Analytic solutions of heat type equations

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1. Introduction

It is well known that harmonic functions posses the mean-value property:

$$\begin{aligned} u(\mathring{x}) &= \frac{1}{\sigma(n)R^n}\int_{B(\mathring{x},R)} u(x)dx \\ &= \frac{1}{n\sigma(n)R^{n-1}}\int_{S(\mathring{x},R)} u(x)dS(x), \end{aligned}$$

where $\sigma(n) = \pi^{n/2}/\Gamma(n/2+1)$.

On the other hand if a continuous function u satisfies the above equality for every ball (for every sphere) in Ω, then u is twice continuously differentiable and harmonic on Ω.

Clearly, for polyharmonic functions, i.e., solutions of the iterated Laplace operator Δ^m, $m \in \mathbb{N}$, or more generally for real analytic functions, the integral means over balls or spheres need not to be equal the value of a function at the center of a ball or sphere. It appears however that this means can be expressed by some polynomials of the radius of the ball or sphere.

In fact, the mean-value formula for polyharmonic functions for spherical means in dimensions $d = 2, 3$ was established already in 1909 by Pizzetti [18, 19]. The inverse to the mean-value property for polyharmonic functions in dimension $n = 2$ was first proved by Sbrana [20]. The Pizzetti mean-value property for polyharmonic functions and its inverse was extended to the case of spherical and solid means in arbitrary dimension by Nicolesco [17].

Afterwards some other theorems on mean-value properties for polyharmonic and real analytic functions have been obtained by Ghermanesco [11], Friedman [10], Bramble and Payne [6], Bojanov [5], Zalcman [21], Ziemian [22] and others.

In the lecture we first recall differential relations between the spherical and solid means of functions. Next we extend the mean-value formulas to the case of real analytic functions and obtain a characterization of such functions in terms of integral means over balls or spheres. We also obtain similar characterization of functions of Laplacian growth.

As an application we study the problem of analyticity in time of solutions of the initial value problem to the heat equations $\partial_t u = \Delta u$ with real analytic initial data $u(0,\cdot) = u_0$. We prove that the solution u is analytic in time at $t = 0$ iff the integral means of u_0 over balls or spheres of radius R can be extended to entire functions of R of exponential order at most 2.

Next we extend the result to the case when Δ is replaced by its affine perturbations and for solutions of heat type equations on a real analytic manifold. Finally we state results on Borel summability of solutions of the heat type equations.

2. Spherical and solid means.

Let $u \in C^0(\Omega)$, $\mathring{x} \in \Omega$, $0 < R < \operatorname{dist}(\mathring{x}, \partial\Omega)$.

$$M(u, \mathring{x}; R) = \frac{1}{\sigma(n) R^n} \int\limits_{B(\mathring{x}, R)} u(x) dx,$$

$$N(u, \mathring{x}; R) = \frac{1}{n\sigma(n) R^{n-1}} \int\limits_{S(\mathring{x}, R)} u(x) dS(x).$$

Lemma 1 *Let $u \in C^0(\Omega)$. Then for any $\mathring{x} \in \Omega$ and $0 < R < \operatorname{dist}(\mathring{x}, \partial\Omega)$,*

$$\Big(\frac{R}{n}\frac{\partial}{\partial R} + 1\Big) M(u, \mathring{x}; R) = N(u, \mathring{x}; R).$$

If $u \in C^2(\Omega)$, then

$$\frac{n}{R}\frac{\partial}{\partial R} N(u, \mathring{x}; R) = M(\Delta u, \mathring{x}; R).$$

3. Mean-value properties for real-analytic functions.

Theorem 1 (Mean-value property). *Let* $u \in \mathcal{A}(\Omega)$, $\mathring{x} \in \Omega$. *Then* $M(u, \mathring{x}; R)$ *and* $N(u, \mathring{x}; R)$ *are analytic functions at the origin and for small* R,

$$M(u, \mathring{x}; R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(\mathring{x})}{4^k \left(\frac{n}{2} + 1\right)_k k!} R^{2k}, \tag{1}$$

$$N(u, \mathring{x}; R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(\mathring{x})}{4^k \left(\frac{n}{2}\right)_k k!} R^{2k}. \tag{2}$$

Here $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhamer symbol.

