A connection formula of a divergent bilateral basic hypergeometric function

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Formal and Analytic Solutions of Differential, Difference and Discrete Equations

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1. Notations

Assumptions:

A complex number $q \in \mathbb{C}^*$ is $0 < |q| < 1$.

The q-shifted operator σ_q: $\sigma_q f(x) = f(qx)$.

The q-shifted factorial $(a;q)_n$

$$(a;q)_n := \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & n \geq 1, \\ [(1-aq^{-1})(1-aq^{-2})\dots(1-aq^n)]^{-1}, & n \leq -1 \end{cases}$$

$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n$

$$(a_1, a_2, \dots, a_m; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \dots (a_m;q)_\infty.$$

The basic hypergeometric series with the base q:

$$
{}_r\varphi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,x) := \sum_{n\geq 0} \frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_s;q)_n(q;q)_n} \left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^{1+s-r} x^n.
$$

Radius of convergence:

$\infty, 1$ or 0 according to whether $r-s<1, r-s=1$ or $r-s>1$.

The bilateral basic hypergeometric series with the base q:

$$
{}_r\psi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,x) := \sum_{n\in\mathbb{Z}} \frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_s;q)_n} \left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^{s-r} x^n.
$$

The series ${}_r\psi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,x)$ converges on:

$r<s \quad |x|>R:=\left|\dfrac{b_1b_2\cdots b_s}{a_1a_2\cdots a_r}\right|$

$r=s \quad R<|x|<1$

$s<r$ *divergent around the origin*

2. Aim and main tools:

2.1 Aim — **Connection formmula for the *divergent* bilateral basic hypergeometric series**

Main Theorem. *For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have*

$$\left(\mathcal{L}^+_{q,\lambda} \circ \mathcal{B}^+_q {}_2\psi_1(a_1, a_2; b_1; q, x)\right)(x)$$
$$= \frac{(1/a_2, qa_1/a_2, b_1/a_1, q; q)_\infty}{(b_1, q/a_1, a_1/a_2, qa_2/a_1; q)_\infty} \frac{\theta(a_1\lambda/q)}{\theta(\lambda/q)} \sum_{n\geq 0} \frac{(qa_1/b_1; q)_n (b_1/a_1a_2x)^n}{(qa_1/a_2; q)_n (q; q)_n}$$
$$+ \frac{(1/a_1, qa_2/a_1, b_1/a_2, q; q)_\infty}{(b_1, q/a_2, a_2/a_1, qa_1/a_2; q)_\infty} \frac{\theta(a_2\lambda/q)}{\theta(\lambda/q)} \sum_{n\geq 0} \frac{(qa_2/b_1; q)_n (b_1/a_1a_2x)^n}{(qa_2/a_1; q)_n (q; q)_n},$$

provided that the set $[\lambda; q]$ is the q-spiral such that $[\lambda; q] := \left\{\lambda q^k | k \in \mathbb{Z}\right\}$ for any fixed $\lambda \notin q^{\mathbb{Z}}$.

Remark. The notation "$\left(\mathcal{L}^+_{q,\lambda} \circ \mathcal{B}^+_q {}_2\psi_1(a_1, a_2; b_1; q, x)\right)(x)$" is the suitable resummation of the *divergent series* ${}_2\psi_1(a_1, a_2; b_1; q, x)$.

1. The q-Borel transformation of the first kind is

$$\left(\mathcal{B}_q^+ f\right)(\xi) := \sum_{n\in\mathbb{Z}} a_n q^{\frac{n(n-1)}{2}} \xi^n \left(=: \varphi(\xi)\right).$$

2. The q-Laplace transformation of the first kind is

$$\left(\mathcal{L}_{q,\lambda}^+ \varphi\right)(x) := \frac{1}{1-q}\int_0^{\lambda\infty} \frac{\varphi(\xi)}{\theta_q\left(\frac{\xi}{x}\right)} \frac{d_q\xi}{\xi} = \sum_{n\in\mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)},$$

   here, this transformation is given by Jackson's q-integral.

