The action of the Kontorovich-Lebedev integral transform on d-orthogonal polynomial sequences

Ana F. Loureiro
University of Kent, UK

(joint work with S. Yakubovich)

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Outline

d-Orthogonal polynomials

Index Integral Transforms

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Kontorovich-Lebedev transform

Ana F. Loureiro and S. Yakubovich, The Kontorovich-Lebedev transform as a map between *d*-orthogonal polynomials, to appear in Stud. Appl. Math.

A Monic Orthogonal Polynomial Sequence (MOPS) $\{P_n\}_{n\geqslant 0}$ is defined by

$$\langle u_0, P_nP_k\rangle = N_n\delta_{n,k}\ ,\ \text{with}\quad N_n \neq 0.$$

where u_0 is the first element of the corresponding dual sequence (canonical form).

- In this case u_0 is said to be **regular**.

Equivalently, u_0 is **regular** iff $\Delta_n := \det\left[(u_0)_{i+j}\right]_{0\leqslant i,j\leqslant n} \neq 0,\ n \geqslant 0$, where $(u_0)_n :=< u_0, x^n >$.

- It always satisfies the second order recurrence relation

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x)$$

with $P_0 = 1$ and $P_{-1} = 0$ and

$$\beta_n = \frac{\langle u_0, xP_n^2\rangle}{\langle u_0, P_n^2\rangle} \qquad \text{and} \qquad \gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2\rangle}{\langle u_0, P_n^2\rangle} \neq 0,\ n \in \mathbb{N}$$

Semiclassical polynomials

Definition. A MOPS $\{P_n\}_{n\geqslant 0} \perp u$ is called **semiclassical** when $\exists \Phi, \Psi \in \mathcal{P}$, with Φ monic and $\deg \Psi \geqslant 1$, such that (Maroni,1988)

$$(\Phi u)' + \Psi u = 0. \tag{1}$$

The pair (Φ, Ψ) is not unique.

- Simplification criteria : $\exists c$ such that $\Phi(c) = 0$ and

$$\left|\Phi'(c) + \Psi(c)\right| + \left| < u, \theta_c^2(\Phi) + \theta_c(\Psi) > \right| = 0 \ , \tag{2}$$

where $\theta_c(f)(x) = \frac{f(x)-f(c)}{x-c}$, for any $f \in \mathcal{P}$, and u would then fulfill

$$(\theta_c(\Phi)u)' + \left(\theta_c^2(\Phi) + \theta_c(\Psi)\right) u = 0.$$

- The class of u is $s = \min_{(\Phi,\Psi)} \left[\max\left(\deg(\Phi) - 2, \deg(\Psi) - 1\right)\right]$

- Moreover, $\Phi(x)P'_{n+1}(x) = \sum_{\nu=n-s}^{n+\deg\Phi} \theta_{n,\nu} P_\nu(x)$ with $\theta_{n,n-s}\theta_{n,n+t} \neq 0, \quad n \geqslant s.$

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The d-orthogonality

Definition. A MPS $\{P_n\}_{n\geqslant 0}$ is d-orthogonal with respect to the vector functional $\mathbf{U} = (u_0, \ldots, u_{d-1})^T$, iff ... (Maroni,1989)(van Iseghem,1987)

$$\begin{cases} \langle u_k, x^m P_n\rangle = 0 & , \quad n \geqslant md + k + 1 \ , \ m \geqslant 0, \\ \langle u_k, x^m P_{md+k}\rangle \neq 0 & , \quad m \geqslant 0. \end{cases} \tag{3}$$

In this case, the d-MOPS $\{P_n\}_{n\geqslant 0}$ necessarily satisfies the $(d+1)$-order recurrence relation

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \sum_{\nu=0}^{d-1} \gamma_{n-\nu}^{d-1-\nu} P_{n-1-\nu}(x) \quad , \quad n \geqslant d+1, \tag{4}$$

where $\gamma_{n+1}^0 \neq 0$ for all $n \geqslant 0$, and $\gamma_k^{-m} = 0$, for $m \geqslant 0$.

