Maillet Type Theorem for Singular First Order Nonlinear Partial Differential Equations of Totally Characteristic Type

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1. Introduction and Main Result

Let $t=(t_1,\cdots,t_d)\in\mathbb{C}^d$, $x=(x_1,\cdots,x_n)\in\mathbb{C}^n$ be d and n dimensional complex variables and $u=u(t,x)$ denote an unknown function.

We consider the following first order nonlinear PDE.

$$(1)\qquad f(t,x,u,\partial_t u,\partial_x u)=0 \quad \text{with} \quad u(0,x)\equiv 0$$

where $\partial_t u=\left(\partial_{t_1}u,\cdots,\partial_{t_d}u\right)$ and $\partial_x u$ is the similar definition with $\partial_t u$, and $f(t,x,u,\tau,\xi)$ $(\tau=(\tau_j)\in\mathbb{C}^d, \xi=(\xi_k)\in\mathbb{C}^n)$ is holomorphic near the origin and an entire function in τ variables.

Assumption 1 (Singular in t)

We assume that (1) is singular in t variables in the sense that

$$f(0,x,0,\tau,0) \equiv 0 \quad \text{for} \quad {}^{\forall}x \in \mathbb{C}^n, \quad {}^{\forall}\tau \in \mathbb{C}^d.$$

Assumption 2 (Existence of Formal Solutions)

(1) has a formal solution of the form

$$u(t,x) = \sum_{j=1}^{d} \varphi_j(x) t_j + \sum_{|\alpha| \geq 2, |\beta| \geq 0} u_{\alpha\beta} t^{\alpha} x^{\beta}$$

where $\varphi_j(x) \in \mathbb{C}\{x\}$ for all $j = 1,2,\cdots,d$.

Assumption 3 (Totally Characteristic Type)

Let $\boldsymbol{\varphi}(x) = (0, x, 0, \{\varphi_j(x)\}, 0)$. We say that (1) is of totally characteristic type, if the following conditions are satisfied for all $k = 1,2,\cdots,n$.

$$f_{\xi_k}(\boldsymbol{\varphi}(x)) \not\equiv 0, \text{but } f_{\xi_k}(\boldsymbol{\varphi}(0)) = 0.$$

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$$f_{\xi_k}\big(\boldsymbol{\varphi}(x)\big) \not\equiv 0, \text{but } f_{\xi_k}\big(\boldsymbol{\varphi}(0)\big) = 0.$$

We define holomorphic functions $a_{ij}(x)$ and $b_k(x)$ by

$$a_{ij}(x) = f_{t_i\tau_j}\big(\boldsymbol{\varphi}(x)\big) + f_{u\tau_j}(\boldsymbol{\varphi}(x))\varphi_i(x) + \sum_{k=1}^{n} f_{\tau_j\xi_k}(\boldsymbol{\varphi}(x))\frac{\partial \varphi_i}{\partial x_k}(x)$$

$$b_k(x) = f_{\xi_k}\big(\boldsymbol{\varphi}(x)\big)(= O(|x|))$$

We put $\{\lambda_j\}$ the eigenvalues of $(a_{ij}(0))_{i,j=1,\cdots,d}$

and put $\{\mu_k\}$ the eigenvalues of $\frac{\partial(b_1,\cdots,b_n)}{\partial(x_1,\cdots,x_n)}(0)$.

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Theorem 1 ([S] Funkcial Ekvac, 45(2002))

Under the assumptions 1, 2 and 3, if $\mathrm{Ch}(\{\lambda_j\},\{\mu_k\}) \not\ni 0$ (this condition is called "Poincaré condition"), then the formal solution is convergent near the origin.

Theorem 2 ([S] Sūrikaiseki kenkyūjo kōkyūroku, 1431(2005))

Under the assumptions 1, 2 and 3, we assume that $\mathrm{Ch}(\{\lambda_j\}) \not\ni 0$

Moreover, $\frac{\partial(b_1,\cdots,b_n)}{\partial(x_1,\cdots,x_n)}(0) \sim \mathrm{diag}(N_1,\cdots,N_I)$ where N_j $(j=1,\cdots,I)$

denotes the nilpotent Jordan block of size n_j and we put $n_0=\max\{n_1,\cdots,n_I\}$.

