On formal power series solutions of inhomogeneous linear moment partial differential equations

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Abstract

We study the formal power series solutions of the initial value problem for general inhomogeneous linear moment partial differential equations in two complex variables with constant coefficients

$$\begin{cases} P(\partial_{m_1,t}, \partial_{m_2,z})\hat{u}(t,z) = \hat{f}(t,z) \in \mathbb{C}[[t,z]]_{\tilde{s}_1,\tilde{s}_2} \\ \partial^j_{t,m_1}\hat{u}(0,z) = 0 \ \text{ for } \ j = 0, \dots, n-1 \end{cases},$$

where $\partial_{m_1,t}$ and $\partial_{m_2,z}$ are moment-differential operators introduced by W. Balser and M. Yoshino, $n \in \mathbb{N}$ and

$$P(\lambda,\zeta) = P_0(\zeta)\lambda^n - \sum_{j=1}^{n} P_j(\zeta)\lambda^{n-j}$$

is a polynomial of order n with respect to λ.

Moment functions

Definition

A pair of functions e_m and E_m is said to be kernel functions of order k ($k > 1/2$) if they have the following properties:

1. $e_m \in \mathcal{O}(S_0(\pi/k))$, $e_m(z)/z$ is integrable at the origin, $e_m(x) \in \mathbb{R}_+$ for $x \in \mathbb{R}_+$ and e_m is exponentially flat of order k in $S_0(\pi/k)$
(i.e. $\forall_{\varepsilon>0}\exists_{A,B>0}$ such that $|e_m(z)| \leq Ae^{-(|z|/B)^k}$ for $z \in S_0(\pi/k - \varepsilon)$).

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(i.e. $\forall_{\varepsilon>0}\exists_{A,B>0}$ such that $|e_m(z)| \leq Ae^{-(|z|/B)^k}$ for $z \in S_0(\pi/k - \varepsilon)$).
2. $E_m \in \mathcal{O}^k(\mathbb{C})$ (i.e. $E_m \in \mathcal{O}(\mathbb{C})$ and $\exists_{A,B>0}$ such that $|E_m(z)| \leq Ae^{B|z|^k}$ for $z \in \mathbb{C}$) and $E_m(1/z)/z$ is integrable at the origin in $S_\pi(2\pi - \pi/k)$.

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2. $E_m \in \mathcal{O}^k(\mathbb{C})$ (i.e. $E_m \in \mathcal{O}(\mathbb{C})$ and $\exists_{A,B>0}$ such that $|E_m(z)| \leq Ae^{B|z|^k}$ for $z \in \mathbb{C}$) and $E_m(1/z)/z$ is integrable at the origin in $S_\pi(2\pi - \pi/k)$.
3. The connection between e_m and E_m is given by the corresponding moment function m of order $1/k$ as follows. The function m is defined in terms of e_m by

$$m(u) := \int_0^\infty x^{u-1} e_m(x)dx \ \text{ for } \ \mathrm{Re}\, u \geq 0$$

and the kernel function E_m has the power series expansion $E_m(z) = \sum_{n=0}^\infty \frac{z^n}{m(n)}$ for $z \in \mathbb{C}$.

Moment functions

In the case $k \leq 1/2$ we must define the kernel functions of order k and the corresponding moment functions in another way.

Definition

A function e_m is called a kernel function of order $k \in (0, 1/2]$ if we can find a pair of kernel functions $e_{\tilde{m}}$ and $E_{\tilde{m}}$ of order $pk > 1/2$ (for some $p \in \mathbb{N}$) so that

$$e_m(z) = e_{\tilde{m}}(z^{1/p})/p \quad \text{for} \quad z \in S_0(\pi/k).$$

For a given kernel function e_m of order $k > 0$ we define the corresponding moment function m of order $1/k > 0$ and the kernel function E_m of order $k > 0$ as in the previous definition.
It means that

$$m(u) = \widetilde{m}(pu) \quad \text{and} \quad E_m(z) = \sum_{j=0}^{\infty} \frac{z^j}{m(j)} = \sum_{j=0}^{\infty} \frac{z^j}{\widetilde{m}(jp)}.$$

Moment functions

Proposition

Let m_1, m_2 be moment functions of orders $s_1, s_2 \in \mathbb{R}_+$ respectively. Then

1. $m_1 m_2$ is a moment function of order $s_1 + s_2$,
2. m_1/m_2 is a moment function of order $s_1 - s_2$ ($s_1 > s_2$).

