On formal power series solutions of inhomogeneous linear moment partial differential equations

Sławomir Michalik

Cardinal Stefan Wyszyński University
Warsaw, Poland

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Abstract

We study the formal power series solutions of the initial value problem for general inhomogeneous linear moment partial differential equations in two complex variables with constant coefficients

\[
\left\{ \begin{array}{l}
P(\partial_{m_1,t}, \partial_{m_2,z}) \hat{u}(t, z) = \hat{f}(t, z) \in \mathbb{C}[[t, z]]_{\tilde{s}_1, \tilde{s}_2}\\
\partial_{t,m_1}^j \hat{u}(0, z) = 0 \text{ for } j = 0, \ldots, n - 1
\end{array} \right.,
\]

where \( \partial_{m_1,t} \) and \( \partial_{m_2,z} \) are moment-differential operators introduced by W. Balser and M. Yoshino, \( n \in \mathbb{N} \) and

\[
P(\lambda, \zeta) = P_0(\zeta)\lambda^n - \sum_{j=1}^{n} P_j(\zeta)\lambda^{n-j}
\]

is a polynomial of order \( n \) with respect to \( \lambda \).
Moment functions

Definition

A pair of functions $e_m$ and $E_m$ is said to be kernel functions of order $k$ ($k > 1/2$) if they have the following properties:

1. $e_m \in O(S_0(\pi/k))$, $e_m(z)/z$ is integrable at the origin, $e_m(x) \in \mathbb{R}_+$ for $x \in \mathbb{R}_+$ and $e_m$ is exponentially flat of order $k$ in $S_0(\pi/k)$ (i.e. $\forall \varepsilon > 0 \exists A, B > 0$ such that $|e_m(z)| \leq Ae^{-|z|/B}^k$ for $z \in S_0(\pi/k - \varepsilon)$).
**Moment functions**

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A pair of functions $e_m$ and $E_m$ is said to be **kernel functions of order $k$** ($k > 1/2$) if they have the following properties:

1. $e_m \in \mathcal{O}(S_0(\pi/k))$, $e_m(z)/z$ is integrable at the origin, $e_m(x) \in \mathbb{R}_+$ for $x \in \mathbb{R}_+$ and $e_m$ is exponentially flat of order $k$ in $S_0(\pi/k)$ (i.e. $\forall \varepsilon > 0 \exists A, B > 0$ such that $|e_m(z)| \leq Ae^{-|z|/B^k}$ for $z \in S_0(\pi/k - \varepsilon)$).

2. $E_m \in \mathcal{O}^k(\mathbb{C})$ (i.e. $E_m \in \mathcal{O}(\mathbb{C})$ and $\exists A, B > 0$ such that $|E_m(z)| \leq Ae^{B|z|^k}$ for $z \in \mathbb{C}$) and $E_m(1/z)/z$ is integrable at the origin in $S_\pi(2\pi - \pi/k)$. 
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2. $E_m \in O^k(\mathbb{C})$ (i.e. $E_m \in O(\mathbb{C})$ and $\exists A,B > 0$ such that $|E_m(z)| \leq Ae^{B|z|^k}$ for $z \in \mathbb{C}$) and $E_m(1/z)/z$ is integrable at the origin in $S_{\pi}(2\pi - \pi/k)$.

3. The connection between $e_m$ and $E_m$ is given by the corresponding moment function $m$ of order $1/k$ as follows. The function $m$ is defined in terms of $e_m$ by

$$m(u) := \int_0^\infty x^{u-1} e_m(x)dx \text{ for } \text{Re } u \geq 0$$

and the kernel function $E_m$ has the power series expansion $E_m(z) = \sum_{n=0}^\infty \frac{z^n}{m(n)}$ for $z \in \mathbb{C}$. 
Moment functions

In the case $k \leq 1/2$ we must define the kernel functions of order $k$ and the corresponding moment functions in another way.

**Definition**

A function $e_m$ is called a kernel function of order $k \in (0, 1/2]$ if we can find a pair of kernel functions $e_{\tilde{m}}$ and $E_{\tilde{m}}$ of order $pk > 1/2$ (for some $p \in \mathbb{N}$) so that

$$e_m(z) = e_{\tilde{m}}(z^{1/p})/p \quad \text{for} \quad z \in S_0(\pi/k).$$

For a given kernel function $e_m$ of order $k > 0$ we define the corresponding moment function $m$ of order $1/k > 0$ and the kernel function $E_m$ of order $k > 0$ as in the previous definition. It means that

$$m(u) = \tilde{m}(pu) \quad \text{and} \quad E_m(z) = \sum_{j=0}^{\infty} \frac{z^j}{m(j)} = \sum_{j=0}^{\infty} \frac{z^j}{\tilde{m}(jp)}.$$
Proposition

Let $m_1, m_2$ be moment functions of orders $s_1, s_2 \in \mathbb{R}_+$ respectively. Then

1. $m_1 m_2$ is a moment function of order $s_1 + s_2$,
2. $m_1 / m_2$ is a moment function of order $s_1 - s_2$ ($s_1 > s_2$).

Using the above proposition we extend the notion of moment functions to real orders as follows

Definition

We say that $m$ is a moment function of order $s < 0$ if $1/m$ is a moment function of order $-s > 0$.