The theta function of Jacobi:

$$\theta_q(x) := \sum_{n\in\mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n, \qquad \forall x \in \mathbb{C}^*.$$

Properties of the theta function:

1. Jacobi's triple product identity is

$$\theta_q(x) = \left(q, -x, -\frac{q}{x}; q\right)_\infty .$$

2. The q-difference equation which the theta function satisfies;

$$\theta_q(q^k x) = q^{-\frac{n(n-1)}{2}} x^{-k} \theta_q(x), \quad \forall k \in \mathbb{Z}.$$

3. The inversion formula;

$$\theta_q\left(\frac{1}{x}\right) = \frac{1}{x}\theta_q(x).$$

$[\lambda; q]$-spiral: For any fixed $\lambda \in \mathbb{C}^* \setminus q^{\mathbb{Z}}$, the set $[\lambda; q]$-spiral is

$$[\lambda; q] := \lambda q^{\mathbb{Z}} = \{\lambda q^k; k \in \mathbb{Z}\}.$$

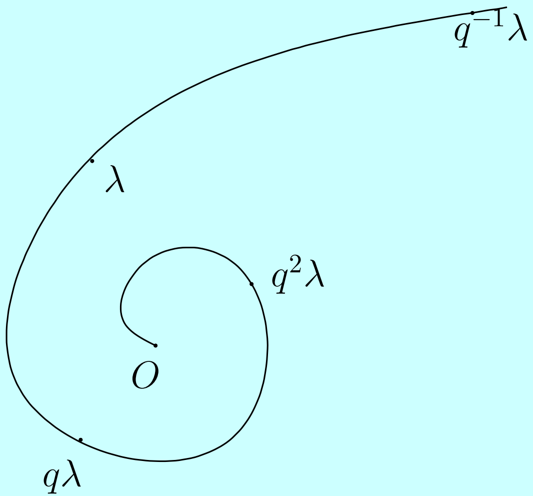


Figure 1. $[\lambda; q]$ – spiral

Relation between the theta function and $[\lambda; q]$-spiral:

Lemma 1. *We have*

$$\theta(\lambda q^k/x) = 0 \overset{iff}{\Longleftrightarrow} x \in [-\lambda; q].$$

3. Linear q-difference equation of the Laplace type

The q-difference equation of the Laplace type:

$$\left\{(a_1x+b_1)\sigma_q^2+(a_2x+b_2)\sigma_q+(a_3x+b_3)\right\}u(x)=0$$

6 parameters: a_1, a_2, a_3, b_1, b_2 and b_3.

By transformations $x \to cx$ and $u \to x^d u$, generic equations reduce to **3 parameters equation:**

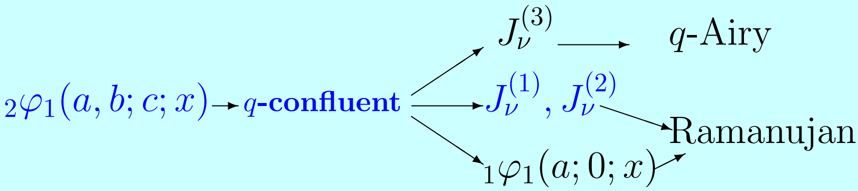
$$\left[(c-abqx)\sigma_q^2-\left\{(c+q)-(a+b)qx\right\}\sigma_q+q(1-x)\right]u(x)=0.$$

A three parameters solution is **Heine's basic hypergeometric series**:

$$u(x)={}_2\varphi_1(a,b;c;q,x)=\sum_{n\geq 0}\frac{(a,b;q)_n}{(c;q)_n(q;q)_n}x^n.$$

The degeneration diagram

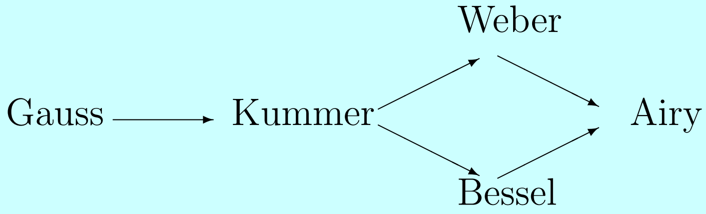
The **degeneration diagram** for ${}_2\varphi_1(a,b;c;q,x)$[Y. Ohyama, 2011]:



1. $J_\nu^{(k)} (k = 1, 2, 3)$ are q-**Bessel functions**.
2. The q-**Airy function** and **the Ramanujan entire function** $A_q(x)$ (Kajiwara, et al., 2004; Ismail, 2005) are q-analogues of the Airy functions.
3. The function ${}_1\varphi_1(a;0;q,x)$ is called **the** q-**Hermite function**.

Remark. $A_q(x)$ is found by Ramanujan in "**the Lost notebook**"

This diagram is a q-analogue of the degeneration diagram for **Gauss' hypergeometric series $_2F_1$**:



Remark. Three q-Bessel functions and two q-Airy functions satisfy different types of q-difference equations.