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Using the above proposition we extend the notion of moment functions to real orders as follows

Definition

We say that m is a moment function of order $s < 0$ if $1/m$ is a moment function of order $-s > 0$.
We say that m is a moment function of order 0 if there exist moment functions m_1 and m_2 of the same order $s > 0$ such that $m = m_1/m_2$.

Moment functions

Example

In the theory of k-summability ($k > 0$) we use the following kernel functions of order k with the corresponding moment function m of order $1/k$:

- $e_m(z) = kz^k e^{-z^k}$
- $m(u) = \Gamma(1 + u/k)$
- $E_m(z) = \sum_{n=0}^{\infty} z^n/\Gamma(1 + n/k) =: \mathbb{E}_{1/k}(z)$, where $\mathbb{E}_{1/k}$ is the Mittag-Leffler function of index $1/k$.

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Analogously, we denote by Γ_s the moment function of order $s \in \mathbb{R}$ defined by

$$\Gamma_s(u) := \begin{cases} \Gamma(1+su) & \text{for} \quad s \geq 0 \\ 1/\Gamma(1-su) & \text{for} \quad s < 0 \end{cases}.$$

Γ_s are the canonical examples of moment functions, since every moment function m of order $s \in \mathbb{R}$ has the same growth as Γ_s. It means that there exist constants $c, C > 0$ such that

$$c^n\Gamma_s(n) \leq m(n) \leq C^n\Gamma_s(n) \quad \text{for every} \quad n \in \mathbb{N}.$$

Moment Borel transform

Definition

Let m be a moment function. Then the linear operator $\mathcal{B}_{m,x} \colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ defined by

$$\mathcal{B}_{m,x}\Big(\sum_{j=0}^{\infty} u_j x^j\Big) := \sum_{j=0}^{\infty} \frac{u_j}{m(j)} x^j$$

is called an m-moment Borel transform.

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is called an *m*-moment Borel transform.

Remark

For every $\hat{u} \in \mathbb{C}[[x]]$ the following properties of moment Borel transforms are satisfied:

1. $\mathcal{B}_{m_1,x}\mathcal{B}_{m_2,x}\hat{u} = \mathcal{B}_{m_1 m_2}\hat{u}$ for every moment functions m_1 and m_2.
2. $\mathcal{B}_{m,x}\mathcal{B}_{1/m,x}\hat{u} = \mathcal{B}_{1/m,x}\mathcal{B}_{m,x}\hat{u} = \mathcal{B}_{1,x}\hat{u} = \hat{u}$ for every moment function m.

Gevrey order

According to the properties of moment functions we may define the Gevrey order of formal power series as follows

Definition

Let $s \in \mathbb{R}$. Then $\hat{u} \in \mathbb{C}[[x]]$ is called a formal power series of Gevrey order s if there exists a disc $D \subset \mathbb{C}$ with centre at the origin such that $\mathcal{B}_{\Gamma_s,x}\hat{u} \in \mathcal{O}(D)$. The space of formal power series of Gevrey order s is denoted by $\mathbb{C}[[x]]_s$.

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Remarks

1. We may replace Γ_s in the above definition by any moment function m of order s.
2. If $\hat{u} \in \mathbb{C}[[x]]_s$ and $s \leq 0$ then $\hat{u}$ is convergent, so its sum u is well defined. Moreover $\hat{u} \in \mathbb{C}[[x]]_0 \Leftrightarrow u \in \mathcal{O}(D)$ and $\hat{u} \in \mathbb{C}[[x]]_s \Leftrightarrow u \in \mathcal{O}^{-1/s}(\mathbb{C})$ for $s < 0$.

Borel summability

Now we are ready to define the k-summability of formal power series

Definition

Let $k > 0$ and $d \in \mathbb{R}$. Then $\hat{u} \in \mathbb{C}[[x]]$ is called k-summable in a direction d if there exists a disc-sector $\hat{S}_d := S_d \cup D$ in a direction d such that $\mathcal{B}_{\Gamma_{1/k},x}\hat{u} \in \mathcal{O}^k(\widehat{S}_d)$.

