Massera type theorems in vector-valued analytic functions and hyperfunctions

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classical Massera theorem

J. L. Massera (1950, Duke Math. J. **17**) studied the existence of a periodic solution to a periodic ordinary differential equation of normal form. In the linear case, he gave

Theorem (Massera, linear case)

Consider an equation

$$\frac{dx}{dt} = A(t)x + f(t),$$

where $A : \mathbb{R} \to \mathbb{R}^{m\times m}$ *and* $f : \mathbb{R} \to \mathbb{R}^m$ *are 1-periodic and continuous. Then, the existence of a bounded solution in the future (i.e., a solution defined and bounded on a set* $\{t > t_0\}$ *with some* t_0*) implies the existence of a 1-periodic solution.*

Note: Since periodic C^1-functions are bounded, we have the equivalence:

$$\exists \text{ a bounded solution in the future.} \iff \exists \text{ a 1-periodic solution.}$$

some generalizations

Question

Do such phenomena appear commonly in periodic linear equations?

After Massera, many generalizations have appeared.
Refer, for ex., to

- functional differential equations with delay: Chow-Hale (1974, FE. **17**), Hino-Murakami (1989, Lect. Notes Pure Appl. Math. **118**), etc.
- Banach valued, abstract settings: Shin-Naito (1999, JDE. **153**), Naito-Nguyen-Miyazaki-Shin (2000, JDE. **160**), etc.
- discrete dynamical systems in reflexive Banach spaces and those in sequentially complete locally convex spaces with the sequential Montel property: Zubelevich (2006, Regul. Chaotic Dyn. **11**).

and also the references therein.

my interest

Interest

Is there any counterpart to the Massera type phenomenon in the framework of hyperfunctions?

"Hyperfunctions": a notion of generalized functions, due to Sato (1959, 1960, J. Fac. Sci. Univ. Tokyo, **8**).
Note: There's **NO** notion of boundedness for hyperfunctions on $]t_0, +\infty[$.

Obstacles

1. What is "a bounded solution in the future"?
2. Does the periodicity imply the boundedness in the future?

previous results

We constructed
$\mathscr{B}_{L^\infty}$: the sheaf of *bounded hyperfunctions at infinity* on $\mathbb{D}^1 := \mathbb{R} \sqcup \{\pm\infty\}$, (an extension of the sheaf $\mathscr{B}$ of hyperfunctions on $\mathbb{R}$).

1. We can interpret "a bounded solution in the future" as a solution u in $\mathscr{B}_{L^\infty}$ at $t = +\infty$.
2. Periodic hyperfunctions can be canonically identified with periodic bounded hyperfunctions at $t = +\infty$.

By using $\mathscr{B}_{L^\infty}$, we gave a Massera type theorem ▸ Statement for a class of equations, (O., 2008.)

previous results

The class contains, for ex.,

$$\frac{du}{dt} = A(t)u + \int_0^r B(t,s)u(t-s)ds + f(t),$$

A, B, f: square matrices and a column vector, continuous, and ω-periodic in t. Moreover, A, B are real-analytic in t,
$r > 0$: a constant, (representing the "finite delay").

Note: This result could be extended to some classes of equations, containing integro-differential equations with infinite delay, under an additional assumption, so-called the "fading memory condition", which we mentioned in FASDE II, Aug. 2011.
But, today, we do not focus on this direction.

vector valued cases

Let E be a sequentially complete locally convex space.

The result can be extended to E-valued case, when E admits the sequential Montel property:

Definition (sequential Montel property)

(M) Any bounded sequence in E has a convergent subsequence.

or, when E is reflexive.

