Kernels of summability for formal power series with coefficients of moderate growth

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Notations

$S(d,\gamma)$ Unbounded sector of opening $\pi\gamma$, bisected by direction $d \in \mathbb{R}$.

S_γ Unbounded sector of opening $\pi\gamma$, bisected by direction $d = 0$.

$G(d,\gamma)$ Sectorial region of opening $\pi\gamma$, bisected by direction $d = 0$.

Strongly regular sequences (following V. Thilliez)

A sequence $\mathbb{M} = (M_n)_{n\in\mathbb{N}_0}$ of positive real numbers, with $M_0 = 1$, is said to be strongly regular if it is:

- logarithmically convex: $M_n^2 \leq M_{n-1}M_{n+1}, \quad n \geq 1.$
- of moderate growth: there exists a constant $A > 0$ such that

$$M_{n+p} \leq A^{n+p} M_n M_p, \quad n, p \in \mathbb{N}_0.$$

- strongly non-quasianalytic: there exists $B > 0$ such that

$$\sum_{k\geq n} \frac{M_k}{(k+1)M_{k+1}} \leq B\frac{M_n}{M_{n+1}}, \quad n \in \mathbb{N}_0.$$

Examples of strongly regular sequences

Examples:

- $\mathbb{M}_\alpha = (n!^\alpha)_{n\in\mathbb{N}_0}$, Gevrey sequence of order $\alpha > 0$.
- $\mathbb{M}_{\alpha,\beta} = \big(n!^\alpha \prod_{m=0}^{n} \log^\beta(e+m)\big)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$.
- For $q > 1$, $\mathbb{M} = (q^{n^2})_{n\in\mathbb{N}_0}$ is logarithmically convex and strongly non-quasianalytic, but not of moderate growth.

Ultraholomorphic classes and the asymptotic Borel map, I

Given $\mathbb{M}$, $A > 0$ and a sector S, we consider

$$\mathcal{A}_{\mathbb{M},A}(S) = \left\{ f \in \mathcal{H}(S) : \|f\|_{\mathbb{M},A} := \sup_{z \in S, n \in \mathbb{N}_0} \frac{|f^{(n)}(z)|}{A^n n! M_n} < \infty \right\}.$$

$(\mathcal{A}_{\mathbb{M},A}(S), \| \ \|_{\mathbb{M},A})$ is a Banach space. $\mathcal{A}_{\mathbb{M}}(S) := \cup_{A>0} \mathcal{A}_{\mathbb{M},A}(S)$.

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$(\mathcal{A}_{\mathbb{M},A}(S),\|\ \|_{\mathbb{M},A})$ is a Banach space. $\mathcal{A}_{\mathbb{M}}(S):=\cup_{A>0}\mathcal{A}_{\mathbb{M},A}(S)$.

We say $f\in\tilde{\mathcal{A}}_{\mathbb{M}}(S)$ if $f\in\mathcal{H}(S)$, and for every bounded proper subsector T of S there exists $A_T>0$ such that $f|_T\in\mathcal{A}_{\mathbb{M},A_T}(T)$. Similarly for sectorial regions.

$f\in\tilde{\mathcal{A}}_{\mathbb{M}}(S)\Leftrightarrow f$ admits the series $\hat{f}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$ as its $\mathbb{M}$-asymptotic expansion at 0, denoted $f\sim_{\mathbb{M}}\hat{f}$: For every T there exist $C,B>0$ such that for every $z\in T$ and every $n\in\mathbb{N}_0$, we have

$$\Big|f(z)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}z^k\Big|\le CB^nM_n|z|^n.$$

Ultraholomorphic classes and the asymptotic Borel map, II

$$\Lambda_{\mathbb{M},A} = \Big\{ \boldsymbol{\mu} = (\mu_n)_{n\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} : |\boldsymbol{\mu}|_{\mathbb{M},A} := \sup_{n\in\mathbb{N}_0} \frac{|\mu_n|}{A^n n! M_n} < \infty \Big\}.$$

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The asymptotic Borel map $\tilde{\mathcal{B}}:\tilde{\mathcal{A}}_{\mathbb{M}}(S)\longrightarrow\Lambda_{\mathbb{M}}$,
$f\mapsto(f^{(n)}(0):=\lim_{z\to0,z\in T}f^{(n)}(z))_{n\in\mathbb{N}_0}$ is well defined.