Borel summability

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Definition

Let $k > 0$ and $d \in \mathbb{R}$. Then $\hat{u} \in \mathbb{C}[[x]]$ is called *k-summable in a direction d* if there exists a disc-sector $\hat{S}_d := S_d \cup D$ in a direction d such that $\mathcal{B}_{\Gamma_{1/k},x}\hat{u} \in \mathcal{O}^k(\widehat{S}_d)$.

Remark

By the general theory of moment summability we may replace $\Gamma_{1/k}$ in the above definition by any moment function m of order $1/k$.

Moment operators

Definition

Let m be a moment function. Then the linear operator $\partial_{m,x}\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ defined by

$$\partial_{m,x}\Big(\sum_{j=0}^{\infty} \frac{u_j}{m(j)} x^j\Big) := \sum_{j=0}^{\infty} \frac{u_{j+1}}{m(j)} x^j$$

is called the *m*-moment differential operator $\partial_{m,x}$.

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is called the *m-moment differential operator* $\partial_{m,x}$.

Moreover, the right-inversion operator $\partial_{m,x}^{-1}\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ given by

$$\partial_{m,x}^{-1}\Big(\sum_{j=0}^{\infty}\frac{u_j}{m(j)}x^j\Big) := \sum_{j=1}^{\infty}\frac{u_{j-1}}{m(j)}x^j$$

is called the *m-moment integration operator* $\partial_{m,x}^{-1}$.

Moment differential operators

Example

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- $\partial_{\Gamma_1,x}\hat{u} = \partial_x \hat{u}$.
- $(\partial_{\Gamma_s,x}\hat{u})(x^s) = \partial_x^s(\hat{u}(x^s))$ $(s > 0)$, where ∂_x^s is the Caputo fractional derivative of order s (i.e. $\partial_x^s(\sum_{j=0}^{\infty} \frac{u_j}{\Gamma_s(j)} x^{sj}) := \sum_{j=0}^{\infty} \frac{u_{j+1}}{\Gamma_s(j)} x^{sj}$).

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- $\partial_{\Gamma_0,x}\hat{u}(x) = \partial_{1,x}\hat{u}(x) = \frac{\hat{u}(x)-u_0}{x}$.

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- $\partial_{\Gamma_0,x}\hat{u}(x) = \partial_{1,x}\hat{u}(x) = \frac{\hat{u}(x)-u_0}{x}$.
- $\partial_{\Gamma_{-1},x}\hat{u}(x) = \frac{1}{x}\int_0^x \frac{\hat{u}(y)-u_0}{y}\,dy$.

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- $\partial_{\Gamma_0,x}\hat{u}(x) = \partial_{1,x}\hat{u}(x) = \frac{\hat{u}(x)-u_0}{x}$.
- $\partial_{\Gamma_{-1},x}\hat{u}(x) = \frac{1}{x}\int_0^x \frac{\hat{u}(y)-u_0}{y}\,dy$.
- $(\partial_{\Gamma_{-s},x}\hat{u})(x^s) = \frac{1}{x^s}\partial_x^{-s}\frac{\hat{u}(x^s)-u_0}{x^s}$ ($s > 0$) where ∂_x^{-s} is the right-inversion operator to ∂_x^s and is defined by $\partial_x^{-s}(\sum_{j=0}^{\infty} \frac{u_j}{\Gamma_s(j)} x^{sj}) := \sum_{j=1}^{\infty} \frac{u_{j-1}}{\Gamma_s(j)} x^{sj}$.

Moment operators

The moment differential operator $\partial_{m,z}$ is well-defined for every $\varphi \in \mathcal{O}(D)$. Moreover, if $\varphi \in \mathcal{O}(D_r)$ and m is a moment function of order $1/k > 0$ then for every $|z| < \varepsilon < r$ and $n \in \mathbb{N}$ we have

$$\partial^n_{m,z}\varphi(z) = \frac{1}{2\pi i}\oint_{|w|=\varepsilon} \varphi(w) \int_0^{\infty(\theta)} \zeta^n E_m(z\zeta)\frac{e_m(w\zeta)}{w\zeta}\, d\zeta\, dw,$$

where $\theta \in (-\arg w - \frac{\pi}{2k}, -\arg w + \frac{\pi}{2k})$.