Statement in reflexive case

boundedness for hyperfunctions

$\mathbb{D}^1 := \mathbb{R} \sqcup \{\pm\infty\}$: a compactification of $\mathbb{R}$. Consider

$$\begin{array}{ccccccc} \mathscr{O} & \cdots & \mathbb{C} = \mathbb{R} + i\mathbb{R} & \subset & \mathbb{D}^1 + i\mathbb{R} & \cdots & \mathscr{O}_{L^\infty} \\ & & \cup & & \cup & & \\ \mathscr{B} & \cdots & \mathbb{R} =]-\infty, +\infty[& \subset & \mathbb{D}^1 = [-\infty, +\infty] & \cdots & \mathscr{B}_{L^\infty} \end{array}$$

Here,

- $\mathscr{O}$ the sheaf of holomorphic functions on $\mathbb{C}$,
- $\mathscr{B}$ the sheaf of hyperfunctions on $\mathbb{R}$,
- $\mathscr{O}_{L^\infty}$ the sheaf of bounded holomorphic functions on $\mathbb{D}^1 + i\mathbb{R}$,
- $\mathscr{B}_{L^\infty}$ the sheaf of bounded hyperfunctions at infinity on $\mathbb{D}^1$,

boundedness for hyperfunctions

$\mathbb{D}^1 := \mathbb{R} \sqcup \{\pm\infty\}$: a compactification of $\mathbb{R}$. Consider

$$\begin{array}{ccccccc} \mathscr{O} & \cdots & \mathbb{C} = \mathbb{R} + i\mathbb{R} & \subset & \mathbb{D}^1 + i\mathbb{R} & \cdots & {}^E\mathscr{O}_{L^\infty} \\ & & \cup & & \cup & & \\ \mathscr{B} & \cdots & \mathbb{R} =]-\infty, +\infty[& \subset & \mathbb{D}^1 = [-\infty, +\infty] & \cdots & {}^E\mathscr{B}_{L^\infty} \end{array}$$

Here,

$\mathscr{O}$ the sheaf of holomorphic functions on $\mathbb{C}$,

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$\mathscr{O}_{L^\infty}$ the sheaf of bounded holomorphic functions on $\mathbb{D}^1 + i\mathbb{R}$,

$\mathscr{B}_{L^\infty}$ the sheaf of bounded hyperfunctions at infinity on $\mathbb{D}^1$,

and for a sequentially complete Hausdorff locally convex space E,

${}^E\mathscr{O}_{L^\infty}$ the E-valued variant of $\mathscr{O}_{L^\infty}$,

${}^E\mathscr{B}_{L^\infty}$ the E-valued variant of $\mathscr{B}_{L^\infty}$.

bounded hyperfunctions at infinity

Definition (sheaves $\mathscr{O}_{L^\infty}$ and $\mathscr{B}_{L^\infty}$)

1. The sheaf $\mathscr{O}_{L^\infty}$ on $\mathbb{D}^1 + i\mathbb{R}$ is defined by
$$\mathscr{O}_{L^\infty}(U) = \{f \in \mathscr{O}(U \cap \mathbb{C}) \mid \forall L \Subset U, f \text{ is bounded on } L \cap \mathbb{C}\}$$
for any open set $U \subset \mathbb{D}^1 + i\mathbb{R}$.
2. The sheaf $\mathscr{B}_{L^\infty}$ on $\mathbb{D}^1$ is defined as the sheaf associated with the presheaf
$$\mathbb{D}^1 \overset{\text{open}}{\supset} \Omega \mapsto \varinjlim_{U} \frac{\mathscr{O}_{L^\infty}(U \setminus \Omega)}{\mathscr{O}_{L^\infty}(U)}.$$
Here U runs through complex neighborhoods of Ω.

The space $\mathscr{O}_{L^\infty}(U)$ is endowed with a natural Fréchet topology.

vector valued variants

E: a sequentially complete Hausdorff locally convex space,
${}^{E}\mathscr{O}$: the sheaf of E-valued holomorphic functions on $\mathbb{C}$.