Flat functions and quasianalyticity in Gevrey classes

Consider the Gevrey case of order $1/k > 0$, i.e. $\mathbb{M}_{1/k} = (n!^{1/k})_{n\in\mathbb{N}_0}$.

We write $\mathcal{A}_{1/k}(S)$, $\tilde{\mathcal{A}}_{1/k}(S)$, $\Lambda_{1/k}$, $f \sim_{1/k} \hat{f}$ and so on for simplicity.

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Theorem

Let S be a sector and $f\in\mathcal{H}(S)$. The following are equivalent:

- $f\in\tilde{\mathcal{A}}_{1/k}(S)$ *and* $f\sim_{1/k}\hat{0}$.
- *For every bounded proper subsector T of S there exist $c_1,c_2>0$ with*

$$|f(z)|\le c_1e^{-c_2|z|^{-k}},\qquad z\in T.$$

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The injectivity of the asymptotic Borel map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{1/k}(S) \longrightarrow \Lambda_{1/k}$ is essential for the definition of k-summability!!

Theorem (Watson's lemma)

$\tilde{\mathcal{B}}$ is injective if, and only if, the opening of S is greater than π/k.

Definition of k-summability in a direction

Let $d \in \mathbb{R}$. A formal series $\hat{f} = \sum_{n\geq 0} \dfrac{f_n}{n!} z^n$ is k-summable in direction d if there exist a sectorial region $G = G(d, \gamma)$, with $\gamma > 1/k$, and $f \in \tilde{\mathcal{A}}_{1/k}(G)$ such that $f \sim_{1/k} \hat{f}$.

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In this case, $(f_n)_{n\in\mathbb{N}_0} \in \Lambda_{1/k}$, f is the unique with the property above, and it is called the k-sum of $\hat{f}$ in direction d.

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What about the general case?

Associated functions

The sequence of quotients, $\boldsymbol{m} = (m_p := M_{p+1}/M_p)_{p\in\mathbb{N}_0}$, is an increasing sequence tending to infinity.

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- $h_{\mathbb{M}}(t) = \inf_{n\geq 0} M_n t^n$, $t>0$; $h_{\mathbb{M}}(0)=0$.

  $$h_{\mathbb{M}}(t) = \begin{cases} M_j t^j & \text{if } t \in \left[\frac{1}{m_j}, \frac{1}{m_{j-1}}\right), \ j=1,2,\dots, \\ 1 & \text{if } t \geq 1/m_0. \end{cases}$$

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Non-decreasing continuous map in $[0,\infty)$, eventually equal to 1.

- $M(t) := \sup_{p\in\mathbb{N}_0} \log\left(\frac{t^p}{M_p}\right) = -\log\big(h_{\mathbb{M}}(1/t)\big), \quad t>0; \qquad M(0)=0$

$$M(t) = \begin{cases} j\log(t) - \log(M_j) & \text{if } t\in[m_{j-1},m_j),\ j=1,2,\dots, \\ 0 & \text{if } t\in[0,m_0). \end{cases}$$

Non-decreasing, continuous, piecewise continuously differentiable map in $[0,\infty)$ with $\lim_{t\to\infty} M(t)=\infty$.

Optimal opening for quasianalyticity, I

Let S be a sector and $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ be a sequence of positive numbers. The class $\mathcal{A}_{\mathbb{M}}(S)$ is quasianalytic if it does not contain nontrivial flat functions.

