Linear differential equations on the Riemann sphere

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§ Special Functions and Fuchsian differential equations

$f(x)$: a special function defined locally on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$

analytically continued along any path in $X = \mathbb{P}^1(\mathbb{C}) \setminus \{c_0, \ldots, c_p\}$

$\forall U$: simply connected open $\subset X$

($\forall$analytic continuations of f) $\in \mathcal{F}(U) \subset \mathcal{O}(U)$

1. $\dim \mathcal{F}(U) = n$
2. V: simply connected $\subset U \Rightarrow \mathcal{F}(U)\big|_V = \mathcal{F}(V)$
3. $c_j = 0 \;\Rightarrow\; |f(x)| < \exists C|x|^{-\exists m} \quad$ ($|\arg x|$: bounded and $x \to 0$)
   (by linear frac. transf.)

$$\boxed{\{\text{Fuchsian differential eq. } Pu = 0 \text{ of order } n\} \xrightarrow{\sim} \{\mathcal{F} \text{ of } \operatorname{rank} n\}}$$

$$P = \Bigl(\prod_{j=1}^{p}(x - c_j)^n\Bigr)\partial^n + a_{n-1}(x)\partial^{n-1} + \cdots + a_1(x)\partial + a_0(x),\ \partial := \tfrac{d}{dx}$$

(order of zeros of $a_\nu(x)$ at c_j) $\geq \nu$ and $\deg a_\nu(x) \leq n(p-1) + \nu$

$u_j(z) = (u_{j,1}, \ldots, u_{j,n})$: local solution at $c_j = 0$

$$u_{j,i} \sim x^{\lambda_{j,i}} \log^{k_{j,i}} x \ (i = 1, \ldots, n), \qquad \{(\lambda_{j,i}, k_{j,i})\,;\, i = 1, \ldots, n\}$$

$\{\lambda_{j,i} \mid i = 1, \ldots, n\}$: characteristic exponents at c_j

Connection problem:

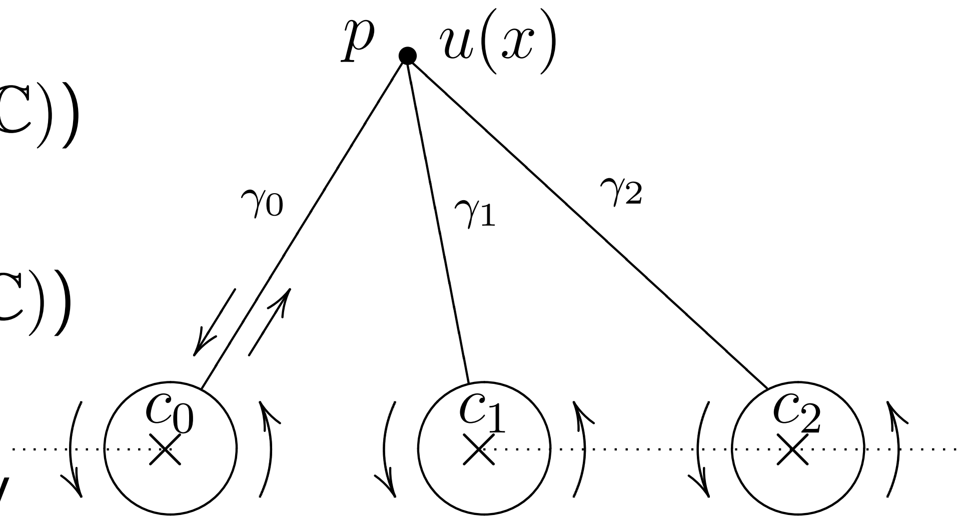
$$u_j(z) = u(z) G_j \quad (G_j \in GL(n, \mathbb{C}))$$

Monodromy:

$$\gamma_j u(z) = u(z) M_j \ (M_j \in GL(n, \mathbb{C}))$$

$$M_j = G_j e^{2\pi\sqrt{-1}A_j} G_j^{-1}$$

$\sim e^{2\pi\sqrt{-1}A_j} \leftarrow$ Local monodromy



Suppose M_j is semisimple. Then $A_j = \operatorname{diag}(\lambda_{j,1}, \ldots, \lambda_{j,n})$.

Deligne-Katz-Simpson problem:

$\boxed{\text{Given local monodromies} \Rightarrow \exists?\ \text{the Fuchsian differential equation}}$

Multiplicities of eigenvalues of M_j $\Rightarrow$ partitions of n

$n = m_{j,1} + \cdots + m_{j,n_j}$ $(j = 0, \ldots, p)$: spectral type

char. exponents at c_j : $\{\lambda_{j,\nu} + i \mid \nu = 1, \ldots, n_j,\ 0 \le i < m_{j,\nu}\}$

$$[\lambda]_{(m)} := \begin{pmatrix} \lambda \\ \lambda+1 \\ \vdots \\ \lambda+m-1 \end{pmatrix}, \quad m = 0, 1, \ldots$$

