k-summability of formal solutions for a class of partial differential equations with polynomial coefficients

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1 Result

$$L = \partial_t - P(t, \partial_x), \quad P(t, \partial_x) = \sum_{i,\alpha \in \mathbb{N}_0}^{\text{finite}} a_{i\alpha} t^i \partial_x^\alpha,$$

where $t, x \in \mathbb{C}$, $a_{i\alpha} \in \mathbb{C}$ and $\mathbb{N}_0 := \{0, 1, \cdots\}$.

We consider the following Cauchy problem

$$\begin{cases} LU(t,x) = (\partial_t - P(t, \partial_x))U(t,x) = 0, \\ U(0,x) = \varphi(x) \in \mathcal{O}_x. \end{cases} \tag{CP}$$

$\exists 1$ formal solution :

$$\hat{U}(t,x) = \sum_{n \geq 0} U_n(x) \frac{t^n}{n!}, \quad U_0(x) = \varphi(x).$$

Assumption 1 (A-1) [non-Kowalevskian]

$$\max\{\alpha;\ a_{i\alpha} \neq 0\} \geq 2.$$

Aim: k-summability of $\hat{U}(t,x)$

Newton polygon

$$N(i,\alpha) := \begin{cases} \{(x,y) \in \mathbb{R}^2; x \leq \alpha, y \geq i, a_{i\alpha} \neq 0\}, \\ \phi \qquad\qquad \text{for } a_{i\alpha} = 0. \end{cases}$$

$$N(L) := \mathrm{Ch}\{N(-1,1) \cup \bigcup_{i,\alpha \in \mathbb{N}_0} N(i,\alpha)\}.$$

Assumption 2 (A-2) [Newton polygon]

$N(L)$ has only one side of a positive slope with $(1,-1)$ and (α_0, i_0).

Assumption 3 (A-3) [modified order]

$$\frac{\alpha_0}{i_0 + 1} \geq \frac{\alpha}{i + 1} \quad \text{for } \forall(i, \alpha) \text{ with } a_{i\alpha} \neq 0.$$

We call $\frac{\alpha_0}{i_0+1}$ the modified order of L.

Theorem 1. Let $\kappa := \dfrac{i_0+1}{\alpha_0-1}$. For a fixed $d \in \mathbb{R}$, we define

$$d_j = \frac{i_0+1}{\alpha_0} d + \frac{\arg a_{i_0 \alpha_0} + 2\pi j}{\alpha_0} \text{ for } j = 0, 1, \ldots, \alpha_0 - 1.$$

$$\varphi(x) \in \mathcal{O}\left(\overline{D}_\sigma \cup \bigcup_{j=0}^{\alpha_0-1} S(d_j, \varepsilon)\right),$$

$$|\varphi(x)| \leq C \exp\left(\delta |x|^{\frac{\alpha_0}{\alpha_0-1}}\right). \quad (1)$$

Then the formal solution $\hat{U}(t,x)$ of the Cauchy problem (CP) is κ-summable in d direction. (We write "$\hat{U}(t,x) \in \mathcal{O}_x\{t\}_{\kappa,d}$").

Remark. [M. Miyake]

$\hat{U}(t,x)$ is convergent at $t = 0 \iff \varphi(x)$ entire, (1).

Comments of (A-3)

Example:(A-3) is satisfied

$$\partial_t U(t,x) = t\partial_x^2 U(t,x) + U(t,x).$$

Example:(A-3) is NOT satisfied

$$\partial_t U(t,x) = t^3\partial_x^7 U(t,x) + \partial_x^2 U(t,x) + t\partial_x^4 U(t,x).$$

We shall give the proof of Theorem 1 by using the method of successive approximation.

2 Decomposition of P

We put

$$\frac{\alpha_0}{i_0+1} = \frac{p}{q}, \quad (p, q) = 1.$$

For $j \geq 0$,

$$K_j := \{(i, \alpha);\ j = p(i+1) - q\alpha,\ a_{i\alpha} \neq 0\},$$

$$P_j(t, \partial_x) := \sum_{(i,\alpha)\in K_j} a_{i\alpha} t^i \partial_x^\alpha.$$

Especially, for $(i, \alpha) \in K_0$, we have $1 \leq \alpha \leq \alpha_0,\ 0 \leq i \leq i_0$.