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Theorem (Korenbljum (1966))

Let $\mathbb{M}$ be strongly regular and $\gamma > 0$. The following statements are equivalent:

- *The class $\mathcal{A}_{\mathbb{M}}(S_\gamma)$ is quasianalytic.*
- $\displaystyle\sum_{n=0}^{\infty}\left(\frac{M_n}{(n+1)M_{n+1}}\right)^{1/(\gamma+1)} = \infty.$

Optimal opening for quasianalyticity, II

We put $C_{\mathbb{M}} = \{\gamma > 0 : \mathcal{A}_{\mathbb{M}}(S_\gamma) \text{ is quasianalytic}\}$ and $\omega(\mathbb{M}) = \inf C_{\mathbb{M}}$.

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Proposition

$$\omega(\mathbb{M}) = \liminf_{n\to\infty} \frac{\log(m_n)}{\log(n)} \in (0,\infty).$$

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$$\omega(\mathbb{M}) = \liminf_{n\to\infty} \frac{\log(m_n)}{\log(n)} \in (0,\infty).$$

Example: $\mathbb{M}_{\alpha,\beta} = \left(n!^{\alpha} \prod_{m=0}^{n} \log^{\beta}(e+m)\right)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$.

- $C_{\mathbb{M}_{\alpha,\beta}} = \begin{cases} [\alpha,\infty) & \text{if } \alpha \geq \beta - 1, \\ (\alpha,\infty) & \text{if } \alpha < \beta - 1. \end{cases}$
- $\omega(\mathbb{M}_{\alpha,\beta}) = \alpha$ (in particular for Gevrey sequences!).

Definition of $\mathbb{M}$-summability in a direction

Theorem (generalized Watson's lemma, partial version)

If the opening of S is greater than $\pi\omega(\mathbb{M})$, then $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S) \longrightarrow \Lambda_{\mathbb{M}}$ is injective.

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In this case, $(f_n)_{n\in\mathbb{N}_0}\in\Lambda_{\mathbb{M}}$, and f is unique with this property, and will be denoted

$$f=\mathcal{S}_{\mathbb{M},d}\hat{f},\ \text{the } \mathbb{M}\text{-sum of } \hat{f} \text{ in direction } d.$$

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We recall the ideas in the theory of general moment summability methods by W. Balser (UTX, 2000).

Definition of kernels of order k (W. Balser)

Let $k > 1/2$. A pair of complex functions e, E are kernel functions of order k if:

1. e is holomorphic in $S_{1/k}$.
2. $z^{-1}e(z)$ is integrable at the origin.
3. For every $\varepsilon > 0$ there exist $C, K > 0$ such that
$$|e(z)| \leq C \exp(-K|z|^k), \qquad z \in S_{1/k-\varepsilon}.$$
4. For $x \in \mathbb{R}$, $x > 0$, the values of $e(x)$ are positive real.
5. If we define $m(\lambda) := \int_0^\infty t^{\lambda-1}e(t)dt$, $\mathrm{Re}(\lambda) \geq 0$, the function E, given by
$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{m(n)}, \qquad z \in \mathbb{C},$$
is entire, and there exist $c_1, c_2 > 0$ such that $|E(z)| \leq c_1 \exp(c_2|z|^k)$, $z \in \mathbb{C}$.
6. $z^{-1}E(1/z)$ is integrable at the origin in the sector $S(\pi, 2-1/k)$.

Null asymptotics

Theorem (V. Thilliez (2010))

Let $\mathbb{M}$ be a strongly regular sequence and S a sector. If $f \in \mathcal{H}(S)$, the following are equivalent:

- $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S)$ *and* $f \sim_{\mathbb{M}} \hat{0}$.
- *For every bounded proper subsector T of S there exist $c_1, c_2 > 0$ with*

$$|f(z)| \leq c_1 h_{\mathbb{M}}(c_2|z|) = c_1 e^{-M(1/(c_2|z|))}, \qquad z \in T.$$

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In the Gevrey case $\mathbb{M}_{1/k}$:

For small $t > 0$, $\exp(-c_1 t^{-k}) \leq h_{1/k}(t) \leq \exp(-c_2 t^{-k})$;

for large t, $c_2 t^k \leq M_{1/k}(t) \leq c_1 t^k$.