Def. P has the generalized Riemann scheme (GRS)

$$\{\lambda_{\mathbf{m}}\} := \left\{ \begin{matrix} x = c_0 = \infty & c_1 & \cdots & c_p & \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} & \\ \vdots & \vdots & \vdots & \vdots & ;\, x \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} & \end{matrix} \right\}$$

$$\mathbf{m} = (\mathbf{m}_0, \ldots, \mathbf{m}_p) = \big((m_{0,1}, \ldots, m_{0,n_0}), \ldots, (m_{p,1}, \ldots, m_{p,n_p})\big)$$

: $(p+1)$-tuples of partitions of $n = \operatorname{ord}\mathbf{m}$

Semisimple local monodromy if $\lambda_{j,\nu} - \lambda_{j,\nu'} \notin \mathbb{Z} \quad (\nu \neq \nu')$

$$\operatorname{rank}(M_j - e^{2\pi\sqrt{-1}\lambda_{j,1}}) \cdots (M_j - e^{2\pi\sqrt{-1}\lambda_{j,k}}) = n - m_{j,1} - \cdots - m_{j,k}$$

$(\Uparrow\ \forall j, k$ if $m_{j,1} \geq m_{j,2} \geq \cdots$ and $\lambda_{j,\nu} - \lambda_{j,\nu'} \notin \mathbb{Z} \setminus \{0\})$

$\boxed{M_j \sim \text{block upper triangular with } \nu\text{-th diagonal elem. } e^{2\pi\sqrt{-1}\lambda_{j,\nu}} \mathrm{I}_{m_{j,\nu}}}$

$$[\lambda]_{(m)} := \begin{pmatrix} \lambda \\ \lambda+1 \\ \vdots \\ \lambda+m-1 \end{pmatrix}, \quad m = 0, 1, \ldots$$

Def. P has the generalized Riemann scheme (GRS)

$$\{\lambda_{\mathbf{m}}\} := \left\{ \begin{matrix} x = c_0 = \infty & c_1 & \cdots & c_p \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots & \vdots & ;\ x \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{matrix} \right\}$$

$$\mathbf{m} = (\mathbf{m}_0, \ldots, \mathbf{m}_p) = \big((m_{0,1}, \ldots, m_{0,n_0}), \ldots, (m_{p,1}, \ldots, m_{p,n_p})\big)$$

: $(p+1)$-tuples of partitions of $n = \operatorname{ord} \mathbf{m}$

Semisimple local monodromy if $\lambda_{j,\nu} - \lambda_{j,\nu'} \notin \mathbb{Z} \quad (\nu \neq \nu')$

Fuchs condition (FC): $|\{\lambda_{\mathbf{m}}\}| := \sum m_{j,\nu}\lambda_{j,\nu} - \operatorname{ord} \mathbf{m} + \frac{1}{2} \operatorname{idx} \mathbf{m} = 0$

$\operatorname{idx} \mathbf{m} := \sum_{j,\nu} m_{j,\nu}^2 - (p-1)(\operatorname{ord} \mathbf{m})^2$ (index of rigidity, Katz)

If the equation exists, is it unique or how to parametrize it?

§ Examples (spectral types)

Gauss HG: $11, 11, 11$ $\quad (2 = 1 + 1 = 1 + 1 = 1 + 1)$

Generalized HG: $1^n, n{-}11, 1^n$ $\quad (n = 1 + \cdots + 1 = (n-1) + 1 = 1 + \cdot$

Jordan-Pochhammer: $\overbrace{1p-1, 1p-1, \ldots, 1p-1}^{p}$

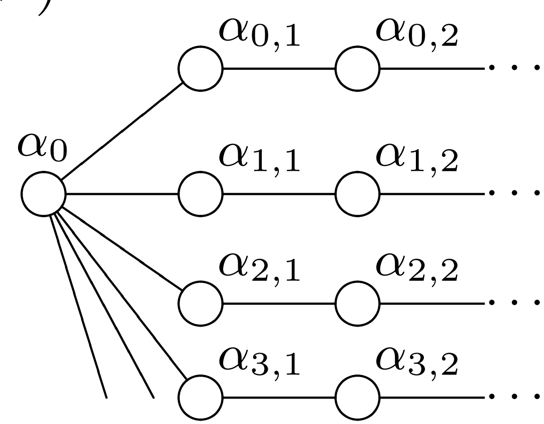
Heun: $11, 11, 11, 11$

$$\text{GHG} : \left\{ \begin{matrix} x = 0 & 1 & \infty & \\ 1-\beta_1 & [0]_{(n-1)} & \alpha_1 & \\ \vdots & & \vdots & ;\ x \\ 1-\beta_{n-1} & & \alpha_{n-1} & \\ {\color{red}0} & -\beta_n & \alpha_n & \end{matrix} \right\}, \quad \sum_{\nu=1}^{n} \alpha_\nu = \sum_{\nu=1}^{n} \beta_\nu$$

$$_nF_{n-1}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1}; x) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_n)_k}{(\beta_1)_k \ldots (\beta_{n-1})_{n-1} k!} x^k$$

with $\quad (\gamma)_k := \gamma(\gamma+1)\cdots(\gamma+k-1)$

§ A Kac-Moody root system (Π, W)



$$(\alpha|\alpha) = 2 \quad (\alpha \in \Pi), \quad (\alpha_0|\alpha_{j,\nu}) = -\delta_{\nu,1},$$

$$(\alpha_{i,\mu}|\alpha_{j,\nu}) = \begin{cases} 0 & (i \neq j \quad \text{or} \quad |\mu - \nu| > 1) \\ -1 & (i = j \quad \text{and} \quad |\mu - \nu| = 1) \end{cases}$$