Definition (sheaves ${}^{E}\mathscr{O}_{L^\infty}$ and ${}^{E}\mathscr{B}_{L^\infty}$)

1. The sheaf ${}^{E}\mathscr{O}_{L^\infty}$ on $\mathbb{D}^1 + i\mathbb{R}$ is defined by
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$$ \mathbb{D}^1 \overset{\text{open}}{\supset} \Omega \mapsto \varinjlim_{U} \frac{{}^{E}\mathscr{O}_{L^\infty}(U \setminus \Omega)}{{}^{E}\mathscr{O}_{L^\infty}(U)}. $$
Here U runs through complex neighborhoods of Ω.

The space ${}^{E}\mathscr{O}_{L^\infty}(U)$ is endowed with a natural locally convex topology.

properties of bounded hyperfunctions

- $\mathscr{B}_{L^\infty}|_{\mathbb{R}} = \mathscr{B}$.
- $\mathscr{B}_{L^\infty}$ is flabby.
- $u \in \mathscr{B}_{L^\infty}(]a, +\infty])$ admits a boundary value representation.
- There exists a natural embedding $L^\infty(]a, +\infty[) \hookrightarrow \mathscr{B}_{L^\infty}(]a, +\infty])$.
- The space $\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ of the global sections of our sheaf $\mathscr{B}_{L^\infty}$ can be identified with the space $\mathcal{B}_{L^\infty}$ of bounded hyperfunctions due to Chung-Kim-Lee (2000, Proc. AMS. **128**).

operators for bounded hyperfunctions

$K = [a, b]$: a closed interval in $\mathbb{R}$ (including the case $K = \{a\}$),
U: an open set in $\mathbb{D}^1 + i\mathbb{R}$,
$P = \{P_V\}_{V \subset U}$: a family of linear continuous maps

$$P_V : {}^E\mathscr{O}_{L^\infty}(V + K) \to {}^E\mathscr{O}_{L^\infty}(V).$$

Definition (Operators of type K)

P is said to be *an operator of type* K for ${}^E\mathscr{O}_{L^\infty}$ on U, if the diagram below commutes for any pair of open sets $V_1 \supset V_2$ in U.

$$\begin{tikzcd} {}^E\mathscr{O}_{L^\infty}(V_1 + K) \arrow[r, "P_{V_1}"] \arrow[d, "\text{restriction}"'] & {}^E\mathscr{O}_{L^\infty}(V_1) \arrow[d, "\text{restriction}"] \\ {}^E\mathscr{O}_{L^\infty}(V_2 + K) \arrow[r, "P_{V_2}"] & {}^E\mathscr{O}_{L^\infty}(V_2) \end{tikzcd}$$

properties of operators of type K

An operator P of type K for ${}^{E}\mathscr{O}_{L^\infty}$ on U induces a family of linear maps

$$P_\Omega : {}^{E}\mathscr{B}_{L^\infty}(\Omega + K) \to {}^{E}\mathscr{B}_{L^\infty}(\Omega), \quad \text{for open sets } \Omega \subset \mathbb{D}^1 \cap U,$$

commuting with restrictions.

- An operator of type $K = \{0\}$ induces a local operator.
- An operator of type $K = [-r, 0]$ induces an operator of finite delay r.

typical examples of our operators

$U := \mathbb{D}^1 + i]{-d}, d[$, $U^\circ := U \cap \mathbb{C} = \mathbb{R}^1 + i]{-d}, d[$: strip domains,
$\omega > 0$: a constant.

Example

- differential operator $\sum_{j=0}^{m} a_j(t)\partial_t^j$ with coefficients $a_j \in \mathscr{O}_{L^\infty}(U)$ is an operator of type $K = \{0\}$.
- translation operator $T_\omega : u(t) \mapsto u(t+\omega)$ is an operator of type $K = \{\omega\}$.
- difference operator $T_\omega - 1 : u(t) \mapsto u(t+\omega) - u(t)$ is an operator of type $K = [0, \omega]$.
- integral operator with finite delay $u(t) \mapsto \int_0^r k(t,s)u(t-s)ds$ is an operator of type $K = [-r, 0]$, if the kernel $k(w,s)$ belongs to $(C \cap L^\infty)(U^\circ \times [0,r])$, and is holomorphic in w.

periodicity for bounded hyperfunctions

T_ω: the ω-translation operator $u(t) \mapsto u(t+\omega)$, $(\omega > 0)$.