Then the formal solution diverges in general, and it belongs to the Gevrey class of order at most $2n_0$, namely, the power series

$$\sum_{|\alpha|\geq 2,|\beta|\geq 0} \frac{u_{\alpha\beta}}{(|\alpha|+|\beta|)!^{2n_0-1}} t^\alpha x^\beta \in \mathbb{C}\{t,x\}.$$

The main theorem in this talk is stated as follows.

Theorem 3 (Main Theorem)

Under the assumptions 1, 2 and 3, we assume that $\mathrm{Ch}(\{\mu_k\}) \not\ni 0$

Moreover, $\left(a_{ij}(0)\right) \sim \mathrm{diag}(N_1, \cdots, N_I)$ where

N_j $(j = 1, \cdots, I)$ denotes the nilpotent Jordan block of size d_j

and we put $d_0 = \max\{d_1, \cdots, d_I\}$. Then the formal solution $u(t,x)$ is divergent in general, and it belongs to the Gevrey class of order at most $(2d_0, d_0 + 1)$, that is, for the formal solution $\sum_{\alpha\in\mathbb{N}^d,\beta\in\mathbb{N}^n} u_{\alpha\beta}\, t^\alpha x^\beta$, the power series

$$\sum_{\alpha\in\mathbb{N}^d,\beta\in\mathbb{N}^n} \frac{u_{\alpha\beta}}{|\alpha|!^{2d_0-1}\,|\beta|!^{d_0}} t^\alpha x^\beta$$

is convergent in a neighborhood of the origin.

2. Refinement of Main Theorem

Let $v(t,x) = u(t,x) - \sum_{j=1}^{d} \varphi_j(x)\, t_j = O(|t|^K)$ $(K \geq 2)$.
Then $v(t,x)$ satisfies the following equation.

$$\text{(2)} \qquad \left(\sum_{i,j=1}^{d} a_{ij}(x) t_i \partial_{t_j} + \sum_{k=1}^{n} b_k(x) \partial_{x_k} + f_u\big(\boldsymbol{\varphi}(x)\big) \right) v(t,x)$$

$$= \sum_{|\alpha|=K} d_\alpha(x)\, t^\alpha + f_{K+1}(t,x,v,\partial_t v, \partial_x v)$$

where

$$f_{K+1}(t,x,v,\tau,\xi) = \sum_{V(\alpha,p,q,r)\geq K+1} f_{\alpha pqr}(x)\, t^\alpha v^p \tau^q \xi^r \in \mathcal{O}_x\{t,v,\tau,\xi\}$$

$V(\alpha,p,q,r) = |\alpha| + Kp + (K-1)|q| + K|r|$ (vanishing order in t)

By linear changes of variables which bring $\left(a_{ij}(0)\right)$ and $\frac{\partial(b_1,\cdots,b_n)}{\partial(x_1,\cdots,x_n)}(0)$ to Jordan canonical forms, the equation (2) is reduced to the following form.

(3) $$(\mathcal{N}+\mathcal{D}+\Delta)v(t,x)=\sum_{i,j,k,l}\alpha_{ijkl}(x)\,t_{i,j}\partial_{t_{k,l}}v+\sum_{k=1}^{n}\beta_k(x)\,\partial_{x_k}v$$
$$+\eta(x)v+\sum_{|\alpha|=K}\zeta_\alpha(x)\,t^\alpha+g_{K+1}(t,x,v,\partial_t v,\partial_x v)$$

where

$$\mathcal{N}=\sum_{j=1}^{I}\sum_{k=1}^{d_j-1}\delta t_{j,k+1}\,\partial_{t_{j,k}}\,,\qquad \mathcal{D}=\sum_{k=1}^{n}\mu_k\,x_k\partial_{x_k}+f_u(\varphi\,(0)),$$

$$\Delta=\sum_{k=1}^{n-1}\nu_k\,x_{k+1}\partial_{x_k}\,,\qquad \alpha_{ijkl}(x),\eta(x)=O(|x|),\qquad \beta_k(x)=O(|x|^2).$$