$W := \langle s_\alpha : x \mapsto x - (\alpha|x)\alpha \mid \alpha \in \Pi \rangle$: Weyl group

Δ_+^{re}: positive real roots $\qquad \Delta_+ = \Delta_+^{re} \cup \Delta_+^{im} \qquad (W\alpha_0 = \Delta_+^{re} \cup \Delta_-^{re})$

Δ_+^{im}: positive imaginary roots $(k\Delta_+^{im} \subset \Delta_+^{im} = W\Delta_+^{im}, k = 2, 3, \ldots)$

$\alpha_{\mathbf{m}} := (\operatorname{ord} \mathbf{m})\alpha_0 + \sum_{j \geq 0,\, k \geq 1} \sum_{\nu > k} m_{j,\nu} \alpha_{j,k}$ (Crawley-Boevey)

$$\Lambda(\lambda) := -\Lambda_0 - \sum_{j \geq 0} \sum_{\nu \geq 1} \Bigl(\sum_{1 \leq i \leq \nu} \lambda_{j,i} \Bigr) \alpha_{j,\nu} \in \mathfrak{h}^\vee := \prod_{\alpha \in \Pi} \mathbb{C}\alpha / \mathbb{C}\Lambda^0$$

$$\begin{array}{ccc}
\{Pu = 0 \text{ with } \{\lambda_{\mathbf{m}}\}, \text{(FC)}\} & \xrightarrow{\pi} & \{(\Lambda(\lambda), \alpha_{\mathbf{m}}) \in \mathfrak{h}^\vee \times \overline{\Delta}_+, \text{(FC)}\} \\
\downarrow u \mapsto \begin{cases} \partial^{-\mu} u \\ (x - c_j)^{\lambda_j} u \end{cases} & \circlearrowright & \downarrow \begin{cases} s_\alpha \ (\alpha \in \Pi) : \text{ reflections} \\ +\lambda_j \Lambda_{0,j}^0 \end{cases} \\
\{Pu = 0 \text{ with } \{\lambda_{\mathbf{m}}\}, \text{(FC)}\} & \xrightarrow{\pi} & \{(\Lambda(\lambda), \alpha_{\mathbf{m}}) \in \mathfrak{h}^\vee \times \overline{\Delta}_+, \text{(FC)}\}
\end{array}$$

$\overline{\Delta}_+ := \{k\alpha\,;\ \alpha \in \Delta_+,\ \operatorname{supp}\alpha \ni \alpha_0,\ k = 1, 2, \ldots\}$

$\Lambda_0 := \frac{1}{2}\alpha_0 + \frac{1}{2}\sum_{j\geq 0}\sum_{\nu\geq 1}(1-\nu)\alpha_{j,\nu}$

$\Lambda_{j,\nu} := \sum_{i>\nu}(\nu - i)\alpha_{j,i} \quad (j = 0, \ldots, p,\ \nu = 0, 1, 2, \ldots)$

$\Lambda^0 := 2\Lambda_0 - 2\Lambda_{0,0}, \qquad \Lambda^0_{j,k} := \Lambda_{j,0} - \Lambda_{k,0}$

$(\partial^{-\mu}u)(x) := \frac{1}{\Gamma(\mu)}\int_{c_j}^{x} u(t)(x-t)^{\mu-1}dt, \quad \partial \mapsto \partial,\ x\partial \mapsto x\partial - \mu$

FC : $|\{\lambda_{\mathbf{m}}\}| = (\Lambda(\lambda) + \frac{1}{2}\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}}) = 0, \quad \operatorname{idx}\mathbf{m} = (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}})$

Theorem. The image of $\xrightarrow{\pi}$ is characterized by $\alpha_{\mathbf{m}} \in \overline{\Delta}_+$ and FC.

Generically irreducible $\Leftrightarrow \alpha_{\mathbf{m}} \in \overline{\Delta}_+$: indivisible or $(\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}}) < 0$

Under this cond. the fiber of $\xrightarrow{\pi}$ is $\mathbb{C}^N$ with $N = 1 - \frac{1}{2}\operatorname{idx}\mathbf{m}$

$\exists_1$universal operator $P_{\mathbf{m}}(\lambda)$ for $\{\lambda_{\mathbf{m}}\}$ with coef. in $\mathbb{C}[\lambda, g_1, \ldots, g_N]$.