If $\mathring{x} = 0$ the proof is done by expending u into Taylor series

$$u(x) = \sum_{\ell \in \mathbb{N}_0^n} \frac{1}{\ell_1! \cdots \ell_n!} \frac{\partial^{|\ell|}}{\partial x^\ell} u(\mathring{x}) x^\ell,$$

and then computing the integral of $x^\ell = x_1^{\ell_1} \cdots x_n^{\ell_n}$ over $B(R)$. Finally, applying Lemma 1 we get (2). $\square$

Theorem 2 (Converse to the mean value property). *Let* $\rho \in C^0(\Omega, \mathbb{R}^+)$, $u \in C^\infty(\Omega)$. *If*

$$\widetilde{N}(x; R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}\right)_k k!} R^{2k}$$

is convergent locally uniformly in $\{(x, R) : x \in \Omega, |R| < \rho(x)\}$, *then* $u \in \mathcal{A}(\Omega)$ *and* $N(u, x; R) = \widetilde{N}(x; R)$ *for* $x \in \Omega$, $R < \min\big(\rho(x), \mathrm{dist}(x, \partial\Omega)\big)$.

Proof. We first derive that for any compact set $K \Subset \Omega$ one can find $C < \infty$ such that for $k \in \mathbb{N}_0$,

$$\sup_{x \in K} |\Delta^k u(x)| \leq C^{2k+1}(2k)!.$$

But by [1, Thm 2.2 in Chapter II] this inequality implies that $u \in \mathcal{A}(\Omega)$. Finally, by Theorem 1 we get $\widetilde{N}(x; R) = N(u, x; R)$. $\square$

4. Functions of Laplacian growth.

In order to control the growth of iterated Laplacians of smooth functions Aronszajn et al. [1] introduced the notion of the Laplacian growth.

Definition. Let $\varrho > 0$ and $\tau \geq 0$. A function u smooth on $\Omega \subset \mathbb{R}^n$ is of *Laplacian growth* (ϱ, τ) if for every $K \Subset \Omega$ and $\varepsilon > 0$ one can find $C = C(K, \varepsilon) < \infty$ such that for $k \in \mathbb{N}_0$,

$$\sup_{x \in K} |\Delta^k u(x)| \leq C(\tau + \varepsilon)^{2k} (2k)!^{1-1/\varrho}. \tag{3}$$

Definition. ([4]) Let $\varrho > 0$ and $\tau \geq 0$. An entire function F is said to be of *exponential growth* (ϱ, τ) if for every $\varepsilon > 0$ one can find C_ε such that for any $R < \infty$

$$\sup_{|z| \leq R} |F(z)| \leq C_\varepsilon \exp\{(\tau + \varepsilon) R^\varrho\}.$$

The exponential growth of an entire function can be also expressed in terms of estimations of its Taylor coefficients.

It appears that a function u of Laplacian growth (ϱ, τ) on Ω is in fact real-analytic on Ω (see [1, Theorem 2.2 in Chapter II]). So the spherical and solid means $N(u, x; R)$ and $M(u, x; R)$ are well defined for $x \in \Omega$ and R small enough. However due to estimation (3) both functions $N(u, x; R)$ and $M(u, x; R)$ can be extended to entire functions of exponential growth.

Theorem 3 *Let* $\varrho > 0$ *and* $\tau \geq 0$. *If* u *is of Laplacian growth* (ϱ, τ), *then* $N(u, x; R)$ *and* $M(u, x; R)$ *extend holomorphically to entire functions of exponential growth* $(\varrho, \tau^{\varrho}/\varrho)$ *locally uniformly in* Ω.

Conversely we have

Theorem 4 *Let $u \in \mathcal{A}(\Omega)$. If $M(u,x;R)$ defined for $x \in \Omega$ and $0 \leq R < \operatorname{dist}(x, \partial\Omega)$ extends to an entire function $\widetilde{M}(u,x;z)$ of exponential growth (ϱ, τ) locally uniformly in Ω,
then u is of Laplacian growth $\left(\varrho, (\varrho\tau)^{1/\varrho}\right)$.
Analogous result holds for $N(u,x;R)$.*

5. Convergent solutions of the heat equation.

Let us consider the initial value problem for the heat equation

$$\left\{\begin{array}{ll} \partial_t u - \Delta_x u & = 0, \\ u_{|t=0} & = u_0, \end{array}\right. \tag{4}$$

where $u_0 \in \mathcal{A}(\Omega)$, $\Omega \subset \mathbb{R}^n$.

Then its formal power series solution is given by

$$\widehat{u}(t,x) = \sum_{k=0}^{\infty} \frac{\Delta^k u_0(x)}{k!} t^k. \tag{5}$$

We ask when the solution u is an analytic function of time variable at $t = 0$. In the dimension $n = 1$ the problem was solved by Kowalevskaya [12].

She proved that the solution u is analytic in time if and only if the initial data u_0 can be analytically extended to an entire function of exponential order 2.

