# On irregular modified $A$-hypergeometric systems and Borel summation method

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Formal and Analytic Solutions of Differential, Difference and Discrete Equations
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- This is a joint work with Francisco-Jesús Castro-Jiménez, María-Cruz Fernández-Fernández and Nobuki Takayama.
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## $A$-hypergeometric systems (GKZ systems)

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- Then the $A$-hypegeometric systm is

$$H_A(\beta) : \begin{cases} \left( \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i \right) \cdot \phi = 0 \quad (i = 1, 2, \dots, d), \\ \left( \prod_{i=1}^{n} \partial_i^{u_i} - \prod_{i=1}^{n} \partial_i^{v_i} \right) \cdot \phi = 0 \end{cases}$$

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- We denote by $M_A(\beta)$ the corresponding left $D_n$-module. It is known that $M_A(\beta)$ is holonomic.

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- It is known that $M_A(\beta)$ is regular holonomic (Hotta, 1998) if

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- Study of Gevrey series solutions along a coordinate space:
  - Castro-Fernández (2011) when $d = 1$.
  - Fernández (2010).

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We consider the irregular $A$-hypergeometric system and stuy the asymptotic properties of

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- We study the Geverey series solutions of the modified $A$-hypergeometric systems.
- Under some condition, Geverey series solutions of the modified $A$-hypergeometric systems is Borel summable (1-summable).

# An example: $A = (1, 2)$, $\beta$: generic

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3. Its Borel transform is
$$f_B(x_1, x_2, \tau) = \sum_{m=0}^{\infty} \frac{[\beta]_{2m}}{m!} x_1^{\beta-2m} x_2^m \frac{\tau^m}{m!}$$
$$= x_1^{\beta}\cdot {}_2F_1(-\frac{\beta}{2}, \frac{1-\beta}{2}, 1;\ \frac{4x_2}{x_1^2}\tau)$$

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- $\Phi(x_1, x_2) = F(x_1, x_2, 1)$ is a solution of $H_A(\beta)$ and

$$\Phi(x_1, t \cdot x_2) \sim \phi(x_1, t \cdot x_2) \quad \text{as} \quad S(0, \pi + \varepsilon) \ni t \to 0.$$

(Note: $F(x_1, x_2, t) = \Phi(x_1, t \cdot x_2)$.)

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$$H_A(\beta): \quad (\theta_1+2\theta_2-\beta)\cdot\phi=0, \quad (\partial_1^2-\partial_2)\cdot\phi=0,$$

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$$\begin{aligned} f_B(x_1,x_2,\tau) &= \sum_{m=0}^{\infty}\frac{[\beta]_{2m}}{m!}x_1^{\beta-2m}x_2^m\cdot\frac{\tau^m}{m!} \\ &= x_1^{\beta}\cdot {}_2F_1\Big(-\frac{\beta}{2},\frac{1-\beta}{2},1;\ \frac{4x_2}{x_1^2}\tau\Big)\end{aligned}$$

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# Modified $A$-hypergeometric sytems (Takayama, 2009)

$f(x_1, \dots, x_n, t) = \phi(t^{w_1} x_1, \dots, t^{w_n} x_n)$ $(w = (w_1, \dots, w_n) \in \mathbb{Z}^n)$ solves

$$H_{A,w}(\beta) : \begin{cases} \left(\sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i\right) \cdot f = 0 \quad (1 \leq i \leq d), \\ \left(\sum_{j=1}^{n} w_j x_j \partial_j - t\partial_t\right) \cdot f = 0, \\ \left(t^{u_{n+1}} \prod_{i=1}^{n} \partial_i^{u_i} - t^{v_{n+1}} \prod_{i=1}^{n} \partial_i^{v_i}\right) \cdot f = 0 \\ \qquad \text{for } \forall u, v \in \mathbb{N}_0^{n+1} \text{ such that } \widetilde{A}u = \widetilde{A}v. \end{cases}$$

Here

$$\widetilde{A} = \begin{pmatrix} A & 0 \\ w & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n & 0 \\ w_1 & w_2 & \cdots & w_n & 1 \end{pmatrix}.$$

# Formal solutions of $M_{A,w}(\beta)$

- $T = \{t = 0\} \subset \mathbb{C}^n_x \times \mathbb{C}_t$
- $\mathcal{O}_{\widehat{X|T}}$: the sheaf of formal series along $T$.

**Theorem**

*Assume $\beta \in \mathbb{C}^d$ is very generic and $w \in \mathbb{Z}^n$. Then*

$$\dim_{\mathbb{C}} \left[ \mathrm{Sol}(M_{A,w}(\beta), \sum_{b(\gamma)=0} t^{\gamma} \mathcal{O}_{\widehat{X|T}})_{(p,0)} \right] = \deg(\mathrm{In}_w(I_A)).$$

Here

- $b(\gamma)$ is the indicial polynomial of $H_{A,w}(\beta)$ along $T$.
- $I_A = \langle \partial^u - \partial^v ; \ u, v \in \mathbb{N}_0^n \text{ with } Au = Av \rangle \subset \mathbb{C}[\partial]$ is the toric ideal and $\mathrm{in}_w(I_A)$ is the initial ideal of $I_A$ with respect to $w$.

