$

where $A_i(x) = \sum_{j=1} A_{i,j} x^{-j}$ *are in* G_{k_i}*-orbit of* B_i*. Here we put* $A_{0,1} = -\sum_{i=1}^{p} A_{i,1}$*.*

Remark 14 (Known results). • *The case* $k_0 = \cdots k_p = 1$ *corresponds to the Fuchsian case.*

- *The case* $k_0 \leq 3$ *and* $k_1 = \cdots k_p = 1$ *is considered by P. Boalch who gave a necessary and sufficient condition of the existence of a solution.*

- *The case; an arbitrary* k_0 *and* $k_1 = \cdots k_p = 1$, *is considered by H-Yamakawa.*

- *The case; each* B_i *has regular semisimple top, is considered by V. Kostov.*

Associated quiver

$$B_i = \mathrm{diag}(q_1^{(i)}(x^{-1})I_{n_1^{(i)}} + R_1^{(i)}x^{-1}, \ldots, q_{m^{(i)}}^{(i)}(x^{-1})I_{n_{m^{(i)}}^{(i)}} + R_{m^{(i)}}^{(i)}x^{-1})$$

for $i = 0, \ldots, p$. Choose $\xi_1^{[i,j]}, \ldots, \xi_{e_{[i,j]}}^{[i,j]} \in \mathbb{C}$ so that

$$\prod_{k=1}^{e_{[i,j]}} (R_j^{(i)} - \xi_k^{[i,j]}) = 0.$$

Set $I_{\mathsf{irr}} = \{i \in \{0, \ldots, p\} \mid m^{(i)} > 0\} \cup \{0\}$ and $I_{\mathsf{reg}} = \{0, \ldots, p\} \backslash I_{\mathsf{irr}}$.

Then let us define the quiver $\mathsf{Q} = (\mathsf{Q}_0, \mathsf{Q}_1)$ as follows.

The set of vertices

$$\mathsf{Q}_0 = \left\{ [i,j] \;\middle|\; \begin{matrix} i \in I_{\mathsf{irr}}, \\ j = 1,\dots,m^{(i)} \end{matrix} \right\} \cup \left\{ [i,j,k] \;\middle|\; \begin{matrix} i = 0,\dots,p, \\ j = 1,\dots,m^{(i)}, \\ k = 1,\dots,e_{[i,j]} - 1 \end{matrix} \right\}$$

The set of arrows (Here $d_i(j,j') = \deg_{\mathbb{C}[x]}(q_j^{(i)}(x) - q_{j'}^{(i)}(x)) - 2$)

$$
\begin{aligned}
\mathsf{Q}_1 = &\left\{ \rho^{[0,j]}_{[i,j']} \colon [0,j] \to [i,j'] \;\middle|\; \begin{array}{l} j = 1,\dots,m^{(0)}, \\ i \in I_{\mathsf{irr}} \backslash \{0\}, \\ j' = 1,\dots,m^{(i)} \end{array} \right\} \\
&\cup \left\{ \rho^{[k]}_{[i,j],[i,j']} \colon [i,j] \to [i,j'] \;\middle|\; \begin{array}{l} i \in I_{\mathsf{irr}},\ 1 \le j < j' \le m^{(i)}, \\ 1 \le k \le d_i(j,j') \end{array} \right\} \\
&\cup \left\{ \rho_{[i,j,1]} \colon [i,j,1] \to [i,j] \mid i \in I_{\mathsf{irr}},\ j = 1,\dots,m^{(i)} \right\} \\
&\cup \left\{ \rho^{[i,1,1]}_{[0,j]} \colon [i,1,1] \to [0,j] \mid i \in I_{\mathsf{reg}},\ j = 1,\dots,m^{(0)} \right\} \\
&\cup \left\{ \rho_{[i,j,k]} \colon [i,j,k] \to [i,j,k-1] \;\middle|\; \begin{array}{l} i = 0,\dots,p, \\ j = 1,\dots,m^{(i)}, \\ k = 2,\dots,e_{[i,j]} - 1 \end{array} \right\}.
\end{aligned}
$$

Associated complex numbers and integers

From HTL normal forms B_i, we define the following numbers. Let us define $\alpha = (\alpha_a)_{a \in \mathsf{Q}_0} \in \mathbb{Z}^{\mathsf{Q}_0}$ and $\lambda = (\lambda_a)_{a \in \mathsf{Q}_0} \in \mathbb{C}^{\mathsf{Q}_0}$ by

$$\alpha_{[i,j]} = n_j^{(i)}, \qquad \alpha_{[i,j,k]} = \dim_{\mathbb{C}}(\mathrm{rank} \prod_{l=1}^{k} (R_j^{(i)} - \xi_l^{[i,j]})),$$

$$\lambda_{[i,j]} = -\xi_1^{[i,j]} \text{ for } i \in I_{\mathsf{irr}} \backslash \{0\}, j = 1, \ldots, m^{(i)},$$
$$\lambda_{[0,j]} = -\xi^{[0,j]} - \sum_{i \in I_{\mathsf{reg}}} \xi_1^{[i,1]} \text{ for } j = 1, \ldots, m^{(0)},$$
$$\lambda_{[i,j,k]} = \xi_k^{[i,j]} - \xi_{k+1}^{[i,j]} \text{ for } i = 0, \ldots, p, j = 1, \ldots, m^{(i)}, k = 1, \ldots, e_{[i,j]} - 1.$$

Theorem 15 (H 2013). *The generalized additive Deligne-Simpson problem has a solution if and only if the following are satisfied.*

1. *α is a positive root of* Q.

2. *For any decomposition $\alpha = \beta_1 + \beta_2 + \cdots$ where $\beta_t \in \mathcal{L}$ are positive roots and satisfy $\lambda \cdot \beta_i = 0$, we have*

$$p(\alpha) > p(\beta_1) + p(\beta_2) + \cdots .$$

Here

$$\mathcal{L} = \left\{ \beta \in \mathbb{Z}^{\mathsf{Q}_0} \,\middle|\, \begin{array}{l} \text{there exists } b \in \mathbb{Z} \text{ such that} \\ \sum_{j=1}^{m^{(i)}} \beta_{[i,j]} = b \text{ for all } i \in I_{irr} \end{array} \right\}.$$

References

W. Crawley-Boevey, *On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero*, Duke Math. J. **118**, no. 2 (2003).

P. Boalch, *Irregular connections and Kac-Moody root systems*, arXiv:0806.1050.

T. Oshima, *Fractional calculus of Weyl algebra and Fuchsian differential equations*, MSJ Memoirs **28** (2012), Math. Soc. Japan.

K. Hiroe, *Linear differential equations on* $\mathbb{P}^1$ *and root systems*, J. Algebra, **382**, (2013).

K. Hiroe, T. Oshima, *A classification of roots of symmetric Kac-Moody root systems and its application*, in Symmetries, integrable systems and representations, Springer-Verlag, 2013.

K. Hiroe, D. Yamakawa, *Moduli spaces of meromorphic connection and quiver varieties*, arXiv:1305.4092, 2013.

K. Hiroe, *Linear differential equations on the Riemann sphere and representations of quivers*, arXiv:1307.7438, 2013.