Here g_i are N accessory parameters satisfying $\deg_{g_1,\ldots,g_N} P_{\mathbf{m}}(\lambda) = 1$

Def. $\mathbf{m}$ is rigid $\overset{\text{def}}{\Leftrightarrow}$ irreducibly realizable and $\operatorname{idx}\mathbf{m} = 2 \quad (\Rightarrow N = 0)$

$(\Leftrightarrow \alpha_{\mathbf{m}} \in \Delta_+^{re}$ with $\operatorname{supp}\alpha_{\mathbf{m}} \ni \alpha_0)$

Rigid spectral types : 9 (ord ≤ 4), 306 (ord $= 10$), 19286 (ord $= 20$)

Rigid spectral types

ord $= 2$ $\quad 11, 11, 11$ ($_2F_1$; Gauss)

ord $= 3$ $\quad 111, 111, 21$ ($_3F_2$) $\qquad 21, 21, 21, 21$(Pochhammer)

ord $= 4$ $\quad 1^4, 1^4, 31$ ($_4F_3$) $\qquad 1^4, 211, 22$ (Even family) $\quad 211, 211, 211$

$31, 31, 31, 31, 31$ (Pochhammer) $\qquad 211, 22, 31, 31 \qquad 22, 22, 22, 31$

Simpson's list 1991: $\quad 1^n, 1^n, n-11 \quad 1^n, [\frac{n}{2}][\frac{n-1}{2}]1, [\frac{n+1}{2}][\frac{n}{2}] \quad 1^6, 42, 2^3$

Remark. The existence of $P_{\mathbf{m}}(\lambda)$ for fixed rigid $\mathbf{m}$ and $\{\lambda_{\mathbf{m}}\}$ was an open problem by N. Katz (Rigid Local Systems, 1995).

Reduction by "fractional operations" $\Leftarrow W$ (Katz's middle convol.)

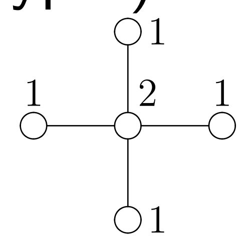
Theorem. $\exists$ only finite basic spectral types with a fixed index

$\mathbf{m} \to$ *trivial*($\Leftarrow$ $\mathbf{m}$: rigid) or basic ($\alpha_{\mathbf{m}} \in$ negative Weyl chamber)

$\operatorname{idx} \mathbf{m} = 0 \ \to \ \tilde{D}_4$ ($\to$ Painlevé VI), $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ (3+1 types)

$\operatorname{idx} \mathbf{m} = -2 \ \to$ 9+3+1 types, etc...

$11, 11, 11, 11 : \ \tilde{D}_4$



§ Fractional calculus of Weyl algebra

Unified and computable interpretation (⇒ a computer program) of

- Existence and Construction of equations
- Integral representation of solutions
- Series expansion of solutions
- Reducibility of the monodromy
- Contiguity relations
- Polynomial solutions
- Connection problem
- Confluences/Unfoldings and Irregular Singularities
- Several variables (PDE) (Appell HG. etc)

$\mathrm{RAd}\big(\partial^{-\mu}\big)P := \partial^{-m}\mathrm{Ad}\,(\partial^{-\mu})\partial^N P$ (Fix N and take maximal m)

$\mathrm{Ad}(\partial^{-\mu})\ : \partial \mapsto \partial,\ x\partial \mapsto x\partial - \mu$ (May take $N = \deg_x P$)

This corresponds to Katz's middle convolution.

Suppose $\mathbf{m}$ is a rigid spectral type (for simplicity)

$\exists_1 w_{\mathbf{m}} \in W$ such that $w_{\mathbf{m}}\alpha_{\mathbf{m}} = \alpha_0$ with the minimal length

$\Delta(\mathbf{m}) := \Delta_+^{re} \cap w_{\mathbf{m}}^{-1}\Delta_-^{re}$

Theorem. $P_{\mathbf{m}}(\lambda)u = 0$ is irred. $\Leftrightarrow$ $\boxed{(\Lambda(\lambda)|\alpha) \notin \mathbb{Z} \quad (\forall \alpha \in \Delta(\mathbf{m}))}$

Theorem. Suppose $m_{0,n_0} = m_{1,n_1} = 1$, $c_0 = 0$, $c_1 = 1$.

$$c(0 : \lambda_{0,n_0} \rightsquigarrow 1 : \lambda_{1,n_1}) = \frac{\displaystyle\prod_{\nu=1}^{n_0-1} \Gamma\big(\lambda_{0,n_0} - \lambda_{0,\nu} + 1\big) \cdot \prod_{\nu=1}^{n_1-1} \Gamma\big(\lambda_{1,\nu} - \lambda_{1,n_1}\big)}{\displaystyle\prod_{\substack{\alpha_{\mathbf{m}'} \in \Delta(\mathbf{m}) \\ m'_{0,n_0}=1 \\ m'_{1,n_1}=0}} \Gamma\big(|\{\lambda_{\mathbf{m}'}\}|\big) \cdot \prod_{\substack{\alpha_{\mathbf{m}'} \in \Delta(\mathbf{m}) \\ m'_{0,n_0}=0 \\ m'_{1,n_1}=1}} \Gamma\big(1 - |\{\lambda_{\mathbf{m}'}\}|\big) \cdot \prod_{j=2}^{p} \big(1 - \tfrac{1}{c_j}\big)^{L_j}}$$

$$\text{Gauss}: \begin{Bmatrix} x = c_0 = 0 & c_1 = 1 & \infty \\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1} \\ \overline{\lambda_{0,2}} & \underline{\lambda_{1,2}} & \lambda_{2,2} \end{Bmatrix} = \begin{Bmatrix} x = 0 & 1 & \infty \\ 1-\gamma & \gamma-\alpha-\beta & \alpha \\ 0 & 0 & \beta \end{Bmatrix}$$

$$\begin{aligned} & 1\overline{1}, 1\underline{1}, 11 \\ = {} & 0\overline{1}, 10, 10 \\ \oplus {} & 10, 0\overline{1}, 01 \end{aligned}$$