Definition 1 ($(\boldsymbol{s},\boldsymbol{\sigma})$-Borel transform and Gevrey class)

Let $\boldsymbol{s}=(s_1,\cdots,s_d)\in(\mathbb{R}_{\geq 1})^d$, $\boldsymbol{\sigma}=(\sigma_1,\cdots,\sigma_n)\in(\mathbb{R}_{\geq 1})^n$.
For a formal power series $u(t,x)=\sum u_{\alpha\beta}\, t^\alpha x^\beta$, we define $(\boldsymbol{s},\boldsymbol{\sigma})$-Borel transform of $u(t,x)$ by

$$\mathfrak{B}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t,x)=\sum\frac{u_{\alpha\beta}|\alpha|!\,|\beta|!}{(\boldsymbol{s}\cdot\alpha)!\,(\boldsymbol{\sigma}\cdot\beta)!}t^\alpha x^\beta$$

where $\boldsymbol{s}\cdot\alpha=s_1\alpha_1+\cdots+s_d\alpha_d$, $\boldsymbol{\sigma}\cdot\beta=\sigma_1\beta_1+\cdots+\sigma_n\beta_n$.
We say that $u(t,x)$ belongs to $\mathcal{G}^{(\boldsymbol{s},\boldsymbol{\sigma})}$, if $\mathfrak{B}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t,x)$ converges in a neighborhood of the origin.

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Remark

By an easy calculation, the following relation holds.

$$\mathfrak{B}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t,x)\in\mathcal{G}^{(\boldsymbol{s}',\boldsymbol{\sigma}')}\Longrightarrow u(t,x)\in\mathcal{G}^{(\boldsymbol{s}+\boldsymbol{s}'-1,\boldsymbol{\sigma}+\boldsymbol{\sigma}'-1)}$$

Proposition 1

Let $\boldsymbol{s}^j = (1,2,\cdots,d_j) \in \mathbb{N}^{d_j}$ $(j = 1,2,\cdots,I)$ and $d_0 = \max\{d_1,\cdots d_I\}$.
Under the assumptions 1, 2, 3 and Poincaré condition for $\{\mu_k\}$,
the formal solution belongs to the Gevrey class of order at most
$(\boldsymbol{s}', \boldsymbol{\sigma}')$ where $\boldsymbol{s}'$ and $\boldsymbol{\sigma}'$ are defined as follows.

$$\boldsymbol{s}' = (\boldsymbol{s}^1, \boldsymbol{s}^2, \cdots, \boldsymbol{s}^I) + (d_0, d_0, \cdots, d_0) \in \mathbb{N}^d$$
$$\boldsymbol{\sigma}' = (d_0 + 1, d_0 + 1, \cdots, d_0 + 1) \in \mathbb{N}^n$$

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$$\boldsymbol{\sigma}' = (d_0+1,d_0+1,\cdots,d_0+1) \in \mathbb{N}^n$$

Proof of Theorem 3 (Proof of Mail Theorem)

Theorem 3 is proved by Proposition 1 immediately. Indeed, all the components of $(\boldsymbol{s}^1,\boldsymbol{s}^2,\cdots,\boldsymbol{s}^I)$ are estimated by d_0. Therefore, all the components of $\boldsymbol{s}'$ are estimated by $2d_0$, which is the desired estimate for Theorem 3.

3. Sketch of the Proof of Proposition 1 in the Example

In this section, we give the sketch of the proof of Proposition 1 in the typical example case. The equation we consider is as follows.

Let $(t,x) = (t_1, t_2, x) \in \mathbb{C}^2 \times \mathbb{C}$. We consider the following nonlinear PDE.

$$Pu(t,x) = a(x)(t_1 + t_2)^2 + xt_1\partial_{t_2}u + (t_2\partial_{t_2}u)(\partial_x u) \qquad (4)$$

with $u(t,x) = O(|t|^2)$, where $a(x) \in \mathbb{C}\{x\}$ and

$$P = t_2\partial_{t_1} + x\partial_x + 1.$$

Proposition 2

$$u(t_1, t_2, x) \in \mathcal{G}^{(3,4,3)}.$$

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Let $\mathbb{C}[t]_L x^m = \left\{\sum_{\alpha_1+\alpha_2=L} u_{\alpha_1,\alpha_2,m} {t_1}^{\alpha_1} {t_2}^{\alpha_2} x^m\right\}$.