Definition of kernels of $\mathbb{M}$-summability

Let $\mathbb{M}$ be strongly regular with $\omega(\mathbb{M}) < 2$. A pair of complex functions e, E are kernel functions for $\mathbb{M}$-summability if:

1. e is holomorphic in $S_{\omega(\mathbb{M})}$.
2. $z^{-1}e(z)$ is integrable at the origin.
3. For every $\varepsilon > 0$ there exist $C, K > 0$ such that

$$|e(z)| \le C h_{\mathbb{M}}\left(\frac{K}{|z|}\right) = C\exp(-M(|z|/K)), \qquad z \in S_{\omega(\mathbb{M})-\varepsilon}.$$

4. For $x \in \mathbb{R}$, $x > 0$, the values of $e(x)$ are positive real.
5. If we define $m(\lambda) := \int_0^\infty t^{\lambda-1}e(t)dt$, $\mathrm{Re}(\lambda) \ge 0$, the function E given by $E(z) = \sum_{n=0}^\infty \frac{z^n}{m(n)}$, $z \in \mathbb{C}$, is entire, and there exist $c, k > 0$ such that for every $z \in \mathbb{C}$,

$$|E(z)| \le \frac{c}{h_{\mathbb{M}}(k/|z|)} = c\exp(M(|z|/k)).$$

6. $z^{-1}E(1/z)$ is integrable at the origin in the sector $S(\pi, 2-\omega(\mathbb{M}))$.

Proximate orders

Idea: For $|z| = r$, instead of comparing $\log|f(z)|$ to r^k, compare to $M(r) = r^{d(r)}$, where

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Def. (Lindelöf, Valiron):
We say $\rho(r) : (0, \infty) \to \mathbb{R}$ is a proximate order if the following hold:

(1) ρ is continuously differentiable,

(2) $\rho(r) \geq 0$ for every $r > 0$,

(3) $\lim_{r\to\infty} \rho(r) = \rho < \infty$,

(4) $\lim_{r\to\infty} r\rho'(r)\log(r) = 0$.

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Condition (1) may be replaced by: (1') $\rho(r)$ is continuous and piecewise continuously differentiable in $(0,\infty)$.

Proximate order of an entire function

Def.: Let $\rho(r)$ be a proximate order and f be an entire function. The type of f associated to $\rho(r)$ is

$$\sigma_f(\rho(r)) = \sigma_f := \limsup_{r\to\infty} \frac{\log \max_{|z|=r} |f(z)|}{r^{\rho(r)}}.$$

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We say $\rho(r)$ is a proximate order of f if $0 < \sigma_f < \infty$.

Remark

If $\rho(r) \to \rho > 0$ is a proximate order of f, then f is of exponential order ρ and there exists $K > 0$ such that for every $z \in \mathbb{C}$ one has

$$|f(z)| \le \exp(K|z|^{\rho(|z|)}).$$

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Results by V. Bernstein, M. M. Dzhrbashyan, M. A. Evgrafov, A. A. Gol'dberg, I. V. Ostrovskii and, in our regards, mainly L. S. Maergoiz.

Existence of analytic proximate orders for a given proximate order

Theorem (L. S. Maergoiz (2001))

Let $\rho(r)$ be a proximate order with $\rho(r) \to \rho > 0$ as $r \to \infty$. For every $\gamma > 0$ there exists an analytic function $V_0(z)$ in S_γ such that:

(1) *For every $z \in S_\gamma$,*

$$\lim_{r\to\infty} \frac{V_0(zr)}{V_0(r)} = z^\rho,$$

uniformly in the compact sets of S_γ.