$\updownarrow$

$$c(0 : \lambda_{0,2} \rightsquigarrow 1 : \lambda_{1,2}) = \frac{\Gamma(\lambda_{0,2} - \lambda_{0,1} + 1)\,\Gamma(\lambda_{1,1} - \lambda_{1,2})}{\Gamma(\lambda_{0,2} + \lambda_{1,1} + \lambda_{2,1})\,\Gamma(\lambda_{0,2} + \lambda_{1,1} + \lambda_{2,2})}$$

$\Leftrightarrow$

§ Irregular singularities (unramified)

At the singular point $x = c_j \ (j = 0, \dots, p)$:

$$\lambda_{j,\nu} \rightsquigarrow \lambda_{j,\nu}(t) = \lambda_{j,\nu,0} + \lambda_{j,\nu,1}t + \lambda_{j,\nu,2}t^2 + \cdots \in \mathbb{C}[t] \qquad (\lambda_{j,\nu,s} \in \mathbb{C})$$

For simplicity we assume

$$\deg(\lambda_{j,\nu} - \lambda_{j,\nu'}) > 0 \ \text{ or } \ \lambda_{j,\nu} - \lambda_{j,\nu'} \notin \mathbb{Z} \qquad (1 \le \nu < \nu' \le n_j).$$

Under a local coordinate with $c_j = 0$ (by a translation $x \mapsto x = c_j$ or $x \mapsto \frac{1}{x}$) :

$$\begin{aligned} u_{j,\nu,k}(x) &\sim (1 + a_1 x + a_2 x^2 + \cdots) e^{\int (\lambda_{j,\nu}(\frac{1}{x}) + k) \frac{dx}{x}} \\ &\sim e^{\int (\lambda_{j,\nu}(\frac{1}{x}) + k) \frac{dx}{x}} + O\big(e^{\int (\lambda_{j,\nu}(\frac{1}{x}) + m_{j,\nu}) \frac{dx}{x}}\big) \\ &\qquad (k = 0, \dots, m_{j,\nu} - 1, \ \nu = 1, \dots, n_j) \end{aligned}$$

$$e^{\int (\lambda_{j,\nu}(\frac{1}{x}) + k) \frac{dx}{x}} := x^{\lambda_{j,\nu,0} + k} e^{-\frac{\lambda_{j,\nu,1}}{x} - \frac{\lambda_{j,\nu,2}}{2x^2} - \frac{\lambda_{j,\nu,3}}{3x^3} - \cdots}$$

$$\left\{ \begin{matrix} & x = c_j \ (j = 0, \dots, p) & \\ & \vdots & \\ \cdots & [\lambda_{j,\nu}]_{(m_{j,\nu})} & \cdots \\ & \vdots & \end{matrix} \right\} = \left\{ \begin{matrix} & c_j & (1) & \cdots \\ & \vdots & \vdots & \\ \cdots & [\lambda_{j,\nu,0}]_{(m_{j,\nu})} & [\lambda_{j,\nu,1}]_{(m^{(1)}_{j,\nu})} & \cdots \\ & \vdots & \vdots & \end{matrix} \right\}$$

§ Spectral type, Confluence and Unfolding

Fix $\mathbf{m} = (m_{j,\nu})$: a tuple of partitions of n

Fix $0 \le j \le p$, $1 \le i,\ i' \le n$. Choose ν and ν' such that

$$m_{j,1} + \cdots + m_{j,\nu-1} < i \le m_{j,1} + \cdots + m_{j,\nu},$$
$$m_{j,1} + \cdots + m_{j,\nu'-1} < i' \le m_{j,1} + \cdots + m_{j,\nu'}$$

Equivalence relations:

$$i \underset{j,r}{\sim} i' \overset{\text{def}}{\Leftrightarrow} \begin{cases} \nu = \nu' & (r = 0) \\ \deg\bigl(\lambda_{j,\nu} - \lambda_{j,\nu'}\bigr) < r & (r \ge 1), \end{cases} \qquad (\deg 0 = 0)$$

"$\underset{r,j}{\sim}$" defines a partition of $n \Rightarrow \mathbf{m}_j^{(r)}$: $n = m_{j,1}^{(r)} + \cdots + m_{j,n_{j,r}}^{(r)}$

$\mathbf{m}_j^{(r)}$ is a refinement of $\mathbf{m}_j^{(r+1)}$

$n_{j,0} = n_j \ge n_{j,1} \ge \cdots \ge n_{j,r_j} > n_{j,r_j+1} = 1$

$\tilde{\mathbf{m}} := (\mathbf{m}_j^{(r)})_{j=0,\dots,p,\ r=0,\dots,r_j}$: spectral type