We say that $m$ is a moment function of order 0 if there exist moment functions $m_1$ and $m_2$ of the same order $s > 0$ such that $m = m_1 / m_2$. 
Moment functions

**Proposition**

Let $m_1$, $m_2$ be moment functions of orders $s_1, s_2 \in \mathbb{R}_+$ respectively. Then

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Sławomir Michalik (Wyszyński University)  Solutions of inhomogeneous moment-PDEs  FASDE 2013
Moment functions

Example

In the theory of $k$-summability ($k > 0$) we use the following kernel functions of order $k$ with the corresponding moment function $m$ of order $1/k$:

- $e_m(z) = k z^k e^{-z^k}$
- $m(u) = \Gamma(1 + u/k)$
- $E_m(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1 + n/k)} =: E_{1/k}(z)$, where $E_{1/k}$ is the Mittag-Leffler function of index $1/k$. 

Analogously, we denote by $\Gamma_s$ the moment function of order $s \in \mathbb{R}$ defined by

$$
\Gamma_s(u) :=
\begin{cases}
\Gamma(1 + su) & \text{for } s \geq 0 \\
\frac{1}{\Gamma(1 - su)} & \text{for } s < 0
\end{cases}
$$

$\Gamma_s$ are the canonical examples of moment functions, since every moment function $m$ of order $s \in \mathbb{R}$ has the same growth as $\Gamma_s$. It means that there exist constants $c, C > 0$ such that $c n \Gamma_s(n) \leq m(n) \leq C n \Gamma_s(n)$ for every $n \in \mathbb{N}$. 

Sławomir Michalik (Wyszyński University) Solutions of inhomogeneous moment-PDEs FASDE 2013 6 / 22
Moment functions

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Analogously, we denote by $\Gamma_s$ the moment function of order $s \in \mathbb{R}$ defined by

$$\Gamma_s(u) := \begin{cases} 
\Gamma(1 + su) & \text{for } s \geq 0 \\
1/\Gamma(1 - su) & \text{for } s < 0
\end{cases}. $$

$\Gamma_s$ are the canonical examples of moment functions, since every moment function $m$ of order $s \in \mathbb{R}$ has the same growth as $\Gamma_s$. It means that there exist constants $c, C > 0$ such that

$$c^n \Gamma_s(n) \leq m(n) \leq C^n \Gamma_s(n) \quad \text{for every } n \in \mathbb{N}.$$
Moment Borel transform

Definition

Let $m$ be a moment function. Then the linear operator $B_{m,x} : \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]]$ defined by

$$B_{m,x} \left( \sum_{j=0}^{\infty} u_j x^j \right) := \sum_{j=0}^{\infty} \frac{u_j}{m(j)} x^j$$

is called an $m$-moment Borel transform.
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is called an $m$-moment Borel transform.

**Remark**

For every $\hat{u} \in \mathbb{C}[[x]]$ the following properties of moment Borel transforms are satisfied:

1. $B_{m_1,x} B_{m_2,x} \hat{u} = B_{m_1 m_2} \hat{u}$ for every moment functions $m_1$ and $m_2$.
2. $B_{m,x} B_{1/m,x} \hat{u} = B_{1/m,x} B_{m,x} \hat{u} = B_{1,x} \hat{u} = \hat{u}$ for every moment function $m$. 

Sławomir Michalik (Wyszyński University) Solutions of inhomogeneous moment-PDEs FASDE 2013 7 / 22
Gevrey order

According to the properties of moment functions we may define the Gevrey order of formal power series as follows

**Definition**

Let $s \in \mathbb{R}$. Then $\hat{u} \in \mathbb{C}[[x]]$ is called a **formal power series of Gevrey order** $s$ if there exists a disc $D \subset \mathbb{C}$ with centre at the origin such that $B_{\Gamma_s, x} \hat{u} \in \mathcal{O}(D)$. The space of formal power series of Gevrey order $s$ is denoted by $\mathbb{C}[[x]]_s$. 

**Remarks**

1. We may replace $\Gamma_s$ in the above definition by any moment function $m$ of order $s$.
2. If $\hat{u} \in \mathbb{C}[[x]]$ and $s \leq 0$ then $\hat{u}$ is convergent, so its sum $u$ is well defined. Moreover $\hat{u} \in \mathbb{C}[[x]]_0 \iff u \in \mathcal{O}(D)$ and $\hat{u} \in \mathbb{C}[[x]]_s \iff u \in \mathcal{O}^{-1/s}(\mathcal{O})$ for $s < 0$. 

Sławomir Michalik (Wyszyński University)
Solutions of inhomogeneous moment-PDEs
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**Remarks**

1. We may replace $\Gamma_s$ in the above definition by any moment function $m$ of order $s$.
2. If $\hat{u} \in \mathbb{C}[[x]]_s$ and $s \leq 0$ then $\hat{u}$ is convergent, so its sum $u$ is well defined. Moreover $\hat{u} \in \mathbb{C}[[x]]_0 \iff u \in \mathcal{O}(D)$ and $\hat{u} \in \mathbb{C}[[x]]_s \iff u \in \mathcal{O}^{-1/s}(\mathbb{C})$ for $s < 0$. 

Sławomir Michalik (Wyszyński University) Solutions of inhomogeneous moment-PDEs  FASDE 2013  8 / 22
Now we are ready to define the $k$-summability of formal power series.

**Definition**

Let $k > 0$ and $d \in \mathbb{R}$. Then $\hat{u} \in \mathbb{C}[[x]]$ is called $k$-summable in a direction $d$ if there exists a disc-sector $\hat{S}_d := S_d \cup D$ in a direction $d$ such that $B_{\Gamma_{1/k},x} \hat{u} \in \mathcal{O}^k(\hat{S}_d)$.
Borel summability

Now we are ready to define the $k$-summability of formal power series

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Let $k > 0$ and $d \in \mathbb{R}$. Then $\hat{u} \in \mathbb{C}[[x]]$ is called $k$-summable in a direction $d$ if there exists a disc-sector $\hat{S}_d := S_d \cup D$ in a direction $d$ such that $B_{\Gamma_{1/k}, x} \hat{u} \in \mathcal{O}^k(\hat{S}_d)$.