4.Connection problems on second order linear q-difference equations—unilateral cases

G. D. Birkhoff (1914) Connection formulae of second order linear q-difference equations are linear relations in a **matrix form**:

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} C_{11}(x) & C_{12}(x) \\ C_{21}(x) & C_{22}(x) \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}.$$

$(u_1(x), u_2(x))$:a system of solutions **around the origin**
$(v_1(x), v_2(x))$:a system of solutions **around the infinity**

Functions C_{ij} $(1 \le i, j \le 2)$ are **elliptic functions**:

$$\sigma_q C_{ij}(x) = C_{ij}(x), \quad C_{ij}(e^{2\pi i}x) = C_{ij}(x).$$

5.The first example of the connection matrix

Connection matrix for Heine's ${}_2\varphi_1(a,b;c;q,x)$: Watson's formula

Heine's equation

$$\left[(c-abqx)\sigma_q^2-\{(c+q)-(a+b)qx\}\,\sigma_q+q(1-x)\right]u(x)=0.$$

Local solutions around the origin

$$u_1(x)={}_2\varphi_1(a,b;c;q,x),\quad u_2(x)=\frac{\theta(cx)}{\theta(qx)}{}_2\varphi_1\left(\frac{aq}{c}\frac{bq}{c};\frac{q^2}{c};q,x\right).$$

Local solutions around the infinity

$$y_\infty^{(a,b)}(x)=\frac{\theta(-ax)}{\theta(-x)}{}_2\varphi_1\left(a,\frac{aq}{c};\frac{aq}{b};q,\frac{cq}{abx}\right)$$

and

$$y_\infty^{(b,a)}(x)=\frac{\theta(-bx)}{\theta(-x)}{}_2\varphi_1\left(b,\frac{bq}{c};\frac{bq}{a};q,\frac{cq}{abx}\right).$$

Connection matrix for Heine's equation

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}(x) \end{pmatrix} \begin{pmatrix} y_\infty^{(a,b)}(x) \\ y_\infty^{(b,a)}(x) \end{pmatrix},$$

provided that

$$C_{11} = \frac{(b, c/a; q)_\infty}{(c, b/a; q)_\infty}, \quad C_{12} = \frac{(a, c/b; q)_\infty}{(c, a/b; q)_\infty},$$

$$C_{21} = \frac{(bq/c, q/a; q)_\infty}{(q^2/c, b/a; q)_\infty}$$

and

$$C_{22}(x) = \frac{(aq/c, q/b; q)_\infty}{(q^2/c, a/b; q)_\infty} \frac{\theta(-ax)}{\theta\left(-\frac{aq}{c}x\right)} \frac{\theta\left(-\frac{bq}{c}x\right)}{\theta(-bx)}.$$

Remark. C_{11}, C_{12} and C_{21} are constant and $C_{22}(x)$ is a q-elliptic function.

Remark. The first formula has given by **G. N. Watson (1910)**. Other connection formula for q-difference equation with irregular singular points are obtained by the q-Borel-Laplace transformation. (C. Zhang)

6. The q-Borel-Laplace transformations

We assume that $f(x) = \sum_{n\in\mathbb{Z}} a_n x^n$, $a_0 = 1$.

6.1. The q-Borel-Laplace transformations of the first kind

1. The q-Borel transformation of the first kind is

$$\left(\mathcal{B}_q^+ f\right)(\xi) := \sum_{n\in\mathbb{Z}} a_n q^{\frac{n(n-1)}{2}} \xi^n \left(=: \varphi(\xi)\right).$$

2. The q-Laplace transformation of the first kind is

$$\left(\mathcal{L}_{q,\lambda}^+ \varphi\right)(x) := \frac{1}{1-q}\int_0^{\lambda\infty} \frac{\varphi(\xi)}{\theta_q\left(\frac{\xi}{x}\right)}\frac{d_q\xi}{\xi} = \sum_{n\in\mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)},$$

here, this transformation is given by Jackson's q-integral.

6.2. The q-Borel-Laplace transformations of the second kind

1. The q-Borel transformation of the second kind is

$$(\mathcal{B}_q^- f)(\xi) := \sum_{n\geq 0} a_n q^{-\frac{n(n-1)}{2}} \xi^n \, (=: g(\xi)) \, .$$

2. The q-Laplace transformation of the second kind is

$$\left(\mathcal{L}_q^- g\right)(x) := \frac{1}{2\pi i} \int_{|\xi|=r} g(\xi)\theta_q\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi},$$

   where $r > 0$ is enough small number.