# Slopes of $M_{A,w}(\beta)$

Slopes for $D$-module (Laurent (1987)).

- We consider a filtration $L_r = F + rV$ $(r \geq 0)$ of $D_{n+1}$-module, where $F = (0, \cdots, 0; 1, \cdots, 1)$, $V = (0, \cdots, 0, -1; 0, \cdots, 0, 1)$.
- If $\mathrm{Ch}^{L_r}(M_{A,w}(\beta))$ "changes" at $r = r_0$, we call $s = r_0 + 1$ a slope.

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By Fourier transformation w.r.t. $t$ $(t \mapsto -\partial_t, \quad \partial_t \mapsto t)$,

- $M_{A,w}(\beta) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} M_{\widetilde{A}}(\widetilde{\beta})$ with $\widetilde{\beta} = (\beta, -1)$
- $Ch^{L_r}(M_{A,w}(\beta)) = \mathcal{F}^{-1}\left(Ch^{\widetilde{L}_r}(M_{\widetilde{A}}(\widetilde{\beta}))\right)$, where $\widetilde{L}_r = \mathcal{F}L_r$

# Definition of $(\widetilde{A}, \widetilde{L}_r)$-umbrella

We assume

$$\widetilde{A} = (\widetilde{a}_1, \cdots, \widetilde{a}_{n+1})$$

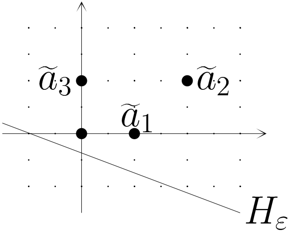
is pointed, i.e., there exists $h \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^n, \mathbb{Q})$ such that $h(\widetilde{a}_i) > 0$. Following Schulze-Walther, we define

**Definition**

(i) The $(\widetilde{A}, \widetilde{L}_r)$-polyhedron is $\Delta_{\widetilde{A}}^{\widetilde{L}_r} = \mathrm{conv}_{H_\varepsilon}\left(\left\{\mathbf{0}, \widetilde{a}_1, \ldots, \widetilde{a}_n, \frac{\widetilde{a}_{n+1}}{-r}\right\}\right)$ in $\mathbb{P}_{\mathbb{R}}^{d+1}$, where $\varepsilon$ is sufficiently small and $H_\varepsilon = h^{-1}(-\varepsilon)$.

(ii) The $(\widetilde{A}, \widetilde{L}_r)$-umbrella $\Phi^{\widetilde{L}_r}$ is the set of faces of $\Delta_{\widetilde{A}}^{\widetilde{L}_r}$ which do not contain the origin.

Let $A = (1, 2)$ and $w = (0, 1)$. Then $\widetilde{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = (\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3)$.



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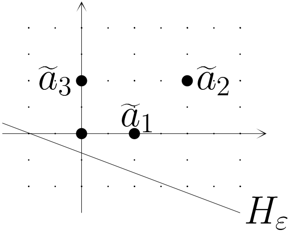


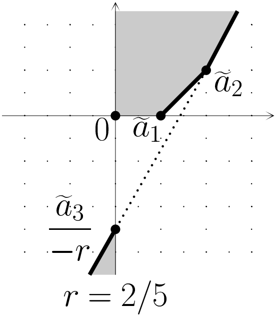

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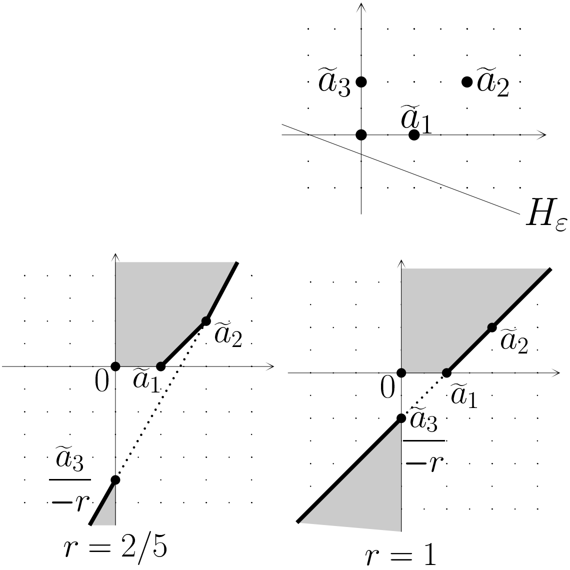

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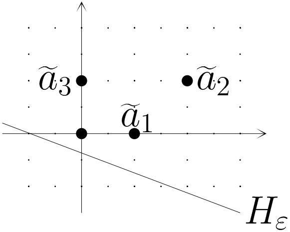