**Remark**

By the general theory of moment summability we may replace $\Gamma_{1/k}$ in the above definition by any moment function $m$ of order $1/k$. 
Moment operators

Definition

Let $m$ be a moment function. Then the linear operator $\partial_{m,x} : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ defined by

$$\partial_{m,x}\left(\sum_{j=0}^{\infty} \frac{u_j}{m(j)} x^j \right) := \sum_{j=0}^{\infty} \frac{u_{j+1}}{m(j)} x^j$$

is called the $m$-moment differential operator $\partial_{m,x}$.
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Moreover, the right-inversion operator $\partial_{m,x}^{-1} : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ given by

$$\partial_{m,x}^{-1} \left( \sum_{j=0}^{\infty} \frac{u_j}{m(j)} x^j \right) := \sum_{j=1}^{\infty} \frac{u_{j-1}}{m(j)} x^j$$

is called the $m$-moment integration operator $\partial_{m,x}^{-1}$. 

Example

Below we present some examples of moment differential operators acting on
\( \hat{u}(x) = \sum_{j=0}^{\infty} u_j x^j. \)
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- \( \partial_{\Gamma,1,x} \hat{u} = \partial_x \hat{u} \).
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Below we present some examples of moment differential operators acting on
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- \( \partial_{\Gamma_1, x} \hat{u} = \partial_x \hat{u} \).

- \( (\partial_{\Gamma_s, x} \hat{u})(x^s) = \partial_x^s (\hat{u}(x^s)) \) (s > 0), where \( \partial_x^s \) is the Caputo fractional derivative of order s (i.e. \( \partial_x^s (\sum_{j=0}^{\infty} \frac{u_j}{\Gamma_s(j)} x^{sj}) := \sum_{j=0}^{\infty} \frac{u_{j+1}}{\Gamma_s(j)} x^{sj} \).
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- \( \partial_{1,x} \hat{u} = \partial_x \hat{u}. \)

- \( (\partial_{s,x} \hat{u})(x^s) = \partial_x^s(\hat{u}(x^s)) \) \( (s > 0), \) where \( \partial_x^s \) is the Caputo fractional derivative of order \( s \) (i.e. \( \partial_x^s \left( \sum_{j=0}^{\infty} \frac{u_j}{\Gamma(s(j))} x^{sj} \right) := \sum_{j=0}^{\infty} \frac{u_{j+1}}{\Gamma(s(j))} x^{sj} \)).

- \( \partial_{0,x} \hat{u}(x) = \partial_{1,x} \hat{u}(x) = \frac{\hat{u}(x)-u_0}{x}. \)
Moment differential operators

Example

Below we present some examples of moment differential operators acting on \( \hat{u}(x) = \sum_{j=0}^{\infty} u_j x^j \).

- \( \partial_{\Gamma,1} x \hat{u} = \partial_x \hat{u} \).
- \( (\partial_{\Gamma,s} x \hat{u})(x^s) = \partial_x^s(\hat{u}(x^s)) \) (\( s > 0 \)), where \( \partial_x^s \) is the Caputo fractional derivative of order \( s \) (i.e. \( \partial_x^s(\sum_{j=0}^{\infty} \frac{u_j}{\Gamma_s(j)} x^j) := \sum_{j=0}^{\infty} \frac{u_{j+1}}{\Gamma_s(j)} x^j \)).
- \( \partial_{\Gamma,0} x \hat{u}(x) = \partial_{1,x} \hat{u}(x) = \frac{\hat{u}(x)-u_0}{x} \).
- \( \partial_{\Gamma,-1} x \hat{u}(x) = \frac{1}{x} \int_0^x \frac{\hat{u}(y)-u_0}{y} \, dy \).
Moment differential operators

Example

Below we present some examples of moment differential operators acting on \( \hat{u}(x) = \sum_{j=0}^{\infty} u_j x^j \).

- \( \partial_{\Gamma_1, x} \hat{u} = \partial_x \hat{u} \).
- \( (\partial_{\Gamma_s, x} \hat{u})(x^s) = \partial^s_x(\hat{u}(x^s)) \) (\( s > 0 \)), where \( \partial^s_x \) is the Caputo fractional derivative of order \( s \) (i.e. \( \partial^s_x(\sum_{j=0}^{\infty} \frac{u_j}{\Gamma_s(j)} x^j) := \sum_{j=0}^{\infty} \frac{u_j+1}{\Gamma_s(j)} x^j \)).
- \( \partial_{\Gamma_0, x} \hat{u}(x) = \partial_{1, x} \hat{u}(x) = \frac{\hat{u}(x)-u_0}{x} \).
- \( \partial_{\Gamma_{-1}, x} \hat{u}(x) = \frac{1}{x} \int_0^x \frac{\hat{u}(y)-u_0}{y} \, dy \).
- \( (\partial_{\Gamma_{-s}, x} \hat{u})(x^s) = \frac{1}{x^s} \partial^{-s}_x \hat{u}(x^s)-u_0 \) (\( s > 0 \)) where \( \partial^{-s}_x \) is the right-inversion operator to \( \partial^s_x \) and is defined by \( \partial^{-s}_x(\sum_{j=0}^{\infty} \frac{u_j}{\Gamma_s(j)} x^j) := \sum_{j=1}^{\infty} \frac{u_{j-1}}{\Gamma_s(j)} x^j \).
Moment operators

The moment differential operator $\partial_{m,z}$ is well-defined for every $\varphi \in \mathcal{O}(D)$. Moreover, if $\varphi \in \mathcal{O}(D_r)$ and $m$ is a moment function of order $1/k > 0$ then for every $|z| < \varepsilon < r$ and $n \in \mathbb{N}$ we have

$$\partial^n_{m,z} \varphi(z) = \frac{1}{2\pi i} \int_{|w| = \varepsilon} \varphi(w) \int_{0}^{\infty(\theta)} \zeta^n E_m(z\zeta) \frac{e_m(w\zeta)}{w\zeta} d\zeta dw,$$

where $\theta \in (-\arg w - \frac{\pi}{2k}, -\arg w + \frac{\pi}{2k})$. 