We introduce the notion of ω-periodicity
for bounded hyperfunction u by the equation $(T_\omega - 1)u = 0$, and
for operators P of type K by the commutativity $P \circ T_\omega = T_\omega \circ P$.

Then, we have,

Fact

- *Every ω-periodic hyperfunction $f \in {}^E\mathscr{B}(\mathbb{R})$ has the unique ω-periodic extension $\hat{f} \in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$.*
- *Every ω-periodic bounded hyperfunction $f \in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ admits an ω-periodic boundary value representation.*

scalar valued case

$\omega > 0$, $K = [a, b] \subset \mathbb{R}$, and $U = \mathbb{D}^1 + i]{-d}, d[$.

P: an ω-periodic operator of type K for $\mathscr{O}_{L^\infty}$ on U,
$f \in \mathscr{B}(\mathbb{R})$: an ω-periodic hyperfunction,
its unique ω-periodic extension in $\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ is also denoted by f,
$(\mathscr{B}_{L^\infty})_{+\infty} = \varinjlim_R \mathscr{B}_{L^\infty}(]R, +\infty])$: the stalk of $\mathscr{B}_{L^\infty}$ at $+\infty$.

Theorem

$Pu = f$ has an ω-periodic $\mathscr{B}(\mathbb{R})$-solution if and only if it has an $(\mathscr{B}_{L^\infty})_{+\infty}$-solution.

back

reflexive valued cases

$\omega > 0$, $K = [a, b] \subset \mathbb{R}$, and $U = \mathbb{D}^1 + i]-d, d[$.

E: a reflexive locally convex space.
P: an ω-periodic operator of type K for ${}^E\mathscr{O}_{L^\infty}$ on U,
$f \in {}^E\mathscr{B}(\mathbb{R})$: an ω-periodic E-valued hyperfunction,
its unique ω-periodic extension in ${}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ is also denoted by f,
$({}^E\mathscr{B}_{L^\infty})_{+\infty} = \varinjlim_R {}^E\mathscr{B}_{L^\infty}(]R, +\infty])$: the stalk of ${}^E\mathscr{B}_{L^\infty}$ at $+\infty$.

Theorem (reflexive case)

$Pu = f$ has an ω-periodic ${}^E\mathscr{B}(\mathbb{R})$-solution if and only if it has an $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solution.

back

analytic solutions

$\omega > 0$, $K = [a, b] \subset \mathbb{R}$, and $U = \mathbb{D}^1 + i]{-d}, d[$.

E: a sequentially complete Hausdorff locally convex space.
P: an ω-periodic operator of type K for ${}^E\mathscr{O}_{L^\infty}$ on U,
$f \in {}^E\mathscr{O}(\mathbb{R})$: an ω-periodic E-valued analytic function,
its unique ω-periodic extension in ${}^E\mathscr{O}_{L^\infty}(\mathbb{D}^1)$ is also denoted by f,
$({}^E\mathscr{O}_{L^\infty})_{+\infty} = \varinjlim_R {}^E\mathscr{O}_{L^\infty}(]R, +\infty])$: the stalk of ${}^E\mathscr{O}_{L^\infty}$ at $+\infty$.