Lemma 1

(1) $P = t_2 \partial_{t_1} + x\partial_x + 1$ is invertible on $\mathbb{C}[t]_L x^m$ for all $L \geq 2$ and $m \geq 0$.

(2) Let $\tilde{\boldsymbol{s}} = (\boldsymbol{s}^1, 1) = (1,2,1)$. For $u(t,x) \in \mathbb{C}[t]_L x^m$, if

$$\mathfrak{B}^{\tilde{s}}(u)(t,x) \ll V_{Lm} T^L X^m \quad (T = t_1 + t_2, X = x, V_{Lm} \geq 0)$$

then $\mathfrak{B}^{\tilde{s}}(P^{-1}u)(t,x) \ll C_0 (X\partial_X + 1)^{-1} V_{Lm} T^L X^m$.

Lemma 2

For $u(t,x) = \sum u_{\alpha\beta}\, t^{\alpha} x^{\beta}$, we define $|u|(t,x) = \sum |u_{\alpha\beta}|\, t^{\alpha} x^{\beta}$. For $\tilde{\boldsymbol{s}} = (1,2,1)$,

(i) $\mathfrak{B}^{\tilde{s}}(uv)(t,x) \ll C_1 \mathfrak{B}^{\tilde{s}}(|u|)(t,x) \cdot \mathfrak{B}^{\tilde{s}}(|v|)(t,x)$

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(i) $\mathfrak{B}^{\tilde{s}}(uv)(t,x) \ll C_1 \mathfrak{B}^{\tilde{s}}(|u|)(t,x) \cdot \mathfrak{B}^{\tilde{s}}(|v|)(t,x)$

(ii) If $\mathfrak{B}^{\tilde{s}}(u)(t,x) \ll V(T,X)$=$\sum_{L\geq 2} V_L(X) T^L = \sum_{L\geq 2, M\geq 0} V_{L,M}\, T^L X^M$, then

$$\mathfrak{B}^{\tilde{s}}\left(\partial_{t_1} P^{-1} u\right) \ll C_2 \partial_T (X\partial_X + 1)^{-1} V(T,X)$$

$$\mathfrak{B}^{\tilde{s}}\left(\partial_{t_2} P^{-1} u\right) \ll C_2 \partial_T (T\partial_T)(X\partial_X + 1)^{-1} V(T,X)$$

$$\mathfrak{B}^{\tilde{s}}(\partial_x P^{-1} u) \ll C_2 \partial_X (X\partial_X + 1)^{-1} V(T,X) \ll C_2 S(V)(T,X)$$

where $S(V)(T,X)$ is called “Shift operator” which is defined by

$$S(V)(T,X) = \sum_{L\geq 2} S(V_L)(X) T^L \coloneqq \frac{V(T,X) - V(T,0)}{X} = \sum_{L\geq 2, M\geq 0} V_{L,M+1}\, T^L X^M$$

Let $U(t,x) = Pu(t,x)$ be a new unknown function. Then $U(t,x)$ satisies

$$U = a(x)(t_1+t_2)^2 + xt_1\partial_{t_2}P^{-1}U + \left(t_2\partial_{t_2}P^{-1}U\right)(\partial_x P^{-1}U).$$

We apply $\tilde{\boldsymbol{s}}$-Borel transform to above equation, we have

$$\begin{gathered}\mathfrak{B}^{\tilde{s}}(U) = \mathfrak{B}^{\tilde{s}}(a(x)(t_1+t_2)^2) + \mathfrak{B}^{\tilde{s}}\left(xt_1\partial_{t_2}P^{-1}U\right)\\ +\mathfrak{B}^{\tilde{s}}\left\{\left(t_2\partial_{t_2}P^{-1}U\right)(\partial_x P^{-1}U)\right\}.\end{gathered}$$

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We apply $\tilde{\boldsymbol{s}}$-Borel transform to above equation, we have