$\lambda_{j,\bar\nu}^{(r)} := \sum_{k=r}^{r_j} \frac{\lambda_{j,\mu,k}}{\prod_{0\le s\le k,\ s\ne r}(t_r - t_s)} \quad (m_{j,1} + \cdots + m_{j,\mu} = m_{j,1}^{(r)} + \cdots + m_{j,\bar\nu}^{(r)})$

$$\left\{\begin{matrix} \cdots \quad x = c_j + t_r \ (j = 0,\dots,p,\ r = 0,\dots,r_j) \quad \cdots \\ [\lambda_{j,\bar\nu}^{(r)}]_{(m_{j,\bar\nu}^{(r)})} \ (\bar\nu = 1,\dots,n_{j,r}) \end{matrix}\right\} \qquad (c_j + t_r \mapsto \tfrac{1}{t_r} \text{ if } c_r = \infty)$$

Operations

$$\mathrm{Ad}\left(\prod_{r=0}^{r_j}(x-c_j-t_r)^{\sum_{k=r}^{r_j}\frac{\lambda_{j,\nu,k}}{\prod_{\substack{0\le s\le k\\ s\ne r}}(t_r-t_s)}}\right)=\mathrm{Ad}\left(e^{\int\sum_{r=0}^{r_j}\frac{\lambda_{j,\nu,r}}{\prod_{k=0}^{r}(x-c_j-t_k)}dx}\right)$$

$$\partial\mapsto\partial-\sum_{r=0}^{r_j}\frac{\lambda_{j,\nu,r}}{\prod_{k=0}^{r}(x-c_j-t_k)},\ x\mapsto x$$

Confluence : $\forall t_r\to 0$

$$\mathrm{idx}\,\tilde{\mathbf{m}}=\sum(m_{j,\nu}^2-n^2)-\sum\deg\left(\lambda_{j,\nu}-\lambda_{j,\nu'}\right)\cdot m_{j,\nu}m_{j,\nu'}+2n^2$$

$$d_{\tilde{\ell}}(\tilde{\mathbf{m}})=\sum(m_{j,\ell_j}-n)-\sum\deg\left(\lambda_{j,\nu}-\lambda_{j,\ell_j}\right)\cdot m_{j,\nu}+2n$$

(FC) $\quad\sum m_{j,\nu}\lambda_{j,\nu,0}-\mathrm{ord}\,\mathbf{m}+\frac{1}{2}\mathrm{idx}\,\tilde{\mathbf{m}}:=\left|\{[\lambda_{j,\nu}^{(r)}]_{(m_{j,\nu}^{(r)})}\}\right|=0$

$\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\}$: rigid $\overset{\text{def}}{\Leftrightarrow}$ irreducibly realizable and $\mathrm{idx}\,\tilde{\mathbf{m}}=2$

$\Rightarrow\exists\tilde{\ell}$ s.t. $d_{\tilde{\ell}}(\tilde{\mathbf{m}})>0$

$\Rightarrow$ constructed by successive applications of the above operators to ∂

$\Rightarrow\exists$versal operator (unfolding)

Rigid Versal Gauss : $11,11,11$

$$(1-c_1x)(1-c_2x)\partial^2+\big((\lambda_2+2c_1c_2)x+\lambda_1-c_1-c_2\big)\partial+\mu\big(\lambda_2+c_1c_2(1-\mu)\big)$$

$$=(1-c_1x)(1-c_2x)\partial^2+(\tilde\lambda_2x+\tilde\lambda_1)\partial+\mu\big(\tilde\lambda_2-c_1c_2(\mu+1)\big)$$

$$\left\{\begin{matrix} x=\frac{1}{c_1} & \frac{1}{c_2} & \infty & \\ 0 & 0 & \mu & ;\ x \\ \frac{\lambda_1}{c_1-c_2}+\frac{\lambda_2}{c_1(c_1-c_2)} & \frac{\lambda_1}{c_2-c_1}+\frac{\lambda_2}{c_2(c_2-c_1)} & \frac{\lambda_2}{c_1c_2}-\mu+1 & \end{matrix}\right\}\quad (0\neq c_1\neq c_2\neq 0)$$

$$\left\{\begin{matrix} x=\frac{1}{c_2} & \infty & (1) & \\ 0 & \mu & 0 & ;\ x \\ \frac{\lambda_2}{c_2^2}+\frac{\lambda_1}{c_2} & -\frac{\lambda_2}{c_2^2}-\frac{\lambda_1}{c_2^2}-\mu+1 & \frac{\lambda_2}{c_2} & \end{matrix}\right\}\quad (0=c_1\neq c_2\neq 0)$$