Theorem

Assume the sequential Montel property or the reflexivity for E. Then $Pu = f$ has an ω-periodic ${}^E\mathscr{O}(\mathbb{R})$-solution if and only if it has an $({}^E\mathscr{O}_{L^\infty})_{+\infty}$-solution.

idea of the proof (simplest case), 1

We give the idea of the proof of the theorem for analytic solutions, of the part

"$\exists$ a solution u_0 in $({}^{E}\mathscr{O}_{L^\infty})_{+\infty} \Rightarrow \exists$ an ω-periodic solution u in ${}^{E}\mathscr{O}(\mathbb{R})$",

in the case that P is of type $K = \{0\}$, (that is, P is a local operator).

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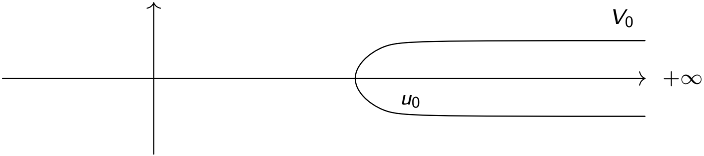
in the case that P is of type $K = \{0\}$, (that is, P is a local operator).

We can find a neighborhood $V_0 \subset \mathbb{D}^1 + i\mathbb{R}$ of $+\infty$, such that $u_0 \in {}^E\mathscr{O}_{L^\infty}(V_0)$ and that $Pu_0 = f$ on V_0.

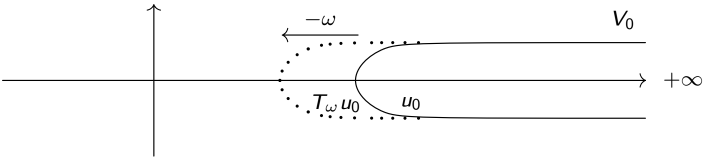
idea of the proof (simplest case), 2

u_0 is a solution on V_0, and bounded there.



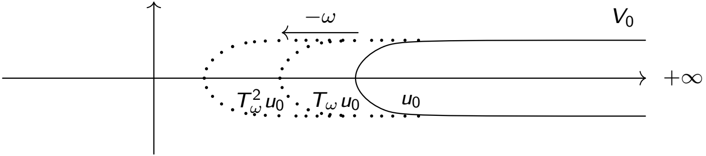
idea of the proof (simplest case), 2

u_0 is a solution on V_0, and bounded there.
$T_\omega u_0$ is a solution on $V_0 + \{-\omega\}$.



idea of the proof (simplest case), 2

u_0 is a solution on V_0, and bounded there.
$T_\omega u_0$ is a solution on $V_0 + \{-\omega\}$.
$T_\omega^2 u_0 = T_{2\omega} u$ is a solution on $V_0 + \{-2\omega\}$.



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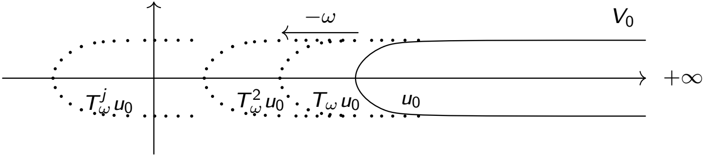
...



idea of the proof (simplest case), 2

u_0 is a solution on V_0, and bounded there.
$T_\omega u_0$ is a solution on $V_0 + \{-\omega\}$.
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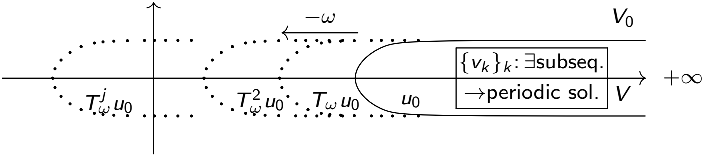
$\cdots$



We define $v_k := \frac{1}{k}\sum_{j=0}^{k-1} T_\omega^j u_0$, and consider a bounded sequence $\{v_k\}_k$ in ${}^E\mathscr{O}_{L^\infty}(V)$ on a domain $V = V_1 + [0, \omega] \subset V_0 \cap \mathbb{C}$ with some convex domain $V_1 \subset V_0 \cap \mathbb{C}$.

idea of the proof (simplest case), 3

Scalar (or sequential Montel) case:

- We can show a Montel type lemma for ${}^{E}\mathscr{O}(V)$.
- By applying it, we can choose a convergent subsequence from $\{v_k\}_k$, and its limit is an ω-periodic solution.

notes on reflexive locally convex spaces

Consider the case that E is a reflexive locally convex space. We denote

- by E_{w}, the space E endowed with the weak topology, and
- by E', the dual space of E endowed with the strong topology.