(2) $\overline{V_0(W)} = V_0(\overline{W})$ *for every $W \in S_\gamma$.*

(3) $V_0(r)$ *is positive, increasing and strictly convex relative to* $\log(r)$ *in* $(0,\infty)$, *and* $V_0(0) = 0$.

(4) $\log(V_0(r))$ *is strictly concave in* $(0,\infty)$.

(5) *The function* $\rho_0(r) := \log(V_0(r))/\log(r)$, $r > 0$, *is a proximate order and*

$$\lim_{r\to\infty} [\rho_0(r) - \rho(r)] \log(r) = 0.$$

Existence of kernels

We denote by $\mathfrak{B}(\gamma,\rho(r))$ the class of such functions V.

Theorem

Suppose $\mathbb{M}$ is a strongly regular sequence with $\omega(\mathbb{M})<2$ such that $d(r)$ is a proximate order. Take $\gamma=\min\{2\omega(\mathbb{M}),2\}$ and $V\in\mathfrak{B}(\gamma,d(r))$. Then:

- *The function defined in $S_{\omega(\mathbb{M})}$ by*

$$e(z)=\frac{1}{\omega(\mathbb{M})}z\exp(-V(z))$$

  is a kernel of $\mathbb{M}$-summability.
- *The corresponding kernel E is (of exponential order $1/\omega(M)$ and) of proximate order $d(r)$.*

Conditions for $d(r)$ to be a proximate order

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Proposition

$d(r)$ is an increasing function and

$$\lim_{r\to\infty} d(r) = \limsup_{n\to\infty} \frac{\log(n)}{\log(m_n)} = \frac{1}{\omega(\mathbb{M})} > 0,$$

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$rd'(r)\log(r) \to 0$ as $r\to\infty$?

Conditions for $d(r)$ to be a proximate order

Proposition

$d(r)$ is a proximate order if, and only if,

$$\lim_{j\to\infty}\frac{j}{M(m_{j-1})}=\lim_{j\to\infty}\frac{j}{\log(m_{j-1}^{j}/M_j)}=\frac{1}{\omega(\mathbb{M})}.$$

Corollary

If $\lim_{j\to\infty} j\log\Big(\frac{m_j}{m_{j-1}}\Big)$ exists, then $d(r)$ is a proximate order.

Comments

(1) The previous conditions hold for every sequence $\mathbb{M}_{\alpha,\beta}$.

(2) Our method provides kernels, but not all. For example, $e(z) = kz^k e^{-z^k}$ gives rise to the standard Laplace and Borel (with Mittag-Leffler kernel) transforms of order k, but it is not provided by the previous method.

(3) In the Gevrey case $\mathbb{M}_{1/k}$, for large r we have $c_2 r^k \leq M_{1/k}(r) \leq c_1 r^k$, so that one can work with the proximate order $\rho(r) \equiv k$, which provides the kernel $\hat{e}(z) = kze^{-z^k}$ with moment function $\hat{m}(\lambda) = \Gamma((\lambda+1)/k)$ and the corresponding $\hat{E}$.

(5) If $d(r)$ is not a proximate order but there exists a proximate order $\rho(r)$ and constants $A, B > 0$ such that eventually $A \leq (d(r) - \rho(r))\log(r) \leq B$, then one may also construct kernels for $\mathbb{M}$-summability by working with $\rho(r)$ instead of $d(r)$.

(6) Consequence of 3. and 5. in the definition of kernels: $\mathbb{M}$ and $(m(n))_{n\in\mathbb{N}_0}$ are equivalent, i.e., there exist $C, D > 0$ such that $C^n M_n \leq m(n) \leq D^n M_n$ for every $n \in \mathbb{N}_0$.