Using the above formula, we may define a moment pseudodifferential operator $\lambda(\partial_{m,z})$: $\mathcal{O}(D) \to \mathcal{O}(D)$ as an operator satisfying $\lambda(\partial_{m,z}) E_m(z\zeta) := \lambda(\zeta) E_m(z\zeta)$ for $|\zeta| \geq r_0$. Hence, if $\lambda(\zeta)$ is an analytic function for $|\zeta| \geq r_0$ of polynomial growth at infinity then $\lambda(\partial_{m,z})$ is defined by $\lambda(\partial_{m,z}) \varphi(z) := \frac{1}{2\pi i} \int_{|w| = \varepsilon} \varphi(w) \int_{0}^{\infty(\theta)} \zeta^n E_m(z\zeta) \frac{e_m(w\zeta)}{w\zeta} d\zeta dw$ for every $\varphi \in \mathcal{O}(D_r)$ and $|z| < \varepsilon < r$, where $\theta \in (-\arg w - \frac{\pi}{2k}, -\arg w + \frac{\pi}{2k})$. 
Moment operators

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Using the above formula, we may define a moment pseudodifferential operator $\lambda(\partial_{m,z}) : \mathcal{O}(D) \rightarrow \mathcal{O}(D)$ as an operator satisfying

$$\lambda(\partial_{m,z}) E_m(\zeta z) := \lambda(\zeta) E_m(\zeta z) \quad \text{for} \quad |\zeta| \geq r_0.$$

Hence, if $\lambda(\zeta)$ is an analytic function for $|\zeta| \geq r_0$ of polynomial growth at infinity then $\lambda(\partial_{m,z})$ is defined by

$$\lambda(\partial_{m,z}) \varphi(z) := \frac{1}{2\pi i} \oint_{|w| = \varepsilon} \varphi(w) \int_{r_0e^{i\theta}}^{\infty(\theta)} \lambda(\zeta) E_m(\zeta z) \frac{e_m(\zeta w)}{\zeta w} d\zeta \, dw$$

for every $\varphi \in \mathcal{O}(D_r)$ and $|z| < \varepsilon < r$, where $\theta \in (\arg w - \frac{\pi}{2k}, \arg w + \frac{\pi}{2k})$. 

Slawomir Michalik (Wyszyński University)
Commutation formula

The operators $B_{m',x}, \partial_{m,x}, P(\partial_{m,x}): \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ and $\lambda(\partial_{m,x}): \mathcal{O}(D) \to \mathcal{O}(D)$ satisfy the following commutation formulas:

1. $B_{m',x} \partial_{m,x} \hat{u} = \partial_{mm',x} B_{m',x} \hat{u},$
2. $B_{m',x} P(\partial_{m,x}) \hat{u} = P(\partial_{mm',x}) B_{m',x} \hat{u}$ for any polynomial $P$ with constant coefficients.
3. $B_{m',x} \lambda(\partial_{m,x}) \hat{u} = \lambda(\partial_{mm',x}) B_{m',x} \hat{u}$ for any moment pseudodifferential operator $\lambda(\partial_{m,x}).$
Commutation formula

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2. $B_{m',x} P(\partial_{m,x}) \hat{u} = P(\partial_{mm',x}) B_{m',x} \hat{u}$ for any polynomial $P$ with constant coefficients.
3. $B_{m',x} \lambda(\partial_{m,x}) \hat{u} = \lambda(\partial_{mm',x}) B_{m',x} \hat{u}$ for any moment pseudodifferential operator $\lambda(\partial_{m,x}).$

By the above commutation formula we may extend the definition of moment pseudodifferential operators:

**Definition**

Let $s \in \mathbb{R}, m$ be a moment function of order $\tilde{s} \in \mathbb{R}$ and $\lambda(\zeta)$ be a holomorphic function for $|\zeta| \geq r_0$ of polynomial growth at infinity. A moment pseudodifferential operator $\lambda(\partial_{m,z}): \mathbb{C}[[z]]_s \to \mathbb{C}[[z]]_s$ is defined by

$$
\lambda(\partial_{m,z}) \hat{\phi}(z) := B_{-\tilde{s},z} \lambda(\partial_{m \tilde{s},z}) B_{\tilde{s},z} \hat{\phi}(z),
$$

where $\hat{\phi} \in \mathbb{C}[[z]]_s$ and $\tilde{s} = \max\{s, 1 - \tilde{s}\}$ and the operator $\lambda(\partial_{m \tilde{s},z})$ was constructed in the previous definition.
Factorization of $P(\partial_{m_1}, t, \partial_{m_2}, z)$

Let $P(\lambda, \zeta) = P_0(\zeta)\lambda^n - \sum_{j=1}^{n} P_j(\zeta)\lambda^{n-j}$ be a general polynomial of two variables, which is of order $n$ with respect to $\lambda$.