Remark. These resummation methods are introduced by **J.-P. Ramis** and **C. Zhang**.

Remark. The q-Borel transformation is **the formal inverse** of the q-Laplace transformation:

The q-Borel transformation $\mathcal{B}_q^+$ is formal inverse of the q-Laplace transformation $\mathcal{L}_{q,\lambda}^+$:

Lemma 2. *For any entire function $f(x)$, we have*

$$\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ f = f.$$

The q-Borel transformation $\mathcal{B}_q^-$ also can be considered as a formal inverse of the q-Laplace transformation $\mathcal{L}_q^-$.

Lemma 3. *We assume that the function f can be q-Borel transformed to the analytic function $g(\xi)$ around $\xi = 0$. Then, we have*

$$\mathcal{L}_q^- \circ \mathcal{B}_q^- f = f.$$

7. Example of connection formulae

Connection matrix for the q-confluent hypergeometric series

q-confluent hypergeometric equation

$$(1 - abqx)u(xq^2) - \{1 - (a+b)qx\}\, u(xq) - qxu(x) = 0.$$

Local solutions around the origin

$$u_1(x) = {}_2\varphi_0(a, b; -; q, x),$$
$$u_2(x) = \frac{(abx;q)_\infty}{\theta(-qx)}{}_2\varphi_1\left(\frac{q}{a}, \frac{q}{b}; 0; q, abx\right)$$

Local solutions around the infinity

$$S_\mu(a, b; q, x) = \frac{\theta(a\mu x)}{\theta(\mu x)}{}_2\varphi_1\left(a, 0; \frac{aq}{b}; q, \frac{q}{abx}\right),$$
$$S_\mu(b, a; q, x) = \frac{\theta(b\mu x)}{\theta(\mu x)}{}_2\varphi_1\left(b, 0; \frac{bq}{a}; q, \frac{q}{abx}\right),$$

Connection matrix for q-confluent equation

Theorem. *For any* $\lambda, \mu \in \mathbb{C}^*$, $x \in \mathbb{C}^* \setminus [1;q] \cup [-\mu/a;q] \cup [-\lambda;q]$, *we have*

$$\begin{pmatrix} {}_2f_0(a,b;\lambda,q,x) \\ {}_2f_1(a,b;q,x) \end{pmatrix} = \begin{pmatrix} C_\mu^\lambda(a,b;q,x) & C_\mu^\lambda(b,a;q,x) \\ C_\mu(a,b;q,x) & C_\mu(b,a;q,x) \end{pmatrix} \begin{pmatrix} S_\mu(a,b;q,x) \\ S_\mu(b,a;q,x) \end{pmatrix}.$$

- The set $[\lambda;q]$ is **the q-spiral.**
- ${}_2f_0(a,b;\lambda,q,x)$ is the q-Borel-Laplace transform (of the first kind) of ${}_2\varphi_0(a,b;-;q,x)$ (given by C. Zhang).
- ${}_2f_1(a,b;q,x)$ is the q-Borel-Laplace transform (of the second kind) of ${}_2\varphi_1(a,b;0;q,x)$ (Morita).
- $S_\mu(a,b;q,x)$ is the solution of around the infinity.
- $C_\mu^\lambda(a,b;q,x)$ and $C_\mu(a,b;q,x)$ are elliptic functions.

Thanks to these resummation methods, we obtain many connection formulae for q-special functions. But we have new questions:

What is *the connection formulae* for *bilateral basic hypergeometric series*?

Thanks to these resummation methods, we obtain many connection formulae for q-special functions. But we have new questions:

What is *the connection formulae* for *bilateral basic hypergeometric series*?

... Slater *knows an answer:*

Theorem. (L. J. Slater, 1952)

For any $|b_1 \cdots b_r/a_1 \ldots a_r| < |x| < 1$, we have

$$\frac{(b_1, \ldots, b_r, q/a_1, \ldots, q/a_r, x, q/x; q)_\infty}{(qa_1, \ldots qa_r, 1/a_1, \ldots, 1/a_r; q)_\infty} {}_r\psi_r(a_1, \ldots, a_r; b_1, \ldots, b_r; q, x)$$

$$= \frac{a_1^{r-1}(q, qa_1/a_2, \ldots qa_1/a_r, b_1/a_1, \ldots, b_r/a_1, a_1x, q/a_1x; q)_\infty}{(qa_1, 1/a_1, a_1/a_2, \ldots, a_1/a_r, qa_2/a_1, \ldots, qa_r/a_1; q)_\infty}$$

$$\times {}_r\varphi_{r-1}\left(qa_1/b_1, \ldots, qa_1/b_r; qa_1/a_2, \ldots, qa_1/a_r; q, \frac{b_1 \cdots b_r}{a_1 \cdots a_r x}\right)$$

$$+ \mathrm{idem}(a_1; a_2, \ldots, a_r).$$

Remark. This Theorem gives the relation between the bilateral basic hypergeometric series ${}_r\psi_r$ and the basic hipergeometric series ${}_r\varphi_{r-1}$.