In this case, $\exists 1 J (\geq 1)$ such that

$$P(t, \partial_x) = \sum_{j=0}^{J} P_j(t, \partial_x).$$

3 The sequence of CPs

$$\begin{cases} \partial_t u_0(t,x) = P_0(t,\partial_x)u_0(t,x) \\ u_0(0,x) = \varphi(x). \end{cases} \tag{E_0}$$

For $k \geq 1$,

$$\begin{cases} \partial_t u_k(t,x) = \displaystyle\sum_{j=0}^{\min\{J,k\}} P_j(t,\partial_x)u_{k-j}(t,x), \\ u_k(0,x) = 0. \end{cases} \tag{E_k}$$

For each k, $\exists 1$ formal solution $\hat{u}_k(t,x) = \displaystyle\sum_{n\geq 0} u_{kn}(x)\frac{t^n}{n!}$.

$\Longrightarrow$ $\hat{U}(t,x) = \displaystyle\sum_{k\geq 0} \hat{u}_k(t,x)$ is the formal solution of (CP).

4 Construction of $\hat{u}_k$

Proposition 1. For each k, the formal solution $\hat{u}_k$ is given by

$$\hat{u}_k(t,x) = \sum_{n\geq 0} u_{kn}(x)\frac{t^n}{n!} = \sum_{n\geq 0} A_k(n)\varphi^{(\frac{p}{q}n-\frac{k}{q})}(x)\frac{t^n}{n!},$$

where $A_k(n) = 0$ if $\frac{p}{q}n - \frac{k}{q} \notin \mathbb{N}_0$ or $n < 0$.

$$\begin{cases} A_0(n+1) = \sum_{K_0} a_{i\alpha}[n]_i A_0(n-i) \\ A_0(0) = 1. \end{cases} \qquad (R_0)$$

Here $[n]_i = \begin{cases} n(n-1)\cdots(n-i+1) & (i \geq 1), \\ 1 & (i = 0). \end{cases}$

For $k \geq 1$,

$$\begin{cases} A_k(n+1) = \sum_{j=0}^{\min\{J,k\}} \sum_{K_j} a_{i\alpha}[n]_i A_{k-j}(n-i) \\ A_k(0) = 0. \end{cases} \qquad (R_k)$$

Remark [The Order of Zeros of $\hat{u}_k$]. For $k \geq 0$, we put

$$k = p\ell - r(k),$$

where $\ell \geq 0$ and $0 \leq r(k) < p$.

Example: $k = 0 \implies \ell = 0,\ r(0) = 0.$

$$k = 1 \implies \ell = 1,\ r(1) = p - 1.$$

For any k, a pair $(\ell, r(k))$ is uniquely decided.

In this case, we have $u_{kn}(x) = A_k(n)\varphi^{(\frac{p}{q}n - \frac{k}{q})}(x)$ if $n \geq \ell$ and

$$\boxed{u_{kn}(x) \equiv 0 \text{ if } n < \ell \iff A_k(n) = 0 \text{ if } n < \ell.}$$

5 Gevrey order of $\hat{u}_k$

Proposition 2. Let $\hat{u}_k(t,x)$ be the formal solution of (E_k). Then for each k, $\hat{u}_k(t,x) \in \mathcal{O}_x[[t]]_{1/\kappa}$, $(\kappa = (i_0+1)/(\alpha_0 - 1))$ which means that

$$\max_{|x|\le\sigma} \left| \frac{u_{kn}(x)}{n!} \right| \le C K^n \Gamma\left(1 + \frac{n}{\kappa}\right).$$

We put $\hat{f}_k(t) := \sum_{n\ge 0} A_k(n) t^n$.

Lemma 1. $\hat{f}_k(t) \in \mathbb{C}[[t]]_{1/\tilde{\kappa}}$, $\tilde{\kappa} = (i_0+1)/i_0$.

$\hat{f}_k$ satisfy the following ordinary differential equations.

$$\left(\sum_{K_0} a_{i\alpha} t^{i+1}[\delta_t + i]_i - 1\right) f_0(t) = 1, \quad \delta_t = d/(dt),$$

$$\left(\sum_{K_0} a_{i\alpha} t^{i+1}[\delta_t + i]_i - 1\right) f_k(t) =$$
$$- \sum_{j=1}^{\min\{J,k\}} \sum_{K_j} a_{i\alpha} t^{i+1}[\delta_t + i]_i f_{k-j}(t)$$

$$\Longleftrightarrow \left(a_{i_0,\alpha_0}(t^{\tilde{\kappa}}\delta_t)^{i_0} - 1\right) f_k(t) = \mathcal{L}_f(t,\delta_t) f_k(t) + \{\cdots\}.$$

Here $\mathcal{L}_f$ is the linear ordinary differential operator which is not the principal part in the sense of the Newton polygon and $\{\cdots\}$ denotes the inhomogeneous term.