Index integral transforms

In 1964, Wimp formally introduced the general index transform over parameters of the Meijer G-function

$$F(\tau) = \int_0^\infty G^{m,n+2}_{p+2,q}\left(x;\, \begin{matrix} 1-\mu+i\tau, 1-\mu-i\tau, (a_p) \\ (b_q) \end{matrix}\right) f(x)dx \ ,$$

whose inversion formula

$$f(x) = \frac{1}{\pi^2}\int_0^\infty \tau \sinh(2\pi\tau) F(\tau)\, G^{q-m,p-n+2}_{p+2,q}\left(x;\, \begin{matrix} \mu+i\tau, \mu-i\tau, -(a_p^{n+1}), -(a_n) \\ -(b_q^{m+1}), -(b_m) \end{matrix}\right) d\tau \ .$$

was established in 1985 by Yakubovich.

Examples.
Kontorovich-Lebedev (KL) · Mehler-Fock · Olevski-Fourier-Jacobi · Whittaker ·

The Kontorovich-Lebedev (KL) transform

For $\alpha > 0$, consider

$$KL_\alpha[f](\tau) = 2\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^{-2}\int_0^\infty x^\alpha K_{i\tau}(2\sqrt{x})f(x)\mathrm{d}x\ ,$$

$$x^{\alpha+1}\ f(x) = \frac{1}{\pi^2}\lim_{\lambda\to\pi-}\int_0^\infty \tau\sinh(\lambda\tau)\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^2 K_{i\tau}(2\sqrt{x})\ KL_\alpha[f](\tau)\mathrm{d}\tau\ ,$$

valid for any continuous function $f \in L_1\left(\mathbb{R}_+, K_0(2\mu\sqrt{x})dx\right)$, $0 < \mu < 1$, in a neighborhood of each $x \in \mathbb{R}_+$ where $f(x)$ has bounded variation.

Here, $K_{i\tau}(2\sqrt{x}) = \int_0^\infty \exp(-2\sqrt{x}\cosh(u))\cos(\tau u)\mathrm{d}u$, $x \in \mathbb{R}_+$, $\tau \in \mathbb{R}_+$.

Thus,

$$KL_\alpha : \quad x^n \quad \longmapsto \quad (\alpha+1-\tfrac{i\tau}{2})_n(\alpha+1+\tfrac{i\tau}{2})_n = \prod_{\sigma=1}^{n}\left((\alpha+1+\sigma)^2+\tfrac{\tau^2}{4}\right)$$

where $(a)_n := a(a+1)\ldots(a+n-1)$

Parseval identity for KL_α

Theorem

The operator KL_α is an isomorphism between Hilbert spaces

$$\begin{array}{rcccl} KL_\alpha & : & L_2(\mathbb{R}_+; x^{2\alpha+1}dx) & \to & L_2\left(\mathbb{R}_+; \tau\sinh(\pi\tau)\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^4 \frac{d\tau}{4\pi^2}\right) \\ & & f & \mapsto & 2\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^{-2}\int_0^\infty x^\alpha K_{i\tau}(2\sqrt{x})f(x)dx \end{array}$$

The following generalized Parseval equality holds

$$\int_0^\infty x^{2\alpha+1}f(x)g(x)dx$$
$$= \frac{1}{4\pi^2}\int_0^\infty \tau\sinh(\pi\tau)\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^4 KL_\alpha[f](\tau)KL_\alpha[g](\tau)d\tau\ ,$$

where $f, g \in L_2(\mathbb{R}_+; x^{2\alpha+1}dx)$.

Proposition. For any polynomial f and $|\mathrm{Im}\mu| < 2\beta$, it is valid the identity

$$\int_0^\infty x^{\alpha+\beta} f(x) K_{i\mu}(2\sqrt{x}) dx$$
$$= \frac{1}{8\pi\Gamma(2\beta)} \int_0^\infty KL_\alpha[f](\tau) \frac{\left|\Gamma\left(\beta + \frac{i(\tau+\mu)}{2}\right)\Gamma\left(\beta + \frac{i(\tau-\mu)}{2}\right)\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\right|^2}{|\Gamma(i\tau)|^2} d\tau \ .$$

Proposition. For any polynomial f and $\beta > 0$, we have

$$\int_0^\infty x^{\alpha+\beta} \mathrm{e}^{-x} f(x) dx$$
$$= \frac{\sqrt{\mathrm{e}}}{4\pi} \int_0^\infty KL_\alpha[f](\tau) \frac{\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\Gamma\left(\beta+\frac{i\tau}{2}\right)\right|^2}{|\Gamma(i\tau)|^2} W_{-\beta+\frac{1}{2},\frac{i\tau}{2}}(1) d\tau \ , \quad (5)$$

holds, where $W_{\gamma,\mu}(x)$ represents the Whittaker function.