Remark 2. The special case ($r = 2, b_2 \mapsto q$) gives **Watson's formula**.

Remark 3. Ramanujan's sum for ${}_1\psi_1$ is the $r = 1$ case of Slater's formula:

Theorem. (**Ramanujan's sum for ${}_1\psi_1$**)

$$
\begin{aligned}
{}_1\psi_1(a; b; q, z) &= \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty} \\
&= \frac{(b/a, q; q)_\infty}{(b, q/a; q)_\infty} \frac{\theta(-az)}{\theta(-z)} {}_1\varphi_0 \left(a; -; q, \frac{q}{az} \right),
\end{aligned}
$$

where $0 < |z| < |1|$.

... but *the degenerated case* have not known.

8. Connection formulae for the bilateral series

q-difference equation:

$$\left(\frac{b_1}{q^2} - a_1a_2x\right)u(q^2x) - \left\{\frac{1}{q} - (a_1+a_2)x\right\}u(qx) - xu(x) = 0.$$

Solution around the origin (**divergent series**):

$${}_2\psi_1(a_1, a_2; b_1; q, x) := \sum_{n\in\mathbb{Z}} \frac{(a_1;q)_n(a_2;q)_n}{(b_1;q)_n(q;q)_n}\left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^{-1} x^n.$$

Solutions around infinity (**convergent series**):

$$v_1(x) = \frac{\theta(a_1x)}{\theta(x)}{}_2\varphi_1\left(\frac{qa_1}{b_1}, 0; \frac{qa_1}{a_2}; q, \frac{b_1}{a_1a_2x}\right),$$

$$v_2(x) = \frac{\theta(a_2x)}{\theta(x)}{}_2\varphi_1\left(\frac{qa_2}{b_1}, 0; \frac{qa_2}{a_1}; q, \frac{b_1}{a_1a_2x}\right)$$

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$$\left(\frac{b_1}{q^2} - a_1a_2x\right) u(q^2x) - \left\{\frac{1}{q} - (a_1+a_2)x\right\} u(qx) - xu(x) = 0.$$

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$$v_2(x) = \frac{\theta(a_2x)}{\theta(x)} {}_2\varphi_1\left(\frac{qa_2}{b_1}, 0; \frac{qa_2}{a_1}; q, \frac{b_1}{a_1a_2x}\right)$$

$\Rightarrow$ We apply the q-Borel transformation.

Proof of main theorem.

$$
{}_2\psi_1(a_1, a_2; b_1; q, x) \xrightarrow{\mathcal{B}_q^+} {}_2\psi_2(a_1, a_2; b_1, 0; q, \xi)
$$

$$
\xrightarrow{\text{Slater's formula}} \xrightarrow{\mathcal{L}_{q,\lambda}^+} \left(\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ {}_2\psi_1(a_1, a_2; b_1; q, x)\right)(x)
$$

$$
\begin{aligned}
&= \frac{(1/a_2, qa_1/a_2, b_1/a_1, q; q)_\infty}{(b_1, q/a_1, a_1/a_2, qa_2/a_1; q)_\infty} \frac{\theta(a_1\lambda/q)}{\theta(\lambda/q)} \frac{\theta(a_1qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_1x)} v_1(x) \\
&+ \frac{(1/a_1, qa_2/a_1, b_1/a_2, q; q)_\infty}{(b_1, q/a_2, a_2/a_1, qa_1/a_2; q)_\infty} \frac{\theta(a_2\lambda/q)}{\theta(\lambda/q)} \frac{\theta(a_2qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_2x)} v_2(x).
\end{aligned}
$$

Remark. The functions $\frac{\theta(a_jqx/\lambda)}{\theta(qx/\lambda)}\frac{\theta(x)}{\theta(a_jx)}$, $(j = 1, 2)$ are **q-elliptic functions**.

Remark. These coefficients are new example of *the Stokes coefficients for q-difference equations.*

Dziękuję!