In the multidimensional case the solution of the problem was given by Aronszajn at al. [1] in terms of the growth of iterates of the Laplacian of the initial data.

Theorem 5 *Let* $0 < T \leq \infty$. *If formal power series solution (5) of the heat equation (4) is convergent for* $|t| < T$ *locally uniformly in* Ω, *then* $N(u_0, x; R)$ *and* $M(u_0, x; R)$ *extend to an entire function of exponential growth* $(2, 1/(4T))$ *locally uniformly in* Ω.

Conversely, if $N(u_0, x; R)$ *or* $M(u_0, x; R)$ *can be extended to an entire function of exponential growth* $(2, 1/(4T))$ *locally uniformly in* Ω*, then the solution* $\widehat{u}$ *of the heat equation (4) is convergent for* $|t| < T$ *locally uniformly in* Ω*.*

Proof. Assume that $\widehat{u}(t,x)$ is convergent for $|t| < T$ loc. unif. in Ω. Then $\forall K \Subset \Omega, \varepsilon > 0\ \exists C$ s.t.

$$\begin{aligned}\sup_{x\in K} |\Delta^k u_0(x)| &\le C\Big(\frac{1}{T}+\varepsilon\Big)^k \cdot k! \\ &\le C_\varepsilon\Big((2T)^{-1/2}+\varepsilon\Big)^{2k} \cdot (2k)!^{1/2}.\end{aligned}$$

Hence u_0 is of Laplacian growth $(2, 1/\sqrt{2T})$ and by Theorem 3, $N(u_0,x;R)$ and $M(u_0,x;R)$ extend to entire functions of exponential growth $(2, 1/(4T))$.

On the other hand let $N(u_0, x; R)$ or $M(u_0, x; R)$ can be extended to an entire function of exponential growth $(2, 1/(4T))$ loc. unif. in Ω. Then by Theorem 4, u_0 is of Laplacian growth $(2, 1/\sqrt{2T})$ loc. unif. in Ω.

Hence for $|t| < T$ and small $\varepsilon > 0$

$$\begin{aligned}
\sup_{x\in K} \sum_{k=0}^{\infty} \frac{|\Delta^k u_0(x)|}{k!}|t|^k &\le C_\varepsilon \sum_{k=0}^{\infty} \frac{\left(1/\sqrt{2T}+\varepsilon\right)^{2k} \cdot (2k!)^{1/2}|t|^k}{k!} \\
&\le \cdots \\
&\le C_\varepsilon \sum_{k=0}^{\infty} \left[\left(\frac{1}{T}+\varepsilon\right)|t|\right]^k < \infty.
\end{aligned}$$

So $\widehat{u}(t,x)$ is convergent for $|t| < T$ loc. unif. in Ω. $\square$

6. A perturbed heat equation

Set $\Delta^{a,b} = \Delta + \langle a, \nabla \rangle + b$ for $a \in \mathbb{R}^n$, $b \in \mathbb{R}$. Then the unique formal power series solution $\widehat{w}(t,x)$ to

$$(6) \qquad \begin{cases} \partial_t w - \Delta^{a,b} w &= 0, \\ w_{|t=0} &= w_0 \in \mathcal{A}(\Omega), \end{cases}$$

is given by

$$\widehat{w}(t,x) = \sum_{k=0}^{\infty} \frac{(\Delta^{a,b})^k w_0(x)}{k!} t^k. \tag{7}$$

On the other hand $w(t,x)$ satisfies (6) iff
$u(t,x) = \exp\{\frac{1}{2}\langle a, x\rangle - ct\}\, w(t,x)$
with $c = \frac{1}{4}a^2 - \frac{1}{2}\sum_{i=1}^n a_i + b$
is a solution of the heat equation (4).

Set

$$M^a(w_0;\, x, R) = \fint_{B(x,R)} u(\xi) \exp\{\tfrac{1}{2}\langle a, \xi\rangle\}\, d\xi,$$

$$N^a(w_0; x, R) = \fint_{S(x,R)} u(\xi) \exp\{\tfrac{1}{2}\langle a, \xi\rangle\}\, dS(\xi).$$

Since the multiplication by an exponential function has no influence on convergence/divergence properties by Theorem 5 we get

Corollary 1 *Let $0 < T \leq \infty$. The formal power series solution* (7) *of the initial value problem* (6) *is convergent for $|t| < T$ locally uniformly in Ω iff $M^a(w_0; x, R)$ and/or $N^a(w_0; x, R)$ as functions of R extend holomorphically to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in Ω.*