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holds, where $W_{\gamma,\mu}(x)$ represents the Whittaker function.

More properties of KL_α

- For any $m, n \in \mathbb{N}_0$ and any $f \in \mathcal{P}$, it is valid

$$KL_\alpha \left[\left(\frac{1}{x}\mathcal{A}x + 2\alpha \frac{d}{dx}x \right)^m x^n f(x) \right](\tau) = (-1)^m \left(\frac{\tau^2}{4} + \alpha^2 \right)^m \left| \left(\alpha + 1 + \frac{i\tau}{2} \right)_n \right|^2 KL_{\alpha+n}[f](\tau).$$

where $\mathcal{A} = x\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x} - x$.

- Let $\{S_n(\cdot;\alpha) := KL_\alpha[P_n](\cdot)\}_{n\geqslant 0}$. If $\{P_n\}_{n\geqslant 0}$ is given by

$$P_n(x) = (-1)^n \frac{\left(\prod_{\nu=1}^{q}(b_\nu)_n\right)}{\left(\prod_{\nu=1}^{p}(a_\nu)_n\right)} {}_{p+1}F_q \left(\begin{array}{c} -n, a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| x \right), \quad n \geqslant 0, \qquad (6)$$

where the coefficients a_j, b_k with $j = 1, \ldots, p$ and $k = 1, \ldots, q$, do not depend on x but possibly depending on n, then

$$S_n\left(\frac{\tau^2}{4}\right) = \frac{(-1)^n \left(\prod_{\nu=1}^{q}(b_\nu)_n\right)}{\left(\prod_{\nu=1}^{p}(a_\nu)_n\right)} {}_{p+3}F_q \left(\begin{array}{c} -n, a_1, \ldots, a_p, \alpha+1-\frac{i\tau}{2}, \alpha+1+\frac{i\tau}{2} \\ b_1, \ldots, b_q \end{array} \middle| 1 \right)$$

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Examples.

1. The MPS $\{P_n\}_{n\geqslant 0}$ is said to be an Appell sequence when

$$P'_{n+1}(x) = (n+1)P_n(x)\ ,\ \ n \geqslant 0.$$

If $\{P_n\}_{n\geqslant 0}$ is also d-orthogonal,

then $\{S_n := KL_\alpha[P_n]\}_{n\geqslant 0}$ is $(2d+2)$-orthogonal and

$$\delta_i S_{n+1}\left(\tfrac{\tau^2}{4};\alpha\right) = (n+1)S_n\left(\tfrac{\tau^2}{4};\alpha+\tfrac{1}{2}\right)\ ,\ \ n \in \mathbb{N}_0,$$

where $\delta_\omega f(x) := \dfrac{f(x+\omega)-f(x-\omega)}{2\omega}$.

Examples.

2. Let $\{R_n\}_{n\geqslant 0}$ be a reversed Appell sequence, i.e.,

$$R_n(x) = \frac{1}{\lambda_n} x^n P_n\left(\frac{1}{x}\right) \ , \ n \in \mathbb{N}_0,$$

where $\{P_n\}_{n\geqslant 0}$ is an Appell sequence and $\lambda_n = P_n(0) \neq 0$, $n \geqslant 0$.

According to Ben Cheikh & Douak (2001), if $\{R_n\}_{n\geqslant 0}$ is d-orthogonal then

$$R_n(x) := R_n(x;\bar{\alpha}_d) = (-1)^n \left(\prod_{\sigma=1}^{d} (\alpha_\sigma + 1)_n\right) {}_1F_d\left(\begin{matrix} -n \\ \alpha_1+1,\ldots,\alpha_d+1 \end{matrix}; x\right) \ , \ n \geqslant 0,$$

The corresponding KL_α-transformed sequence $\{S_n\}_{n\geqslant 0}$ with $S_n(\cdot;\alpha,\bar{\alpha}_d) = KL_\alpha[R_n(x;\bar{\alpha}_d)](\cdot)$ is $\widetilde{d}$-orthogonal where

$$\widetilde{d} = \begin{cases} 2 & , \quad d = 1 \\ 1 & , \quad d = 2 \\ d & , \quad d = 3,4,5,\ldots \end{cases} ,$$

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$$\widetilde{d} = \begin{cases} 2 & , \quad d=1 \\ 1 & , \quad d=2 \longrightarrow \text{Continuous Dual Hahn} \\ d & , \quad d=3,4,5,\ldots \end{cases} ,$$

For instance, ...