Laplace-like operator associated to e

Let $S=S(d,\gamma)$ be an unbounded sector. We say a holomorphic function $f:S\to\mathbb{C}$ belongs to $\mathcal{A}^{\mathbb{M}}(S)$ if it is continuous at 0, and for every unbounded subsector T of S there exist $k_1,k_2>0$ such that for every $z\in T$,

$$|f(z)|\le \frac{k_1}{h_{\mathbb{M}}(k_2/|z|)}. \qquad (f \text{ is of } \mathbb{M}\text{-growth in } S).$$

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Given kernels e,E for $\mathbb{M}$, $f\in\mathcal{A}^{\mathbb{M}}(S)$ and a direction τ in S, the e-Laplace transform (in direction τ) of f, $T_e f$, is defined by

$$(T_e f)(z):=\int_0^{\infty(\tau)} e(u/z)f(u)\frac{du}{u}.$$

$\{T_e f\}_{\tau \text{ in } S}$ defines a holomorphic function in a sectorial region $G(d,\gamma+\omega(\mathbb{M}))$.

Borel-like operator associated to e

Let $G=G(d,\gamma)$ be a sectorial region with $\gamma>\omega(\mathbb{M})$, $f:G\to\mathbb{C}$ holomorphic in G and continuous at 0, and $\gamma_{\omega(\mathbb{M})}(\tau)$ a path as the ones used in the classical Borel transform.

Given kernels e,E for $\mathbb{M}$, the e-Borel transform (in direction τ) of f, $T_e^- f$, is defined by

$$(T_e^- f)(u):=\frac{-1}{2\pi i}\int_{\gamma_{\omega(\mathbb{M})}(\tau)}E(u/z)f(z)\frac{dz}{z},\quad u\in S(\tau,\varepsilon),\quad \varepsilon \text{ small enough.}$$

$\{T_e^- f\}_\tau$ defines a holomorphic function in $S(d,\gamma-\omega(\mathbb{M}))$ and it is of $\mathbb{M}$-growth there.

Formal operators associated to e and definition of moment summability

For $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, one has

$$T_e(z^\lambda) = m(\lambda) z^\lambda, \qquad T_e^-(z^\lambda) = \frac{1}{m(\lambda)} z^\lambda.$$

The formal e-Laplace and Borel operators, $\hat{T}_e$ and $\hat{T}_e^-$, are accordingly defined in $\mathbb{C}[[z]]$ by linearity.

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Definition

Let e be a kernel of $\mathbb{M}$-summability and T_e the Laplace operator.
We say $\hat{f} = \sum_{n \geq 0} \frac{f_n}{n!} z^n$ is T_e-summable in direction $d \in \mathbb{R}$ if:

- $(f_n)_{n \in \mathbb{N}_0} \in \Lambda_{\mathbb{M}}$, so that $g := \hat{T}_e^- \hat{f} = \sum_{n \geq 0} \frac{f_n}{n! m(n)} z^n$ converges in a disc, and
- g admits analytic continuation $\mathcal{S}(\hat{T}_e^- \hat{f})$ in a sector $S = S(d, \varepsilon)$ for some $\varepsilon > 0$, and $g \in \mathcal{A}^{\mathbb{M}}(S)$.

Main result for moment summability

Theorem

Given $\mathbb{M}$, d and $\hat{f}$, the following are equivalent:

- $\hat{f}$ *is* $\mathbb{M}$*-summable in direction* d.
- *For every kernel* e *of* $\mathbb{M}$*-summability,* $\hat{f}$ *is* T_e*-summable in direction* d.
- *For some kernel* e *of* $\mathbb{M}$*-summability,* $\hat{f}$ *is* T_e*-summable in direction* d.