6 Preliminary for κ-summability

Lemma 2. Let $\kappa > 0$, $d \in \mathbb{R}$ and $\hat{v}(t,x) = \sum_{n\geq 0} v_n(x) t^n \in \mathcal{O}_x[[t]]_{1/\kappa}$. Then the following statements are equivalent.

i) $\hat{v}(t,x) \in \mathcal{O}_x\{t\}_{\kappa,d}$

ii) Let $(v_B)(s,x) = (\hat{\mathcal{B}}_\kappa \hat{v})(s,x) := \displaystyle\sum_{n\geq 0} v_n(x) \frac{s^n}{\Gamma(1+n/\kappa)}$.

Then $(v_B)(s,x) \in \mathrm{Exp}_s(\kappa; S(d,\varepsilon) \times \overline{D_\sigma})$, which means that

$$\max_{|x|\leq\sigma} |(v_B)(s,x)| \leq C \exp\left(\delta |s|^\kappa\right), \quad s \in S(d,\varepsilon).$$

$$\left(\hat{v}(t,x) \in \mathcal{O}_x[[t]]_{1/\kappa} \Leftrightarrow \max_{|x|\leq\sigma} |v_n(x)| \leq C K^n \Gamma\left(1+\tfrac{n}{\kappa}\right)\right)$$

Summability of $\hat{f}_k$

Lemma 3. $(f_k)_B(s) = (\hat{\mathcal{B}}_{\tilde{\kappa}} \hat{f}_k)(s)$ has (i_0+1) singular points which are given by $s = c \cdot (a_{i_0,\alpha_0})^{-1/(i_0+1)} \omega_{i_0+1}^{-n}$ for $n = 0, 1, \dots, i_0$, where $c = (1/\tilde{\kappa})^{1/\tilde{\kappa}}$ and $\omega_q = e^{2\pi i/q}$. Moreover, $(f_k)_B(s) \in \mathrm{Exp}_s(\tilde{\kappa}; S(d, \varepsilon))$ where

$$d \not\equiv -(\arg a_{i_0,\alpha_0} + 2\pi n)/(i_0+1) \pmod{2\pi} \qquad \text{(SD)}$$

for $n = 0, 1, \dots, i_0$.

Corollary 1. $\hat{f}_k(t) \in \mathbb{C}\{t\}_{\tilde{\kappa},d}$ where d satisfies (SD).

$$\left(\sum\nolimits_{K_0} a_{i,\alpha} t^{i+1} [\delta_t + i]_i - 1\right) f_k(t) =$$

$$\begin{cases} 1, & (k=0) \\ -\displaystyle\sum_{j=1}^{\min\{J,k\}} \sum_{K_j} a_{i\alpha} t^{i+1} [\delta_t + i]_i f_{k-j}(t), & (k \geq 1) \end{cases}$$

$$\Longleftrightarrow \left(a_{i_0,\alpha_0} (t^{\tilde{\kappa}} \delta_t)^{i_0} - 1\right) f_k(t) = \mathcal{L}_f(t, \delta_t) f_k(t) + \{\cdots\}.$$

$$\left((\tilde{\kappa})^{i_0} a_{i_0,\alpha_0} s^{i_0+1} - 1\right) \frac{d}{ds} f_{k\,B}(s) = \frac{d}{ds} \hat{\mathcal{B}}_{\tilde{\kappa}} \, (\text{R.H.S.}),$$

※ $\hat{\mathcal{B}}_\kappa (t^\kappa \delta_t)^i = (\kappa D_s^{-1} s^\kappa D_s)^i = \kappa^i D_s^{-1} s^{\kappa i} D_s$, where $D_s^{-1} = \int_0^s$ and $D_s = d/(ds)$.

Summability of $\hat{G}_k$

Let $k = p\ell - r(k)$. We define

$$\hat{G}_k(t) := \sum_{n\geq 0} A_k(n+\ell) w_k(n) t^n, \quad w_k(n) = \frac{\Gamma\left(\frac{pn+r(k)}{q}\right)}{n!}. \tag{2}$$

Lemma 4. $(G_k)_B(s) = (\hat{\mathcal{B}}_\kappa \hat{G}_k)(s)$ has (i_0+1) singular points which are given by $s = c_G \cdot (a_{i_0,\alpha_0})^{-1/(i_0+1)} \omega_{i_0+1}^{-n}$ for $n = 0, 1, \ldots, i_0$, where c_G is some positive constant. Moreover, $(G_k)_B(s) \in \mathrm{Exp}_s(\kappa; S(d, \varepsilon))$ where d satisfies (SD).