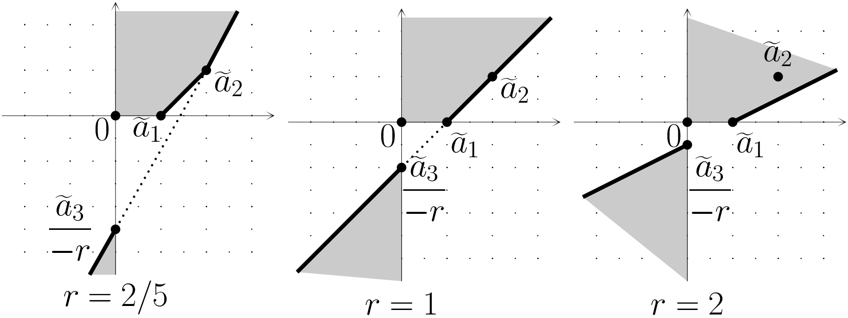

For the modified $A$-hypergeometric systems with very generic $\beta$, we can construct a Gevrey series solution

$$f(x,t) = \sum_{m=0}^{\infty} f_m(x) t^{\gamma+m}$$

along $T = \{t = 0\}$, where

- $f_m(x)$ is a holomoprphic solution near $p$ with $(p, 0) \in T$,
- $\gamma$ is a root of the indicial polynomial $b(\gamma)$ of $H_{A,w}(\beta)$ along T,
- the Gevrey order of it is $s = r + 1$ with

$$r = \max\left\{ -\frac{|b_i|}{w \cdot b_i};\ i \notin \sigma,\ w \cdot b_i > 0 \right\}.$$

  Here $\sigma$ is choosen so that $(a_i)_{i\in\sigma}$ is a basis of $\mathbb{R}^d$, and $\{b_i\}_{i\notin\sigma}$ is a basis of the kernel of $A$ such that

$$(b_i)_j = 0 \quad \text{for all } j \notin \sigma \cup \{i\}, \quad (b_i)_i = 1.$$

# Borel sum of a formal solution

- In the following we assume that the row span of the matrix

$$A_w = \begin{pmatrix} a_1 & \cdots & a_n \\ w_1 & \cdots & w_n \end{pmatrix}$$

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- Or, the weight vector $w$ is in the intersection of the set of the secondary cones of $\overline{A}$.

In this case $r = -\dfrac{|b_i|}{w \cdot b_i}$ holds for any $i \in \sigma$, and $s = 1 + r$ is a slope of $M_{A,w}(\beta)$.

Let $f(x,t) = \sum_{m=0}^{\infty} f_m(x) t^{m+\gamma}$ be a formal solutions of the modified hypergeometric system $H_{A,w}(\beta)$ whose Gevrey index along $T$ is $s = r + 1$.

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**Theorem**

*The formal solution $f(x,t)$ is $1/r$-summable (w.r.t $t$) in all direction but finitely many directions. Furthermore its $1/r$-sum is a solution of $H_{A,w}(\beta)$.*

An important properties to prove Theorem is the following. We first set

$$\varphi(x, z) := \psi(x, t)\Big|_{t=z^r} = \sum_{m=0}^{\infty} C_m(x) z^{r(m+\gamma)}$$

Then

$$\mathcal{B}_1[\varphi](x, \zeta) = \sum_{m=0}^{\infty} \frac{C_m(x)}{\Gamma(1 + r(m+\gamma))} \zeta^{r(m+\gamma)} = \mathcal{B}_{1/r}[\psi](x, \zeta^r)$$

satisfies $H_{A_B}(\beta_B)$, where

$$A_B = \begin{pmatrix} A & 0 \\ w & -1/r \end{pmatrix}, \quad \beta_B = \begin{pmatrix} \beta \\ 0 \end{pmatrix}.$$

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By using this fact we can study the analytic continuation of the Borel transform of $f(t, x)$ and estimate its growth order.

## Theorem

*Assume $\beta$ is very generic,*

- $\widetilde{A} = \begin{pmatrix} a_1 & \cdots & a_n & 0 \\ w_1 & \cdots & w_n & 1 \end{pmatrix}$ *is pointed.*
- *the row span of* $A_w = \begin{pmatrix} a_1 & \cdots & a_n \\ w_1 & \cdots & w_n \end{pmatrix}$ *contains* $(1, \dots, 1)$.

*Then*

- $\exists$ *a formal solution* $f(x,t) = \sum_{m=0}^{\infty} f_m(x) t^{m+\gamma}$ *of* $H_{A,w}(\beta)$.
- $f(x,t)$ *is* $1/r$*-summable with* $r = -|b_i|/w \cdot b_i$.
- *If* $f(x,t)$ *is* $1/r$*-summable in the direction* $0$, $\Phi(x) = \mathcal{S}[f](x,t)\big|_{t=1}$ *gives a solution of* $H_A(\beta)$.
- $\Phi(t^{w_1}x_1, \cdots, t^{w_n}x_n) \sim f(x,t)$ *as* $t \to 0$.

# Conclustion and discussion

1. Asymptotic behavior of $\phi(t^w \cdot x)$ as $t \to 0$ is discussed, where $\phi(x)$ is a solution of $A$-hypergeometric systems.
2. To study this problem, we introduced the modified $A$-hypergeometic systems which $f(t, x) = \phi(t^w \cdot x)$ is satisfied.
3. To determine the Gevrey oorder of $f(t, x)$, we apply Schulze-Walther theory to the Fourier transform of the modified $A$-hypergeometic system.
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**Thank you!**