- $(d=1)$: if $u_0 := u_0(\alpha_1)$ is the regular form associated to the Laguerre polynomials $\{\widehat{L}_n(\cdot;\alpha_1)\}_{n\geqslant 0}$, then

$$< u_0(\alpha_1), f >= \int_0^\infty f(x) \frac{e^{-x}x^{\alpha_1}}{\Gamma(\alpha_1+1)} dx \ , \ f \in \mathcal{P}.$$

Necessarily, the canonical form $s_0(\alpha_1,\alpha)$ corresponding to the KL_α-transform of $\{\widehat{L}_n(\cdot;\alpha_1)\}_{n\geqslant 0}$ admits the representation

$$\langle s_0(\alpha_1,\alpha), g_\alpha\rangle = \frac{\sqrt{e}}{4\pi} \int_0^\infty g_\alpha(\tau) \frac{\left|\Gamma\left(\alpha+1+\frac{i\tau}{2}\right)\Gamma\left(\alpha_1-\alpha+\frac{i\tau}{2}\right)\right|^2}{\Gamma(\alpha_1+1)\ \left|\Gamma\left(i\tau\right)\right|^2} W_{\alpha-\alpha_1+\frac{1}{2},\frac{i\tau}{2}}(1)\, d\tau$$

as long as $\alpha_1 > \alpha$, where $g_\alpha(\tau) = KL_\alpha[f](\tau)$.

For instance, ...

- $(d=2)$: if $u_0 := u_0(\alpha_1,\alpha_2)$ is the regular form associated to the 2-orthogonal Laguerre type polynomials $\{\widehat{B}_n(\cdot;\alpha_1,\alpha_2)\}_{n\geqslant 0}$, then

$$< u_0(\alpha_1,\alpha_2), f >= \int_0^{\infty} f(x)\, \frac{x^{\frac{\alpha_1+\alpha_2}{2}} K_{\alpha_1-\alpha_2}(2\sqrt{x})}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\, dx\ ,\ f\in\mathcal{P}.$$

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as long as $\alpha_2>\alpha$, where $g_\alpha(\tau)=KL_\alpha[f](\tau)$.

¿ Can we determine all the orthogonal polynomial sequences

that are mapped by the KL_α-transform

to d-orthogonal sequences ?

MOPSs whose KL_α-transformed sequence is a d-MOPS

Theorem. Let $\{B_n\}_{n\geqslant 0}$ be a MOPS with respect to u_0.

If $\{S_n\}_{n\geqslant 0}$ is a d-MOPS, then d is even $\geqslant 2$

and $\{B_n\}_{n\geqslant 0}$ is semiclassical of class $s \in \left\{\max(0, \frac{d}{2}-2)\ ,\ \frac{d}{2}-1\ ,\ \frac{d}{2}\right\}$ insofar as there exist two polynomials ϕ, ψ such that u_0 fulfills

$$D(\phi u_0) + \psi u_0 = 0\ .$$

Precisely, there is a monic polynomial ρ, with $\deg \rho(x) = \frac{d}{2}$, and $N \neq 0$, such that (ϕ, ψ) is given by

a) $\phi(x) = x^2$ and $\psi(x) = x(N\rho(x) - (3+2\alpha))$ with $\rho(0) = 0$, $\langle u_0, \rho(x)\rangle \neq N^{-1}(2+2\alpha)$ and $\alpha \neq -\frac{n+3}{2}$, $n \in \mathbb{N}_0$ (u_0 is of class $s = \frac{d}{2}$);

b) $\phi(x) = x$ and $\psi(x) = N\rho(x) - (2+2\alpha)$ with $\langle u_0, \rho(x)\rangle \neq N^{-1}(2+2\alpha)$ and $\alpha \neq -\frac{n}{2} - 1$, for $n \geqslant 1$ (u_0 is of class $s = \frac{d}{2} - 1$);

c) $\phi(x) = 1$ and $\psi(x) = N\theta_0\rho(x)$ with $N\rho(0) = 1 + 2\alpha$ (u_0 is of class $s = \frac{d}{2} - 2$, as long as $d \geqslant 4$).