Corollary 2. $\hat{G}_k(t) \in \mathbb{C}\{t\}_{\kappa,d}$ where d satisfies (SD).

7 Proof of Theorem 1

By Lemma 2 and 4, we obtain the following results for the formal solutions $\hat{u}_k(t,x)$ of the Cauchy problems (E_k).

Proposition 3. Let $d \in \mathbb{R}$ and $k = p\ell - r(k)$. We assume that $\varphi(x)$ satisfies the same conditions as in Theorem 1. Then we have for $s \in S(d,\varepsilon)$

$$\max_{|x|\le\sigma} |(u_k)_B(s,x)| \le C \cdot \frac{|s|^\ell}{\ell!\Gamma(1+\ell/\kappa)} \exp\left(\delta|s|^\kappa\right).$$

Corollary 3. Assume $\varphi(x)$ satisfies the same assumptions as in Theorem 1. Then

$$\hat{u}_k(t,x) \in \mathcal{O}_x\{t\}_{\kappa,d}.$$

Proof of Theorem 1. We remark that $\hat{U}(t,x) = \sum_{k\geq 0} \hat{u}_k(t,x)$ is the formal power series solution of the original Cauchy problem (CP). By Proposition 3, we obtain the desired exponential growth estimate of $U_B(s,x)$ for $s \in S(d,\varepsilon)$.

$$\max_{|x|\leq\sigma} |U_B(s,x)| \leq \sum_{k\geq 0} \max_{|x|\leq\sigma} |(u_k)_B(s,x)|$$

$$\leq \sum_{r(k)=0}^{p-1} \sum_{\ell\geq 0} \max_{|x|\leq\sigma} |(u_{p\ell-r(k)})_B(s,x)|$$

$$\leq pC\exp(\delta|s|^\kappa) \sum_{\ell\geq 0} \frac{|s|^\ell}{\ell!\Gamma(1+\ell/\kappa)} \leq C_1 \exp(\delta|s|^\kappa)\exp(\gamma|s|^K)$$

$$\leq C_2 \exp(\tilde{\delta}|s|^\kappa) \text{ where } K = \frac{\kappa}{\kappa+1} < \kappa.$$

Outline of proof of Proposition 3. Let $k = p\ell - r(k)$.

$$(u_k)_B(s,x) = \sum_{n \geq \ell} A_k(n)\varphi^{(pn/q-k/q)}(x)\frac{s^n}{n!\Gamma(1+n/\kappa)}$$

$$= \sum_{n \geq 0} A_k(n+\ell)\frac{\varphi^{((pn+r(k))/q)}(x)}{(n+\ell)!}\frac{s^{n+\ell}}{\Gamma(1+(n+\ell)/\kappa)}$$

$$= \frac{s^\ell}{2\pi i}\int_0^1 \frac{(1-r_1)^{\ell-1}}{(\ell-1)!}dr_1 \int_0^1 \frac{(1-r_2)^{\ell/\kappa-1}}{\Gamma(\ell/\kappa)}dr_2$$

$$\times \oint \frac{\varphi(x+\zeta)}{\zeta^{1+r(k)/q}} \times (G_k)_B\left(r_1 r_2^{1/\kappa}\frac{s}{\zeta^{p/q}}\right)d\zeta.$$

$$\left(\frac{1}{(n+\ell)!} = \frac{B(n+1,\ell)}{n!(\ell-1)!},\ \frac{1}{\Gamma(1+(n+\ell)/\kappa)} = \frac{B(1+n/\kappa,\ell/\kappa)}{\Gamma(1+n/\kappa)\Gamma(\ell/\kappa)}\right)$$

$$(G_k)_B(X) = \sum_{n \geq 0} A_k(n+\ell) \frac{\left(\frac{pn+r(k)}{q}\right)!}{n!} \frac{X^n}{\Gamma(1+n/\kappa)}.$$

Lemma. $(G_k)_B(s)$ has (i_0+1) singular points which are given by $s = c_G \cdot (a_{i_0,\alpha_0})^{-1/(i_0+1)} \omega_{i_0+1}^{-n}$ $(0 \leq n \leq i_0)$, where $\omega_q = e^{2\pi i/q}$ and c_G is some positive constant.
Moreover, $(G_k)_B(s) \in \mathrm{Exp}_s(\kappa; S(d,\varepsilon))$ where d satisfies

$$d \not\equiv -(\arg a_{i_0,\alpha_0} + 2\pi n)/(i_0+1) \pmod{2\pi}.$$