6. The heat type equations on a real analytic manifold

Let $\mathcal{M}$ be a real analytic manifold of dimension n and $X_1, \ldots, X_d$ real analytic linearly independent vector fields on $\mathcal{M}$.
Define a Laplace type operator on $\mathcal{M}$ by

$$\widetilde{\Delta} = X_1^2 + \cdots + X_n^2.$$

Let us consider the initial value problem

$$\left\{\begin{array}{ll} \partial_t v - \widetilde{\Delta} v = 0, & \\ v_{|t=0} = v_0, & v_0 \in \mathcal{A}(\mathcal{M}). \end{array}\right. \tag{8}$$

The formal power series solution of (8) is given by

$$\widehat{v}(t,y) = \sum_{k=0}^{\infty} \frac{\widetilde{\Delta}^k v_0(y)}{k!} t^k. \tag{9}$$

It is well known that if vector fields X_i commute,

$$(C) \qquad [X_i, X_j] = 0 \quad \text{for} \quad i, j = 1, \dots, n,$$

then for a fixed $\mathring{y} \in \mathcal{M}$ one can find a real analytic diffeomorphism $\Phi : \mathbb{R}^n \supset \Omega \xrightarrow{onto} V \subset \mathcal{M}$ s. t. $\mathring{y} \in V = \Phi(\Omega)$ and $\Phi_*^{-1}(X_i) = \frac{\partial}{\partial x_i}$ for $i = 1, \dots, n$.
Set $B_\Phi(y, R) = \Phi\big(B(x, R)\big)$, $S_\Phi(y, R) = \Phi\big(S(x, R)\big)$, with $x = \Phi^{-1}(y)$, $0 < R < \operatorname{dist}(x, \partial\Omega)$.
Define a measure $\mu_\Phi(A) = \int_{\Phi^{-1}(A)} d\xi$ for $A \subset V$.

Theorem 6 *Let $0 < T \leq \infty$. The formal power series solution (9) of the heat type equation (8) is convergent for $|t| < T$ locally uniformly in V if and only if the solid integral mean*

$$M_\Phi(v_0, y; R) = \frac{1}{\mu_\Phi\big(B_\Phi(y, R)\big)} \int\limits_{B_\Phi(y,R)} v(\eta) d\mu_\Phi(\eta)$$

extends to an entire function of exponential growth $(2, 1/(4T))$ locally uniformly in V.

Proof. Assume that the formal power series solution (9) of (8) is convergent for $|t| < T$ locally uniformly in V. Denote its sum as $v(t, y)$ and set $u(t, x) = v\big(t, \Phi(x)\big)$, $u_0(x) = v_0\big(\Phi(x)\big)$ for $|t| < T$, $x \in \Omega$. Then u satisfies the heat equation (4) and is given by (5) with the series convergent for $|t| < T$ locally uniformly in Ω. Hence by Theorem 5, $M(u_0, x; R)$ extends to entire functions of exponential growth $\big(2, 1/(4T)\big)$ loc. uni. in Ω.

But for $x \in \Omega$ and $y = \Phi(x)$ we have

$$\begin{aligned} \sigma(n)R^n = |B(x,R)| &= \int_{\Phi^{-1}(B_\Phi(y,R))} d\xi \\ &= \mu_\Phi\big(B_\Phi(y,R)\big), \\ \int_{B(x,R)} u_0(\xi)d\xi &= \int_{\Phi^{-1}(B_\Phi(y,R))} v_0\big(\Phi(\xi)\big)d\xi \\ &= \int_{B_\Phi(y,R)} v_0(\eta)d\mu_\Phi(\eta). \end{aligned}$$

So

$$M(u_0, x; R) = M_\Phi(v_0, y; R).$$

Hence $M_\Phi(v_0, y; R)$ as a function of R extends to entire functions of exponential growth $\big(2, 1/(4T)\big)$ locally uniformly in V.

The proof of the converse statement is done in the same way. $\square$

Remark. An analogue of Theorem 6 holds for solutions of heat type equations with $\widetilde{\Delta}$ perturbed by $\sum_{i=1}^{n} a_i X_i + b$. In that case the measure μ_Φ should be replaced by

$$\mu_\Phi^a(A) = \int_{\Phi^{-1}(A)} \exp\{\tfrac{1}{2}\langle a, \xi\rangle\}\, d\xi, \qquad A \subset V.$$