Using the moment-pseudodifferential operators we factorize the moment-differential operator $P(\partial_{m_1}, t, \partial_{m_2}, z)$ as follows

$$P(\partial_{m_1}, t, \partial_{m_2}, z) = P_0(\partial_{m_2}, z)(\partial_{m_1}, t - \lambda_1(\partial_{m_2}, z))^{n_1} \cdots (\partial_{m_1}, t - \lambda_l(\partial_{m_2}, z))^{n_l}$$

$$=: P_0(\partial_{m_2}, z)\tilde{P}(\partial_{m_1}, t, \partial_{m_2}, z)$$

where $\lambda_1(\zeta), \ldots, \lambda_l(\zeta)$ are the characteristic roots of $P(\lambda, \zeta) = 0$ with multiplicities $n_1, \ldots, n_l$ ($n_1 + \ldots + n_l = n$) respectively.
Factorization of $P(\partial_{m_1}, t, \partial_{m_2}, z)$

Let $P(\lambda, \zeta) = P_0(\zeta)\lambda^n - \sum_{j=1}^n P_j(\zeta)\lambda^{n-j}$ be a general polynomial of two variables, which is of order $n$ with respect to $\lambda$.

Using the moment-pseudodifferential operators we factorize the moment-differential operator $P(\partial_{m_1}, t, \partial_{m_2}, z)$ as follows

$$P(\partial_{m_1}, t, \partial_{m_2}, z) = P_0(\partial_{m_2}, z)(\partial_{m_1}, t - \lambda_1(\partial_{m_2}, z))^{n_1} ... (\partial_{m_1}, t - \lambda_l(\partial_{m_2}, z))^{n_l} =: P_0(\partial_{m_2}, z)\tilde{P}(\partial_{m_1}, t, \partial_{m_2}, z)$$

where $\lambda_1(\zeta), ..., \lambda_l(\zeta)$ are the characteristic roots of $P(\lambda, \zeta) = 0$ with multiplicities $n_1, ..., n_l$ ($n_1 + ... + n_l = n$) respectively.

In general $\lambda_j(\zeta)$ are algebraic functions, hence they are holomorphic functions of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ (for some $\kappa \in \mathbb{N}$ and for sufficiently large $r_0$).
Factorization of $P(\partial_{m_1}, t, \partial_{m_2}, z)$

Let $P(\lambda, \zeta) = P_0(\zeta)\lambda^n - \sum_{j=1}^n P_j(\zeta)\lambda^{n-j}$ be a general polynomial of two variables, which is of order $n$ with respect to $\lambda$.

Using the moment-pseudodifferential operators we factorize the moment-differential operator $P(\partial_{m_1}, t, \partial_{m_2}, z)$ as follows

$$P(\partial_{m_1}, t, \partial_{m_2}, z) = P_0(\partial_{m_2}, z)(\partial_{m_1}, t - \lambda_1(\partial_{m_2}, z))^{n_1} \ldots (\partial_{m_1}, t - \lambda_l(\partial_{m_2}, z))^{n_l} =: P_0(\partial_{m_2}, z)\tilde{P}(\partial_{m_1}, t, \partial_{m_2}, z)$$

where $\lambda_1(\zeta), \ldots, \lambda_l(\zeta)$ are the characteristic roots of $P(\lambda, \zeta) = 0$ with multiplicities $n_1, \ldots, n_l$ ($n_1 + \ldots + n_l = n$) respectively.

In general $\lambda_j(\zeta)$ are algebraic functions, hence they are holomorphic functions of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ (for some $\kappa \in \mathbb{N}$ and for sufficiently large $r_0$).

For this reason we use

**Proposition**

Let $v(t, z) = u(t, z^\kappa)$ and $\tilde{m}_2(u) = m_2(u/\kappa)$. Then $P(\partial_{m_1}, t, \partial_{m_2}, z)u = 0$ if and only if $P(\partial_{m_1}, t, \partial_{m_2}, z)\tilde{v} = 0$.

Hence without loss of generality we may assume that $\kappa = 1$, $\lambda_j(\zeta)$ is a holomorphic function for $|\zeta| \geq r_0$ and the moment pseudodifferential operators $\lambda_j(\partial_{m_2}, z)$ are well defined.
Uniqueness of formal solution

If $P_0(\zeta) \neq \text{const.}$ then the formal solution is not uniquely determined. For this reason we choose a formal power series $\hat{g} \in \mathbb{C}[[t, z]]_{s_1, s_2}$ satisfying the equation $P_0(\partial_{m_2, z}) \hat{g} = \hat{f}$. For such $\hat{g}$ we may construct the uniquely determined solution $\hat{u}$ of

\[
\left\{ \begin{array}{l}
\tilde{P}(\partial_{m_1, t}, \partial_{m_2, z}) \hat{u} = \hat{g} \\
\partial_{m_1, t}^j \hat{u}(0, z) = 0 \text{ for } j = 0, \ldots, n - 1
\end{array} \right.
\]

which is also a formal solution of

\[
\left\{ \begin{array}{l}
P(\partial_{m_1, t}, \partial_{m_2, z}) \hat{u} = \hat{f} \\
\partial_{m_1, t}^j \hat{u}(0, z) = 0 \text{ for } j = 0, \ldots, n - 1
\end{array} \right.
\tag{1}
\]

and is called the formal solution of (1) determined by $\hat{g}$. 
Decomposition of equation

By the factorization of operator $\tilde{P}(\partial_{m_1,t}, \partial_{m_2,z})$ we obtain that the formal solution $\hat{u}$ determined by $\hat{g}$ satisfies the decomposition $\hat{u} = \sum_{\alpha=1}^{l} \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$, where $\hat{u}_{\alpha\beta}$ is a formal solution of

$$
\begin{cases}
(\partial_{m_1,t} - \lambda_\alpha(\partial_{m_2,z}))^\beta \hat{u}_{\alpha\beta} = \hat{g}_{\alpha\beta} \\
\partial_{m_1,t}^j \hat{u}_{\alpha\beta}(0, z) = 0 \ (j = 0, ..., \beta - 1)
\end{cases},
$$

where $\hat{g}_{\alpha\beta}(t, z) := d_{\alpha\beta}(\partial_{m_2,z})\hat{g}(t, z) \in \mathbb{C}[[t, z]]_{\tilde{s}_1, \tilde{s}_2}$ and $d_{\alpha\beta}(\zeta)$ is a holomorphic function of polynomial growth.
Gevrey order of formal solution