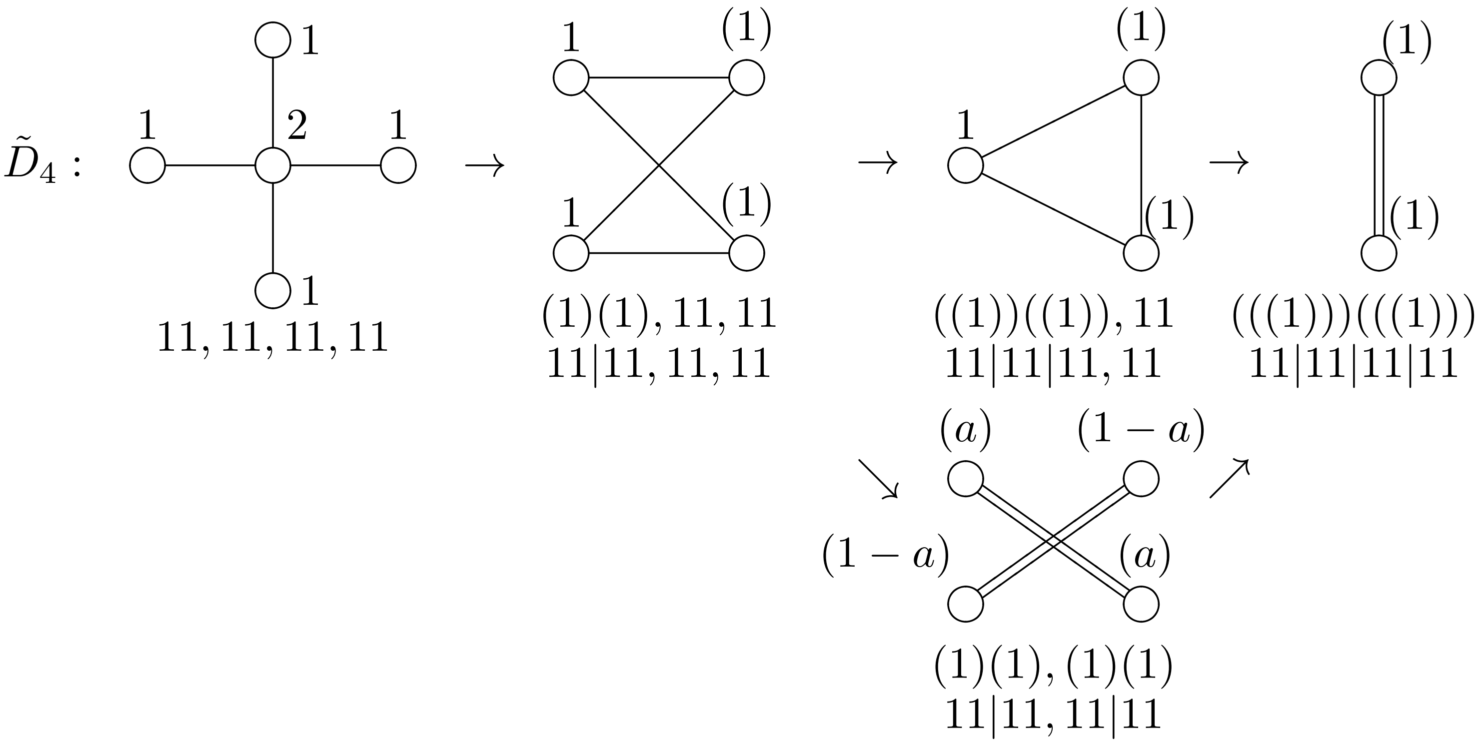
$$\left\{\begin{matrix} x=\frac{1}{c_1} & (1) & \infty & \\ 0 & 0 & \mu & ;\ x \\ -\frac{\lambda_2}{c_1^2} & \frac{\lambda_2}{c_1^3}+\frac{\lambda_1}{c_1^2} & \frac{\lambda_2}{c_1^2}-\mu+1 & \end{matrix}\right\}\quad (c_1=c_2\neq 0)$$

$$\left\{\begin{matrix} x=\infty & (1) & (2) & \\ \mu & 0 & 0 & ;\ x \\ 1-\mu & \lambda_1 & \lambda_2 & \end{matrix}\right\}\quad (c_1=c_2=0)\quad u\sim x^{-\mu},\ x^{\mu-1}e^{-\lambda_1x-\frac{\lambda_2}{2}x^2}$$

$$\boxed{u(x)=\int_{\frac{1}{c_1}}^{x}e^{-\int\left(\frac{\lambda_1'}{1-c_1t}+\frac{\lambda_2't}{(1-c_1t)(1-c_2t)}\right)dt}(x-t)^{\mu-1}dt=\int_{\infty}^{x}e^{-\lambda_1't-\frac{\lambda_2't^2}{2}}(x-t)^{\mu-1}dt}$$

Versal Heun : $11, 11, 11, 11$

$$(1-c_1x)(1-c_2x)(1-c_3x)\partial^2+(\lambda_3x^2+\lambda_2x+\lambda_1)\partial+\mu\big(\lambda_3+c_1c_2c_3(\mu+1)\big)+r$$