Definition. Let $d \in \mathbb{R}$, $U \subset \mathbb{C}^n$ and $\widehat{\varphi}_j \in \mathcal{O}(U)$. A formal series

$$\widehat{\varphi}(t, z) = \sum\nolimits_{j=0}^{\infty} \frac{\varphi_j(z)}{j!} t^j$$

is *Borel summable with respect to t in the direction d* if its Borel transform defined on $D_\varepsilon \times U$ by

$$\left(\widehat{\mathcal{B}}\widehat{\varphi}\right)(s,z)=\sum_{j=0}^{\infty}\frac{\varphi_j(z)}{(j!)^2}s^j$$

extends holomorphically to a domain $\left(D_\epsilon\cup S(d,\epsilon)\right)\times U$ with some $0<\epsilon$ and the extension satisfies for any $U_1 \Subset U$ and $0<\epsilon_1<\epsilon$,

$$\sup_{z\in U_1}\left|\left(\widehat{\mathcal{B}}\widehat{\varphi}\right)(s,z)\right|\le Ae^{B|s|}\quad\text{for}\quad s\in S(d,\epsilon_1)$$

with some $A, B < \infty$.

If so, then the function

$$\varphi^{\theta}(t,z) = \frac{1}{t}\int_0^{\infty(\theta)} \widehat{B}\widehat{\varphi}(s,z)e^{-s/t}ds$$

is called the Borel sum (or 1-sum) of $\widehat{\varphi}$.

S. Michalik obtained a characterization of Borel summable solutions of the heat equation (4).

Theorem M([16]). *Let* $d \in \mathbb{R}$, $U \subset \mathbb{C}^n$ *and* $\widehat{u}$ *be the formal power series solution (5) of the heat equation (4) with* $u_0 \in \mathcal{O}(U)$.
Then the following conditions are equivalent

- $\widehat{u}$ *is* 1*-summable in the direction* d*;*
- $M(u_0; z, R) \in \mathcal{O}^2\big(U \times (\widehat{S}_{d/2} \cup \widehat{S}_{d/2+\pi})\big)$*;*
- $N(u_0; z, R) \in \mathcal{O}^2\big(U \times (\widehat{S}_{d/2} \cup \widehat{S}_{d/2+\pi})\big)$*.*

Furthermore, the 1*-sum of* $\widehat{u}$ *is given by*

$$u^\theta(t, z) = \frac{1}{(4\pi t)^{n/2}} \int_{(e^{i\theta/2}\mathbb{R})^n} \exp\Big\{ -\frac{e^{i\theta}|x|^2}{4t} \Big\} u_0(x + z)dx.$$

if $u_0(x + z)$ *is well defined.*

Theorem 7 *Let $\mathcal{M}$ be a real analytic manifold, $v_0 \in \mathcal{A}(\mathcal{M})$ and $X_1, \ldots, X_n$ real analytic linearly independent commuting vector fields on $\mathcal{M}$. Fix $\mathring{y} \in \mathcal{M}$ and let Φ, Ω, V, B_Φ, μ_Φ and dS_Φ be as in Theorem 6. Set $u_0 = v_0 \circ \Phi$ and assume that u_0 and Φ extend to a complex neighborhood $U \subset \mathbb{C}^n$ of Ω. Then v_0 extends to the neighborhood $\Phi(U)$ of V in the complexification of $\mathcal{M}$.*

Let $d \in \mathbb{R}$ and let $\widehat{v}$ be the formal solution (9) *of the heat type equation* (8)*. Then TFCAE:*

1. *$\widehat{v}$ is Borel summable in d loc. uni. in $\Phi(U)$;*
2. *$M_\Phi(v_0; z, R)$ extends to $\Phi(U) \times \big(D_\epsilon \cup S(d/2, \epsilon) \cup S(d/2+\pi, \epsilon)\big)$ with $0 < \epsilon$ and for any $U_1 \Subset U$, $0 < \epsilon_1 < \epsilon$ and $R \in S(d/2, \epsilon_1) \cup S(d/2+\pi, \epsilon_1)$, $\sup_{z\in\Phi(U_1)} |M_\Phi(v_0; z, R)| \leq Ae^{B|R|^2}$;*
3. *The same holds for $N_\Phi(v_0; z, R)$.*

8. Final remarks and open problems

1. The results on convergence and Borel summability are local. It would be interesting to obtain global analogues. In case of the one dimensional heat equation on S^1 the problems can be easily solved. The general case seems to be open.

2. It would be also interesting to obtain conditions for convergence and Borel summability of formal solutions to (8) in cases when vector fields X_i do not commute and/or are not linearly independent. Of special interest here are the cases when $\widetilde{\Delta}$ is the Grushin operator $\partial_x^2 + x^2\partial_y^2$ or the Laplace operator on the Heisenberg group.

Thank you for your attention!

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