By the above decomposition it is sufficient to study the moment-pseudodifferential equation

\[
\begin{cases}
(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^\beta u = \hat{g} \in \mathbb{C}[[t, z]]\tilde{s}_1,\tilde{s}_2 \\
\partial_{m_1,t}^j u(0, z) = 0 \quad (j = 0, \ldots, \beta - 1).
\end{cases}
\]

For simplicity we assume that \(\beta = 1\). In this case the formal solution \(\hat{u}\) is given by

\[
\hat{u}(t, z) = \sum_{n=0}^{\infty} \left( \partial_{m_1,t}^{-1} \right)^n (\partial_{m_2,z})^n \hat{g}(t, z).
\]

We will study the Gevrey order of formal solution \(\hat{u}\), which depends on the orders \(s_1\) and \(s_2\) of moment functions, on the Gevrey orders \(\tilde{s}_1, \tilde{s}_2\) and depends on the characteristic root \(\lambda(\zeta)\sim \lambda_0\xi^q\) (i.e. \(\lim_{\zeta \to \infty} \lambda(\zeta)\zeta^q = \lambda_0 \neq 0\)).

Estimating the coefficients of the formal solution \(\hat{u}\) we have

**Proposition**

If \(q \geq 0\) then \(\hat{u} \in \mathbb{C}[[t, z]]\max\{q(s_2 + \tilde{s}_2) - s_1, \tilde{s}_1, \tilde{s}_2\}\).
Gevrey order of formal solution

By the above decomposition it is sufficient to study the moment-pseudodifferential equation

\[
\begin{align*}
\left\{(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^\beta u &= \mathring{g} \in \mathbb{C}[[t, z]]\tilde{s}_1, \tilde{s}_2 \\
\partial_{m_1,t}^j u(0, z) &= 0 \quad (j = 0, \ldots, \beta - 1)\end{align*}
\]

For simplicity we assume that $\beta = 1$. In this case the formal solution $\mathring{u}$ is given by

\[
\mathring{u}(t, z) = \sum_{n=0}^{\infty} (\partial_{m_1,t}^{-1})^{n+1} \lambda^n (\partial_{m_2,z}) \mathring{g}(t, z).
\]
Gevrey order of formal solution

By the above decomposition it is sufficient to study the moment-pseudodifferential equation

\[
\begin{cases}
(\partial_{m_1, t} - \lambda(\partial_{m_2, z}))^\beta u = \hat{g} \in \mathbb{C}[[t, z]]\tilde{s}_1, \tilde{s}_2 \\
\partial_{m_1, t}^j u(0, z) = 0 \ (j = 0, \ldots, \beta - 1).
\end{cases}
\]

For simplicity we assume that $\beta = 1$. In this case the formal solution $\hat{u}$ is given by

\[
\hat{u}(t, z) = \sum_{n=0}^{\infty} \left( \partial_{m_1, t}^{-1} \right)^{n+1} \lambda^n (\partial_{m_2, z}) \hat{g}(t, z).
\]

We will study the Gevrey order of formal solution $\hat{u}$, which depends on the orders $s_1$ and $s_2$ of moment functions, on the Gevrey orders $\tilde{s}_1$, $\tilde{s}_2$ and depends on the characteristic root $\lambda(\zeta) \sim \lambda_0 \zeta^q$ (i.e. $\lim_{\zeta \to \infty} \frac{\lambda(\zeta)}{\zeta^q} = \lambda_0 \neq 0$.)
Gevrey order of formal solution

By the above decomposition it is sufficient to study the moment-pseudodifferential equation

\[
\begin{cases}
(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^\beta u = \hat{g} \in \mathbb{C}[[t, z]]\tilde{s}_1,\tilde{s}_2 \\
\partial^j_{m_1,t} u(0, z) = 0 \ (j = 0, \ldots, \beta - 1).
\end{cases}
\]

For simplicity we assume that \(\beta = 1\). In this case the formal solution \(\hat{u}\) is given by

\[
\hat{u}(t, z) = \sum_{n=0}^{\infty} (\partial^{-1}_{m_1,t})^{n+1} \lambda^n (\partial_{m_2,z}) \hat{g}(t, z).
\]

We will study the Gevrey order of formal solution \(\hat{u}\), which depends on the orders \(s_1\) and \(s_2\) of moment functions, on the Gevrey orders \(\tilde{s}_1, \tilde{s}_2\) and depends on the characteristic root \(\lambda(\zeta) \sim \lambda_0 \zeta^q\) (i.e. \(\lim_{\zeta \to \infty} \frac{\lambda(\zeta)}{\zeta^q} = \lambda_0 \neq 0\)). Estimating the coefficients of the formal solution \(\hat{u}\) we have

\[\text{Proposition}\]

If \(q \geq 0\) then \(\hat{u} \in \mathbb{C}[[t, z]]_{\text{max}\{q(s_2+\tilde{s}_2) - s_1, \tilde{s}_1\} , \tilde{s}_2}\).
Integral representation of solution

Let $m$ be a moment function of order 0. Then by the general theory of moment summability $\hat{u}$ is $k$-summable in a direction $d$ iff $B_{m,t}\hat{u}$ is $k$-summable in a direction $d$. Analogously $u \in \mathcal{O}(\hat{S}_d \times D)$ iff $B_{m,t}u \in \mathcal{O}(\hat{S}_d \times D)$. Hence to study analytic continuation or $k$-summability of $\hat{u}$ we may replace $m_1$ and $m_2$ by $\Gamma_{s_1}$ and $\Gamma_{s_2}$, where $s_1$ and $s_2$ are orders of $m_1$ and $m_2$. So, we may assume that $\hat{u}$ satisfies $(\partial_{\Gamma_{s_1},t} - \lambda(\partial_{\Gamma_{s_2},z}))\hat{u} = \hat{g}$, $\hat{u}(0,z) = 0$.