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where $\theta \in (-\arg w - \frac{\pi}{2k}, -\arg w + \frac{\pi}{2k})$.

Using the above formula, we may define a moment pseudodifferential operator $\lambda(\partial_{m,z})\colon \mathcal{O}(D) \to \mathcal{O}(D)$ as an operator satisfying

$$\lambda(\partial_{m,z}) E_m(\zeta z) := \lambda(\zeta) E_m(\zeta z) \quad \text{for} \quad |\zeta| \geq r_0.$$

Hence, if $\lambda(\zeta)$ is an analytic function for $|\zeta| \geq r_0$ of polynomial growth at infinity then $\lambda(\partial_{m,z})$ is defined by

$$\lambda(\partial_{m,z})\varphi(z) := \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_{r_0 e^{i\theta}}^{\infty(\theta)} \lambda(\zeta) E_m(\zeta z) \frac{e_m(\zeta w)}{\zeta w}\, d\zeta\, dw$$

for every $\varphi \in \mathcal{O}(D_r)$ and $|z| < \varepsilon < r$, where $\theta \in (-\arg w - \frac{\pi}{2k}, -\arg w + \frac{\pi}{2k})$.

Commutation formula

The operators $\mathcal{B}_{m',x}, \partial_{m,x}, P(\partial_{m,x})\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ and $\lambda(\partial_{m,x})\colon \mathcal{O}(D) \to \mathcal{O}(D)$ satisfy the following commutation formulas:

1. $\mathcal{B}_{m',x}\partial_{m,x}\hat{u} = \partial_{mm',x}\mathcal{B}_{m',x}\hat{u}$,
2. $\mathcal{B}_{m',x}P(\partial_{m,x})\hat{u} = P(\partial_{mm',x})\mathcal{B}_{m',x}\hat{u}$ for any polynomial P with constant coefficients.
3. $\mathcal{B}_{m',x}\lambda(\partial_{m,x})\hat{u} = \lambda(\partial_{mm',x})\mathcal{B}_{m',x}\hat{u}$ for any moment pseudodifferential operator $\lambda(\partial_{m,x})$.

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3. $\mathcal{B}_{m',x}\lambda(\partial_{m,x})\hat{u} = \lambda(\partial_{mm',x})\mathcal{B}_{m',x}\hat{u}$ for any moment pseudodifferential operator $\lambda(\partial_{m,x})$.

By the above commutation formula we may extend the definition of moment pseudodifferential operators:

Definition

Let $s \in \mathbb{R}$, m be a moment function of order $\tilde{s} \in \mathbb{R}$ and $\lambda(\zeta)$ be a holomorphic function for $|\zeta| \geq r_0$ of polynomial growth at infinity. A moment pseudodifferential operator $\lambda(\partial_{m,z})\colon \mathbb{C}[[z]]_s \to \mathbb{C}[[z]]_s$ is defined by

$$\lambda(\partial_{m,z})\hat{\varphi}(z) := \mathcal{B}_{\Gamma_{-\bar{s}},z}\lambda(\partial_{m\Gamma_{\bar{s}},z})\mathcal{B}_{\Gamma_{\bar{s}},z}\hat{\varphi}(z),$$

where $\hat{\varphi} \in \mathbb{C}[[z]]_s$ and $\bar{s} = \max\{s, 1 - \widetilde{s}\}$ and the operator $\lambda(\partial_{m\Gamma_{\bar{s}},z})$ was constructed in the previous definition.

Factorization of $P(\partial_{m_1,t},\partial_{m_2,z})$

Let $P(\lambda,\zeta)=P_0(\zeta)\lambda^n-\sum_{j=1}^n P_j(\zeta)\lambda^{n-j}$ be a general polynomial of two variables, which is of order n with respect to λ.