In case any of the previous holds, we have (after analytic continuation)

$$\mathcal{S}_{\mathbb{M},d}\hat{f} = T_e(\mathcal{S}(\hat{T}_e^- \hat{f})).$$

Comments

- In case $\mathbb{M} = \mathbb{M}_{1/k}$, we obtain classical k-summability and T_e-summability for kernels e of order k.
- In case $\omega(\mathbb{M}) \geq 2$, modifications are needed as in W. Balser (UTX, 2000).
- If $\mathbb{M}$ and $\mathbb{M}'$ are equivalent strongly regular sequences, the ultraholomorphic classes, classes of functions of respective growth, families of kernels of summability and the summability methods described are all the same, as well as the sums provided for every summable series.

Moment partial differential equations, I

In a recent paper by W. Balser and M. Yoshino (2010), formal moment-differential operators were introduced corresponding to the sequence of moments $(m(n))_{n\in\mathbb{N}_0}$ of a given kernel function e of order $k>0$:

$$\partial_{m,z}\Big(\sum_{j=0}^{\infty}\frac{u_j}{m(j)}z^j\Big):=\sum_{j=0}^{\infty}\frac{u_{j+1}}{m(j)}z^j.$$

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For two sequences of moments $(m_1(n))_n$ and $(m_2(n))_n$, they study the formal power series solutions of an inhomogeneous moment partial differential equation with constant coefficients in two variables,

$$p(\partial_{m_1,t},\partial_{m_2,z})\hat{u}(t,z)=\hat{f}(t,z),$$

where $p(\lambda,\xi)$ is a given polynomial.

Moment partial differential equations, II

Subsequently, S. Michalik (2010-12) has studied similar problems in this context.
For example, given $(m_1(n))_n$ and $(m_2(n))_n$ as before, he considers an initial value problem for a linear moment partial differential equation with constant coefficients in two variables,

$$P(\partial_{m_1,t}, \partial_{m_2,z})u(t,z) = 0, \quad \partial^j_{m_1,t}u(0,z) = \varphi_j(z) \text{ for } j = 0, \dots, n-1,$$

where $P(\lambda, \xi)$ is a polynomial of degree n with respect to λ and the Cauchy data are analytic in a neighbourhood of $0 \in \mathbb{C}$.

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Results: in both cases, similar statements have been obtained for moment differential PDE's associated to moment sequences corresponding to summability kernels for strongly regular sequences.

Application, I

We assume that P can be written in the form

$$P(\lambda, \xi) = P_0(\lambda)(\lambda - \lambda_1(\xi))^{n_1} \ldots (\lambda - \lambda_\ell(\xi))^{n_\ell}$$

for some $\ell \in \mathbb{N}$ and $n_1, \ldots, n_\ell \in \mathbb{N}$ such that $n_1 + \ldots + n_\ell = n$.

Moreover, for every $j = 1, \ldots, n_\ell$, $\lambda_j(\xi)$ is a holomorphic function for $|\xi| \geq r_0$ for some $r_0 > 0$, with polynomial growth at infinity, meaning there exist $\lambda_j \in \mathbb{C} \setminus \{0\}$ and $q_j \in \mathbb{Q}$ such that

$$\lim_{\xi \to \infty} \frac{\lambda_j(\xi)}{\xi^{q_j}} = \lambda_j.$$

Application, II

For each $\lambda_j(\xi)$ one introduces suitable moment-pseudodifferential operators, defined in terms of kernels e_{m_2}, E_{m_2} corresponding to the sequence of moments m_2. These allows one to factorize the moment-differential operator as follows:

$$P(\partial_{m_1,t}, \partial_{m_2,z}) = P_0(\partial_{m_2,z})(\partial_{m_1,t} - \lambda_1(\partial_{m_2,z}))^{n_1} \cdots (\partial_{m_1,t} - \lambda_l(\partial_{m_2,z}))^{n_l}.$$

Application, III

Hence, one can prove that it suffices to study the moment-pseudodifferential equation

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^{\beta} u = 0$$

with initial conditions

$$\partial_{m_1,t}^{j} u(0,z) = 0 \ (n = 0, \ldots, \beta - 2),$$
$$\partial_{m_1,t}^{\beta-1} u(0,z) = \varphi(z) \in \mathcal{O}(D).$$

Here λ is of polynomial growth of order q with $\lim_{\xi\to\infty} \frac{\lambda(\xi)}{\xi^q} = \lambda$.