Since $(\partial^{-1}_{\Gamma_{s_1},x}) \phi(x) = \int x_1/s_0 k_\lambda(x_1/y) k_{s_1-1}(\lambda/z) \phi(y) dy (s > 0)$, using the definition of moment pseudodifferential operators and the power series representation of $\hat{u}$ we can find the integral representation of solution $u(t,z) = -1/2\pi i \int t_1/s_1 0 \oint |w| = \varepsilon g(\tau,w) \int \infty (\theta) r_0 e^{i\theta}(\partial_{\tau} E_{\Gamma_{s_1}}((t_1/s_1 - \tau)s_1) \lambda(\zeta)) \times \times E_{\Gamma_{s_2}}(\zeta z) e^{\Gamma_{s_2}(\zeta w)} \zeta w d\zeta dw d\tau =: I[g]$, where $g \in \mathcal{O}(D^2)$, $s_1, s_2 > 0$ and $s_1 \geq q s_2$. 
Integral representation of solution

Let $m$ be a moment function of order 0. Then by the general theory of moment summability $\hat{u}$ is $k$-summable in a direction $d$ iff $B_{m,t}\hat{u}$ is $k$-summable in a direction $d$. Analogously $u \in O(\hat{S}_d \times D)$ iff $B_{m,t}u \in O(\hat{S}_d \times D)$.

Hence to study analytic continuation or $k$-summability of $\hat{u}$ we may replace $m_1$ and $m_2$ by $\Gamma_{s_1}$ and $\Gamma_{s_2}$, where $s_1$ and $s_2$ are orders of $m_1$ and $m_2$. So, we may assume that $\hat{u}$ satisfies

$$(\partial_{\Gamma_{s_1}}, t - \lambda(\partial_{\Gamma_{s_2}}, z))\hat{u} = \hat{g}, \quad \hat{u}(0, z) = 0.$$
Integral representation of solution

Let \( m \) be a moment function of order 0. Then by the general theory of moment summability \( \hat{u} \) is \( k \)-summable in a direction \( d \) iff \( B_{m,t} \hat{u} \) is \( k \)-summable in a direction \( d \). Analogously \( u \in \mathcal{O}(\hat{S}_d \times D) \) iff \( B_{m,t} u \in \mathcal{O}(\hat{S}_d \times D) \).

Hence to study analytic continuation or \( k \)-summability of \( \hat{u} \) we may replace \( m_1 \) and \( m_2 \) by \( \Gamma_{s_1} \) and \( \Gamma_{s_2} \), where \( s_1 \) and \( s_2 \) are orders of \( m_1 \) and \( m_2 \). So, we may assume that \( \hat{u} \) satisfies

\[
(\partial_{\Gamma_{s_1}} t - \lambda(\partial_{\Gamma_{s_2}} z)) \hat{u} = \hat{g}, \quad \hat{u}(0, z) = 0.
\]

Since

\[
(\partial_{\Gamma_s}^{-1})^k \varphi(x) = \int_0^{x^{1/s}} \frac{ks(x^{1/s} - y)^{ks-1}}{\Gamma_s(k)} \varphi(y^s) \, dy \quad (s > 0),
\]

using the definition of moment pseudodifferential operators and the power series representation of \( \hat{u} \) we can find the integral representation of solution \( u \)

\[
u(t, z) = \frac{-1}{2\pi i} \int_0^{t^{1/s_1}} \int_{|w|=\varepsilon} g(\tau, w) \int_{r_0 e^{i\theta}}^{\infty(\theta)} (\partial_{\tau} E_{\Gamma_{s_1}}((t^{1/s_1} - \tau)^{s_1})\lambda(\zeta)) \times \\
\times E_{\Gamma_{s_2}}(\zeta z) \frac{e_{\Gamma_{s_2}}(\zeta w)}{\zeta w} \, d\zeta \, dw \, d\tau =: I[g],
\]

where \( g \in \mathcal{O}(D^2) \), \( s_1, s_2 > 0 \) and \( s_1 \geq q s_2 \).
Deforming the path of integration with respect to $w$ in the integral representation of solution we conclude that

**Theorem**

Let $\lambda(\zeta) \sim \lambda_0 \zeta^q$, $s_1, s_2 > 0$, $s_1 = qs_2$, $K > 0$ and $d \in \mathbb{R}$. We assume that $u$ is a solution of

$$(\partial_{r_{s_1}}, t - \lambda(\partial_{r_{s_2}}, z))u = g \in \mathcal{O}(D^2), \quad u(0, z) = 0.$$

Then $u(t, z) \in \mathcal{O}^{K,qK}(\hat{S}_d \times \hat{S}_{(d+\arg \lambda_0 + 2k\pi)/q})$ ($k \in \mathbb{N}$) iff $g(t, z) \in \mathcal{O}^{K,qK}(\hat{S}_d \times \hat{S}_{(d+\arg \lambda_0 + 2k\pi)/q})$ ($k \in \mathbb{N}$).
Summable solution