Using the moment-pseudodifferential operators we factorize the moment-differential operator $P(\partial_{m_1,t},\partial_{m_2,z})$ as follows

$$\begin{aligned} P(\partial_{m_1,t},\partial_{m_2,z}) &= P_0(\partial_{m_2,z})(\partial_{m_1,t}-\lambda_1(\partial_{m_2,z}))^{n_1}...(\partial_{m_1,t}-\lambda_l(\partial_{m_2,z}))^{n_l} \\ &=: P_0(\partial_{m_2,z})\tilde{P}(\partial_{m_1,t},\partial_{m_2,z}) \end{aligned}$$

where $\lambda_1(\zeta),...,\lambda_l(\zeta)$ are the characteristic roots of $P(\lambda,\zeta)=0$ with multiplicities $n_1,...,n_l$ ($n_1+...+n_l=n$) respectively.

Factorization of $P(\partial_{m_1,t}, \partial_{m_2,z})$

Let $P(\lambda,\zeta) = P_0(\zeta)\lambda^n - \sum_{j=1}^n P_j(\zeta)\lambda^{n-j}$ be a general polynomial of two variables, which is of order n with respect to λ.

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$$\begin{aligned} P(\partial_{m_1,t}, \partial_{m_2,z}) &= P_0(\partial_{m_2,z})(\partial_{m_1,t} - \lambda_1(\partial_{m_2,z}))^{n_1} \ldots (\partial_{m_1,t} - \lambda_l(\partial_{m_2,z}))^{n_l} \\ &=: P_0(\partial_{m_2,z})\tilde{P}(\partial_{m_1,t}, \partial_{m_2,z}) \end{aligned}$$

where $\lambda_1(\zeta), \ldots, \lambda_l(\zeta)$ are the characteristic roots of $P(\lambda,\zeta) = 0$ with multiplicities $n_1, \ldots, n_l$ $(n_1 + \ldots + n_l = n)$ respectively.

In general $\lambda_j(\zeta)$ are algebraic functions, hence they are holomorpic functions of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ (for some $\kappa \in \mathbb{N}$ and for sufficiently large r_0).

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where $\lambda_1(\zeta),\ldots,\lambda_l(\zeta)$ are the characteristic roots of $P(\lambda,\zeta)=0$ with multiplicities $n_1,\ldots,n_l$ $(n_1+\ldots+n_l=n)$ respectively.

In general $\lambda_j(\zeta)$ are algebraic functions, hence they are holomorpic functions of the variable $\xi=\zeta^{1/\kappa}$ for $|\zeta|\geq r_0$ (for some $\kappa\in\mathbb{N}$ and for sufficiently large r_0).

For this reason we use

Proposition

Let $v(t,z)=u(t,z^\kappa)$ and $\tilde{m}_2(u)=m_2(u/\kappa)$. Then $P(\partial_{m_1,t},\partial_{m_2,z})u=0$ if and only if $P(\partial_{m_1,t},\partial^\kappa_{\tilde{m}_2,z})v=0$.

Hence without loss of generality we may assume that $\kappa=1$, $\lambda_j(\zeta)$ is a holomorphic function for $|\zeta|\geq r_0$ and the moment pseudodifferential operators $\lambda_j(\partial_{m_2,z})$ are well defined.

Uniqueness of formal solution

If $P_0(\zeta) \neq \mathrm{const.}$ then the formal solution is not uniquely determined. For this reason we choose a formal power series $\hat{g} \in \mathbb{C}[[t,z]]_{\tilde{s}_1,\tilde{s}_2}$ satisfying the equation $P_0(\partial_{m_2,z})\hat{g} = \hat{f}$. For such $\hat{g}$ we may construct the uniquely determined solution $\hat{u}$ of

$$\begin{cases} \tilde{P}(\partial_{m_1,t},\partial_{m_2,z})\hat{u} = \hat{g} \\ \partial^j_{m_1,t}\hat{u}(0,z) = 0 \text{ for } j = 0,\dots,n-1 \end{cases},$$

which is also a formal solution of

$$\begin{cases} P(\partial_{m_1,t},\partial_{m_2,z})\hat{u} = \hat{f} \\ \partial^j_{m_1,t}\hat{u}(0,z) = 0 \text{ for } j = 0,\dots,n-1 \end{cases} \tag{1}$$

and is called the formal solution of (1) determined by $\widehat{g}$.