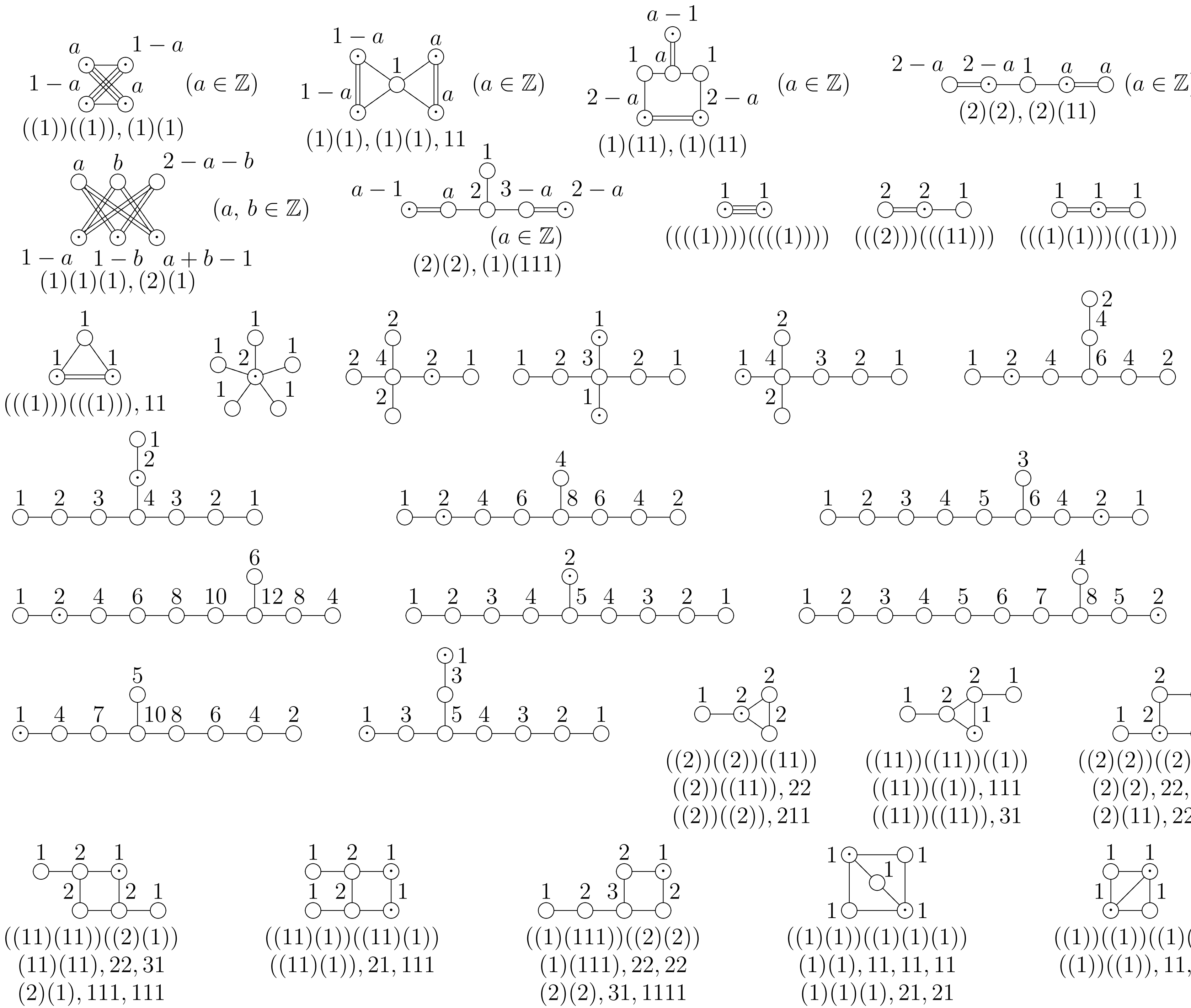
Theorem [Hiroe -O].

i) {Spectral types} $\to$ A universal symmetric Kac-Moody root system

ii) 'Finiteness' of orbits of Weyl group in the space of roots

iii) Reduction and construction as in the Fuchsian case

iv) Classification when the index of rigidity ≥ -2



((1))((1)), (1)(1)
(a ∈ ℤ)
(1)(1), (1)(1), 11
(a ∈ ℤ)
(1)(11), (1)(11)
(a ∈ ℤ)
(2)(2), (2)(11)
(a ∈ ℤ)
(a, b ∈ ℤ)
(1)(1)(1), (2)(1)
(a ∈ ℤ)
(2)(2), (1)(111)
((((1))))((((1))))
(((2)))(((11)))
(((1)(1)))(((1)))
(((1)))(((1))), 11
((2))((2))((11))
((2))((11)), 22
((2))((2)), 211
((11))((11))((1))
((11))((1)), 111
((11))((11)), 31
((2)(2))((2)(11))
(2)(2), 22, 211
(2)(11), 22, 22
((11)(11))((2)(1))
(11)(11), 22, 31
(2)(1), 111, 111
((11)(1))((11)(1))
((11)(1)), 21, 111
((1)(111))((2)(2))
(1)(111), 22, 22
(2)(2), 31, 1111
((1)(1))((1)(1)(1))
(1)(1), 11, 11, 11
(1)(1)(1), 21, 21
((1))((1))((1)(1))
((1))((1)), 11, 11

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Thank you!   End!