We can use the following facts, ($V \subset \mathbb{C}$):

Fact

- ${}^{E}\mathscr{O} = {}^{E_{\mathrm{w}}}\mathscr{O}$, *algebraically.*
- ${}^{E}\mathscr{O}$ *admits (a weak form of) the Köthe duality.*
- P_V *is sequentially closed as a map* ${}^{E_{\mathrm{w}}}\mathscr{O}(V+K) \to {}^{E_{\mathrm{w}}}\mathscr{O}(V)$.
- ${}^{E}\mathscr{O}(V)$ *admits (a weak form of) Montel type lemma*

Here ...

the Köthe duality

E: a reflexive locally convex space, E': its strong dual.

For a compact $L \subset \mathbb{C}$ and an open $V \supset L$, we define a bilinear form

$$\langle \cdot, \cdot \rangle_L : {}^{E'}\mathscr{O}(V \setminus L) \times {}^{E}\mathscr{O}(L) \to \mathbb{C},\ \langle \varphi, v \rangle_L := \int_\gamma \varphi(w)(v(w))dw,$$

by taking a suitable contour γ.

Theorem (a weak form of the Köthe duality)

The bilinear form $\langle \cdot, \cdot \rangle_L$ induces the isomorphisms between vector spaces:

$$\frac{{}^{E'}\mathscr{O}(V \setminus L)}{{}^{E'}\mathscr{O}(V)} \xrightarrow{\sim} ({}^{E}\mathscr{O}(L))', \quad {}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus L) \xrightarrow{\sim} ({}^{E}\mathscr{O}(L))'.$$

Here ${}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus L)$ denotes $\{\varphi \in {}^{E'}\mathscr{O}(\mathbb{C} \setminus L) \mid \lim_{|w| \to \infty} \varphi(w) = 0\}$.

Montel type lemma

E: reflexive, $V \subset \mathbb{C}$: open, $\langle \cdot, \cdot \rangle_L : {}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus L) \times {}^{E}\mathscr{O}(L) \to \mathbb{C}$: a bilinear form in the Köthe duality.

Lemma (a weak form of the Montel type lemma)

Let $\{f_n\}_n$ be a bounded sequence in ${}^{E}\mathscr{O}(V)$. Then, there exists $f \in {}^{E}\mathscr{O}(V)$ satisfying the following property: For any compact $L \Subset V$ and any $F \in {}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus L)$, we can take a subsequence $\{n(k)\}_k$ such that

$$\lim_{k \to \infty} \langle F, f_{n(k)} \rangle_L = \langle F, f \rangle_L.$$

Note: When E is a reflexive Banach space, the subsequence $\{n(k)\}_k$ can be taken independently of L and F. But in the general case, it may depend on the choice of L and F.

idea of the proof (reflexive case)

Reflexive Banach case:

- E_{w} admits the sequential Montel property.
- P_V is sequentially closed in ${}^{E_{\mathrm{w}}}\mathscr{O}$ topologies.

Therefore, a similar proof for ${}^{E_{\mathrm{w}}}\mathscr{O}(V)$ instead of ${}^{E}\mathscr{O}(V)$ can be applied.

Reflexive case:
We can not expect the sequential Montel property for E_{w}. But nevertheless we can bypass that part by using the weak form of Montel type lemma.

Thank you for your attention.

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