Decomposition of equation

By the factorization of operator $\tilde{P}(\partial_{m_1,t}, \partial_{m_2,z})$ we obtain that the formal solution $\hat{u}$ determined by $\hat{g}$ satisfies the decomposition $\hat{u} = \sum_{\alpha=1}^{l} \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$, where $\hat{u}_{\alpha\beta}$ is a formal solution of

$$\begin{cases} (\partial_{m_1,t} - \lambda_\alpha(\partial_{m_2,z}))^\beta \hat{u}_{\alpha\beta} = \hat{g}_{\alpha\beta} \\ \partial^j_{m_1,t} \hat{u}_{\alpha\beta}(0,z) = 0 \quad (j = 0, \dots, \beta-1) \end{cases},$$

where $\hat{g}_{\alpha\beta}(t,z) := d_{\alpha\beta}(\partial_{m_2,z})\hat{g}(t,z) \in \mathbb{C}[[t,z]]_{\tilde{s}_1,\tilde{s}_2}$ and $d_{\alpha\beta}(\zeta)$ is a holomorphic function of polynomial growth.

Gevrey order of formal solution

By the above decomposition it is sufficient to study the moment-pseudodifferential equation

$$\begin{cases} (\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^{\beta} u = \hat{g} \in \mathbb{C}[[t,z]]_{\tilde{s}_1,\tilde{s}_2} \\ \partial^j_{m_1,t} u(0,z) = 0 \ \ (j = 0, \dots, \beta - 1). \end{cases}$$

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For simplicity we assume that $\beta = 1$. In this case the formal solution $\hat{u}$ is given by

$$\hat{u}(t,z) = \sum_{n=0}^{\infty} (\partial_{m_1,t}^{-1})^{n+1} \lambda^{n}(\partial_{m_2,z}) \hat{g}(t,z).$$

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We will study the Gevrey order of formal solution $\hat{u}$, which depends on the orders s_1 and s_2 of moment functions, on the Gevrey orders $\tilde{s}_1$, $\tilde{s}_2$ and depends on the characteristic root $\lambda(\zeta) \sim \lambda_0 \zeta^q$ (i.e. $\lim_{\zeta\to\infty} \frac{\lambda(\zeta)}{\zeta^q} = \lambda_0 \neq 0$.)

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For simplicity we assume that $\beta = 1$. In this case the formal solution $\hat{u}$ is given by

$$\hat{u}(t,z) = \sum_{n=0}^{\infty} (\partial^{-1}_{m_1,t})^{n+1} \lambda^n(\partial_{m_2,z}) \hat{g}(t,z).$$

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Proposition

If $q \geq 0$ then $\hat{u} \in \mathbb{C}[[t,z]]_{\max\{q(s_2+\tilde{s}_2)-s_1,\tilde{s}_1\},\tilde{s}_2}$.

Integral representation of solution

Let m be a moment function of order 0. Then by the general theory of moment summability $\hat{u}$ is k-summable in a direction d iff $\mathcal{B}_{m,t}\hat{u}$ is k-summable in a direction d. Analogously $u \in \mathcal{O}(\hat{S}_d \times D)$ iff $\mathcal{B}_{m,t}u \in \mathcal{O}(\hat{S}_d \times D)$.

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Hence to study analytic continuation or k-summability of $\hat{u}$ we may replace m_1 and m_2 by Γ_{s_1} and Γ_{s_2}, where s_1 and s_2 are orders of m_1 and m_2. So, we may assume that $\hat{u}$ satisfies

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Since

$$(\partial_{\Gamma_s,x}^{-1})^k \varphi(x) = \int_0^{x^{1/s}} \frac{ks(x^{1/s}-y)^{ks-1}}{\Gamma_s(k)} \varphi(y^s)\,dy \quad (s>0),$$

using the definition of moment pseudodifferential operators and the power series representation of $\hat{u}$ we can find the integral representation of solution u

$$\begin{aligned} u(t,z) &= \frac{-1}{2\pi i}\int_0^{t^{1/s_1}} \oint_{|w|=\varepsilon} g(\tau,w) \int_{r_0 e^{i\theta}}^{\infty(\theta)} (\partial_\tau E_{\Gamma_{s_1}}((t^{1/s_1}-\tau)^{s_1})\lambda(\zeta)) \times \\ &\times \quad E_{\Gamma_{s_2}}(\zeta z)\frac{e_{\Gamma_{s_2}}(\zeta w)}{\zeta w}\,d\zeta\,dw\,d\tau =: I[g], \end{aligned}$$

where $g \in \mathcal{O}(D^2)$, $s_1, s_2 > 0$ and $s_1 \geq q s_2$.