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Precisely, there is a monic polynomial ρ, with $\deg \rho(x) = \frac{d}{2}$, and $N \neq 0$, such that (ϕ, ψ) is given by

a) $\phi(x) = x^2$ and $\psi(x) = x(N\rho(x) - (3+2\alpha))$ with $\rho(0) = 0$, $\langle u_0, \rho(x)\rangle \neq N^{-1}(2+2\alpha)$ and $\alpha \neq -\frac{n+3}{2}$, $n \in \mathbb{N}_0$ (u_0 is of class $s = \frac{d}{2}$);

b) $\phi(x) = x$ and $\psi(x) = N\rho(x) - (2+2\alpha)$ with $\langle u_0, \rho(x)\rangle \neq N^{-1}(2+2\alpha)$ and $\alpha \neq -\frac{n}{2} - 1$, for $n \geqslant 1$ (u_0 is of class $s = \frac{d}{2} - 1$);

c) $\phi(x) = 1$ and $\psi(x) = N\theta_0\rho(x)$ with $N\rho(0) = 1 + 2\alpha$ (u_0 is of class $s = \frac{d}{2} - 2$, as long as $d \geqslant 4$).

MOPSs whose KL_α-transformed sequence is a d-MOPS

Moreover, the MOPS $\{B_n\}_{n\geqslant 0}$ fulfills

$$x^2 B_n''(x) + x\Big(N\rho(x) - (3+2\alpha)\Big)B_n'(x)$$
$$-\Big\{N\rho(x)\Big(N\rho(x) - (2+2\alpha)\Big) - Nx\rho'(x) - x + (1+2\alpha)\Big\}B_n = -\sum_{\nu=n-1}^{n+d} \rho_{n,\nu}^d B_\nu(x)$$

where

$$\rho_{n,\mu}^d = \begin{cases} \dfrac{\langle u_0, B_n^2\rangle}{\langle u_0, B_\mu^2\rangle}\alpha_{n+1}^{n+d-\mu} & \text{if} \quad n+1 \leqslant \mu \leqslant n+d, \text{ with } \; n \geqslant 0, \\ \zeta_n - \alpha^2 & \text{if} \quad \mu = n, \text{ with } \; n \geqslant 0, \\ \gamma_1 & \text{if} \quad \mu = n-1 \text{ with } \; n \geqslant 1 \; . \end{cases}$$

MOPSs whose KL_α-transformed sequence is a 2-MOPS

Two situations arise :

Case a. The form u_0 is a semiclassical form of class $s = 1$, insofar as it fulfills

$$D(x^2 u_0) + x(Nx - (3 + 2\alpha))u_0 = 0$$

and, therefore the corresponding MOPS $\{B_n\}_{n\geqslant 0}$ can be expressed as

$$\widetilde{B}_{n+1}(N^{-1}x) = \widehat{L}_{n+1}(N^{-1}x; 2\alpha + 2) + a_n \widehat{L}_n(N^{-1}x; 2\alpha + 2)$$
$$x\widehat{L}_{n+1}(N^{-1}x; 2\alpha + 2) = \widetilde{B}_{n+1}(N^{-1}x) - (a_n - (2n - 2\alpha - 3))\widetilde{B}_n(N^{-1}x)$$

where $\{\widehat{L}_n\}_{n\geqslant 0}$ represents the (monic) Laguerre polynomials and

$$a_n = \frac{\lambda(n+1)! + (2\alpha - \lambda + 2)(2\alpha + 3)_{n+1}}{\lambda\, n! + (2\alpha - \lambda + 2)(2\alpha + 3)_n}\ ,\ n \geqslant 0.$$

Case b. The MOPS $\{B_n\}_{n\geqslant 0}$ is, up to a linear change of variable, a Laguerre sequence of parameter α_1.

thank you !