We restrict to the case $\beta = 1$.

Application, IV

Let us assume that u is a solution of

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0, \qquad u(0,z) = \varphi(z) \in \mathcal{O}(D),$$

and $\omega(m_1) = q\omega(m_2)$. Suppose the corresponding kernels are related by

$$e_{m_1}(z) = \frac{1}{q} e_{m_2}(z^{1/q}), \qquad z \in S_{\omega(m_1)}.$$

Then, for every strongly regular sequence $\mathbb{M} = (M_n)_{n\in\mathbb{N}_0}$ (for which kernels of summability exist) and $d \in \mathbb{R}$, the following are equivalent:

- φ admits analytic continuation in (a small sector around) every direction of the form $(d + \arg(\lambda) + 2p\pi)/q$ for every $p = 0, \dots, q-1$, and it is of exponential $\mathbb{M}$-growth there.
- u admits analytic continuation to $\tilde{S} \times D$, where $\tilde{S}$ is the union of a sector S bisected by d and a disk around 0, and u is uniformly (in z) of exponential $\tilde{\mathbb{M}}$-growth in $\tilde{S}$, where $\tilde{\mathbb{M}} = (M_{qn})_{n\in\mathbb{N}_0}$.

Consequences for the asymptotic Borel map: Injectivity

Remark

Suppose $d(r)$ is a proximate order. Given a kernel e for $\mathbb{M}$-summability, the function

$$G(z) = \exp(-V(1/z)), \qquad z \in S_{\omega(\mathbb{M})},$$

is a (nontrivial) flat function in $\tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$.

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Theorem (generalized Watson's lemma, second partial version)

Suppose $d(r)$ is a proximate order. Then, the map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S) \longrightarrow \Lambda_{\mathbb{M}}$ is injective if, and only if, the opening of S is greater than $\pi\omega(\mathbb{M})$.

Consequences for the asymptotic Borel map: Surjectivity

Theorem (Borel–Ritt–Gevrey, J. P. Ramis (1978))

For $\alpha > 0$ and $\gamma > 0$, $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_\alpha(S_\gamma) \longrightarrow \Lambda_\alpha$ is surjective if, and only if, $\gamma \leq \alpha$.

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Suppose $d(r)$ is a proximate order. Then, the map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma) \longrightarrow \Lambda_{\mathbb{M}}$ is surjective if, and only if, $\gamma \leq \omega(\mathbb{M})$.

This is based in the classical truncated Laplace transform technique, which also allows to obtain linear and continuous right inverses for the asymptotic Borel map.

Consequences for the asymptotic Borel map: Right inverses

Theorem

Let $\mathbb{M}$ be strongly regular and such that $d(r)$ is a proximate order. Then:

(1) The Borel map $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})}) \to \Lambda_{\mathbb{M}}$ admits a linear continuous right inverse.

(2) For every $\gamma < \omega(\mathbb{M})$, there exists $d \geq 1$ such that for every $A > 0$ there is a linear continuous operator

$$T_{\mathbb{M},A,\gamma} : \Lambda_{\mathbb{M},A} \to \mathcal{A}_{\mathbb{M},dA}(S_\gamma)$$

such that $\tilde{\mathcal{B}} \circ T_{\mathbb{M},A,\gamma} = Id_{\Lambda_{\mathbb{M},A}}$.

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V. Thilliez obtained (2) for sectors of opening less than $\pi\gamma(\mathbb{M})$, where $\gamma(\mathbb{M})$ is a growth index about which we only know that $\gamma(\mathbb{M}) \leq \omega(\mathbb{M})$. (1) is a new result.

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