Let $\hat{u}$ be a formal solution of

$$
(\partial_{r_{s_1}}, t - \lambda(\partial_{r_{s_2}}, z))\hat{u} = \hat{g} \in \mathbb{C}[[t, z]]^{s_1, s_2}, \quad \hat{u}(0, z) = 0.
$$

(2)

By the Gevrey estimates $\hat{u} \in \mathbb{C}[[t, z]]_{\max\{q(s_2 + \bar{s}_2) - s_1, \bar{s}_1\}, \bar{s}_2}$. Our aim is a characterisation of summable solutions $\hat{u}$ in terms of inhomogeneity $\hat{g}$.
Summable solution

Let $\hat{u}$ be a formal solution of

$$(\partial_{\Gamma s_1}, t - \lambda(\partial_{\Gamma s_2}, z))\hat{u} = \hat{g} \in \mathbb{C}[[t, z]]^{\tilde{s}_1, \tilde{s}_2}, \quad \hat{u}(0, z) = 0. \quad (2)$$

By the Gevrey estimates $\hat{u} \in \mathbb{C}[[t, z]]_{\max\{q(s_2 + \tilde{s}_2) - s_1, \tilde{s}_2\}, \tilde{s}_2}$. Our aim is a characterisation of summable solutions $\hat{u}$ in terms of inhomogeneity $\hat{g}$.

We assume that $q(s_2 + \tilde{s}_2) \geq s_1 + \tilde{s}_1$ and $q(s_2 + \tilde{s}) > s_1$.

Applying the moment Borel transforms $B_{\Gamma q(s_2 + \tilde{s}_2) / \Gamma s_1, t}$ and $B_{\Gamma s_2 + \tilde{s}_2 / \Gamma s_2, z}$ to (2) we obtain

$$(\partial_{\Gamma q(s_2 + \tilde{s}_2)}, t - \lambda(\partial_{\Gamma s_2 + \tilde{s}_2}, z))U = G \in \mathcal{O}(D^2), \quad U(0, z) = 0,$$

where $U(t, z) := B_{\Gamma q(s_2 + \tilde{s}_2) / \Gamma s_1, t} B_{\Gamma s_2 + \tilde{s}_2 / \Gamma s_2, z} \hat{u}(t, z)$ and $G(t, z) := B_{\Gamma q(s_2 + \tilde{s}_2) / \Gamma s_1, t} B_{\Gamma s_2 + \tilde{s}_2 / \Gamma s_2, z} \hat{g}(t, z)$.
Summable solution

Let \( \hat{u} \) be a formal solution of

\[
(\partial_{r_{s_1}}, t - \lambda(\partial_{r_{s_2}}, z))\hat{u} = \hat{g} \in \mathbb{C}[[t, z]]_{\tilde{s}_1, \tilde{s}_2}, \quad \hat{u}(0, z) = 0.
\]  

(2)

By the Gevrey estimates \( \hat{u} \in \mathbb{C}[[t, z]]_{\max\{q(s_2 + \tilde{s}_2) - s_1, \tilde{s}_1\}, \tilde{s}_2} \). Our aim is a characterisation of summable solutions \( \hat{u} \) in terms of inhomogeneity \( \hat{g} \).

We assume that \( q(s_2 + \tilde{s}_2) \geq s_1 + \tilde{s}_1 \) and \( q(s_2 + \tilde{s}) > s_1 \).

Applying the moment Borel transforms \( B_{r_{q(s_2 + \tilde{s}_2)/r_{s_1}}, t} \) and \( B_{r_{s_2 + \tilde{s}_2}/r_{s_2}, z} \) to (2) we obtain

\[
(\partial_{r_{q(s_2 + \tilde{s}_2)/r_{s_1}}}, t - \lambda(\partial_{r_{s_2 + \tilde{s}_2}, z}))U = G \in \mathcal{O}(D^2), \quad U(0, z) = 0,
\]

where \( U(t, z) := B_{r_{q(s_2 + \tilde{s}_2)/r_{s_1}}, t} B_{r_{s_2 + \tilde{s}_2}/r_{s_2}, z} \hat{u}(t, z) \) and

\[
G(t, z) := B_{r_{q(s_2 + \tilde{s}_2)/r_{s_1}}, t} B_{r_{s_2 + \tilde{s}_2}/r_{s_2}, z} \hat{g}(t, z).
\]

By the previous theorem we conclude that

**Theorem**

Let \( \hat{u} \) be a formal solution of (2), \( q > 0, s_2 + \tilde{s}_2 > 0, q(s_2 + \tilde{s}_2) \geq s_1 + \tilde{s}_1, K := \frac{1}{q(s_2 + \tilde{s}_2)} \)

and \( d \in \mathbb{R} \). If \( G \in \mathcal{O}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q}) \) \( (k \in \mathbb{N}) \) then \( \hat{u} \) is \( K \)-summable in a direction \( d \).
It is trivial to find the necessary condition for $K$-summability of $\hat{u}$:

**Remark 1**

If $\hat{u}$ is $K$-summable in a direction $d$ then also $\hat{g}$ is $K$-summable in the same direction.
Remarks

1. It is trivial to find the necessary condition for $K$-summability of $\hat{u}$:

Remark 1
If $\hat{u}$ is $K$-summable in a direction $d$ then also $\hat{g}$ is $K$-summable in the same direction.

2. Using the integral representation of solution we can also obtain trivially the characterisation of $K$-summable solutions $\hat{u}$:

Remark 2
$\hat{u}$ is $K$-summable in a direction $d$ iff $I[B_{\frac{q(s_2+s_2)}{r_s}, t} B_{\frac{s_2+s_2}{r_s}, z} \hat{g}] \in \mathcal{O}^K(\hat{S}_d \times D)$. 
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