Analytic solution

Deforming the path of integration with respect to w in the integral representation of solution we conclude that

Theorem

Let $\lambda(\zeta) \sim \lambda_0\zeta^q$, $s_1, s_2 > 0$, $s_1 = qs_2$, $K > 0$ and $d \in \mathbb{R}$. We assume that u is a solution of

$$(\partial_{\Gamma_{s_1},t} - \lambda(\partial_{\Gamma_{s_2},z}))u = g \in \mathcal{O}(D^2), \qquad u(0,z) = 0.$$

Then $u(t,z) \in \mathcal{O}^{K,qK}(\hat{S}_d \times \hat{S}_{(d+\arg\lambda_0+2k\pi)/q})$ $(k \in \mathbb{N})$ iff $g(t,z) \in \mathcal{O}^{K,qK}(\hat{S}_d \times \hat{S}_{(d+\arg\lambda_0+2k\pi)/q})$ $(k \in \mathbb{N})$.

Summable solution

Let $\hat{u}$ be a formal solution of

$$(\partial_{\Gamma_{s_1},t} - \lambda(\partial_{\Gamma_{s_2},z}))\hat{u} = \hat{g} \in \mathbb{C}[[t,z]]_{\tilde{s}_1,\tilde{s}_2}, \qquad \hat{u}(0,z) = 0. \tag{2}$$

By the Gevrey estimates $\hat{u} \in \mathbb{C}[[t,z]]_{\max\{q(s_2+\tilde{s}_2)-s_1,\tilde{s}_1\},\tilde{s}_2}$. Our aim is a characterisation of summable solutions $\hat{u}$ in terms of inhomogeneity $\hat{g}$.

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We assume that $q(s_2+\tilde{s}_2) \geq s_1 + \tilde{s}_1$ and $q(s_2+\tilde{s}) > s_1$.

Applying the moment Borel transforms $\mathcal{B}_{\Gamma_{q(s_2+\tilde{s}_2)}/\Gamma_{s_1},t}$ and $\mathcal{B}_{\Gamma_{s_2+\tilde{s}_2}/\Gamma_{s_2},z}$ to (2) we obtain

$$(\partial_{\Gamma_{q(s_2+\tilde{s}_2)},t} - \lambda(\partial_{\Gamma_{s_2+\tilde{s}_2},z}))U = G \in \mathcal{O}(D^2), \qquad U(0,z) = 0,$$

where $U(t,z) := \mathcal{B}_{\Gamma_{q(s_2+\tilde{s}_2)}/\Gamma_{s_1},t}\mathcal{B}_{\Gamma_{s_2+\tilde{s}_2}/\Gamma_{s_2},z}\hat{u}(t,z)$ and
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Theorem

Let $\hat{u}$ be a formal solution of (2), $q > 0$, $s_2 + \tilde{s}_2 > 0$, $q(s_2+\tilde{s}_2) \geq s_1 + \tilde{s}_1$, $K := \frac{1}{q(s_2+\tilde{s}_2)}$ and $d \in \mathbb{R}$. If $G \in \mathcal{O}^{K,qK}(\hat{S}_d \times \hat{S}_{(d+\arg\lambda_0+2k\pi)/q})$ $(k \in \mathbb{N})$ then $\hat{u}$ is K-summable in a direction d.

Remarks

1. It is trivial to find the necessary condition for K-summability of $\hat{u}$:

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If $\hat{u}$ is K-summable in a direction d then also $\hat{g}$ is K-summable in the same direction.

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If $\hat{u}$ is K-summable in a direction d then also $\hat{g}$ is K-summable in the same direction.

2. Using the integral representation of solution we can also obtain trivially the characterisation of K-summable solutions $\hat{u}$:

Remark 2

$\hat{u}$ is K-summable in a direction d iff $I[\mathcal{B}_{\Gamma_{q(s_2+\tilde{s}_2)/\Gamma_{s_1}},t}\mathcal{B}_{\Gamma_{s_2+\tilde{s}_2}/\Gamma_{s_2},z}\hat{g}] \in \mathcal{O}^K(\hat{S}_d \times D)$.

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