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By Lemma 1 and 2,

$$\mathfrak{B}^{\tilde{s}}(a(x)(t_1+t_2)^2) \ll |a|(X)T^2$$

$$\mathfrak{B}^{\tilde{s}}\left(xt_1\partial_{t_2}P^{-1}U\right) \ll CX(T\partial_T)^2(X\partial_X+1)^{-1}V(T,X)$$

$$\begin{aligned}&\mathfrak{B}^{\tilde{s}}\left\{\left(t_2\partial_{t_2}P^{-1}U\right)(\partial_x P^{-1}U)\right\} \\ &\qquad \ll C(T\partial_T)^2(X\partial_X+1)^{-1}V\cdot S(V)(T,X)\end{aligned}$$

Therefore, we consider the equation

$$
\begin{aligned}
&V = |a|(X)T^2 + CX(T\partial_T)^2(X\partial_X + 1)^{-1}V \\
&\qquad\qquad +C(T\partial_T)^2(X\partial_X + 1)^{-1}V \cdot S(V)(T,X), \\
&V(T,X) = O(T^2).
\end{aligned}
$$

By the construction of this equation, we have $\mathfrak{B}^{\tilde{s}}(U)(t,x) \ll V(T,X)$.

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&V(T, X) = O(T^2).
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By the construction of this equation, we have $\mathfrak{B}^{\tilde{s}}(U)(t, x) \ll V(T, X)$.

Let $V = \sum_{L\geq 2} V_L(X)\, T^L$ be an unknown function. By substituting this into the equation, we have the following recurrence formula for $\{V_L(X)\}_{L=2,3,\cdots}$

$$
V_2(X) = |a|(X) + 2^2 CX(X\partial_X + 1)^{-1}V_2(X) \qquad \text{(r.f. 2)}
$$

and for $L \geq 3$,

$$
\begin{aligned}
V_L(X) &= CL^2 X(X\partial_X + 1)^{-1}V_L(X) \\
&\qquad + \sum_{L_1+L_2=L} {L_1}^2 X(X\partial_X + 1)^{-1}V_{L_1}(X) \cdot S(V_{L_2})(X). \qquad \text{(r.f. } L\text{)}
\end{aligned}
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Let $V_L(X) = \sum_{M\geq 0} V_{L,M} X^M$ be the Taylor expansion. By substituting this into (r.f. L), we have the following recurrence formula for $\{V_{L,M}\}_{M=0,1,2,\cdots}$

$$V_{L,M} = \frac{CL^2}{M} V_{L,M-1} + C \sum_{L_1+L_2=L} \sum_{M_1+M_2=M} \frac{{L_1}^2}{M_1+1} V_{L_1,M_1} V_{L_2,M_2+1}$$

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We replace $V_{L,M}$ by $W_{L,M} = \frac{V_{L,M}}{(L+M)!^2}$ (this is equivalent to (3,3)-Borel transform), $\{W_{L,M}\}_{M=0,1,2,\cdots}$ satisfies the following recurrence formula.

$$\begin{aligned} &W_{L,M} \\ &= \frac{CL^2(L+M-1)!^2}{M(L+M)!^2} W_{L,M-1} \\ &+ C \sum_{L_1+L_2=L} \sum_{M_1+M_2=M} \frac{{L_1}^2 (L_1+M_1)!^2\,(L_2+M_2+1)!^2}{(M_1+1)(L+M)!^2} W_{L_1,M_1} W_{L_2,M_2+1} \end{aligned}$$

Here we can estimate as follows.

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$$\frac{{L_1}^2(L_1+M_1)!^2\,(L_2+M_2+1)!^2}{(M_1+1)(L+M)!^2}\leq\frac{{L_1}^2 2!^2\ (L+M-1)!^2}{(M_1+1)(L+M)!^2}$$
$$=\frac{4{L_1}^2}{(M_1+1)(L+M)^2}\leq 4$$

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$$= \frac{4{L_1}^2}{(M_1+1)(L+M)^2} \leq 4$$

Here we consider the following recurrence formula

$$Y_{L,M} = a_M + CY_{L,M-1} + 4C \sum_{L_1+L_2=L} \sum_{M_1+M_2=M} Y_{L_1,M_1} Y_{L_2,M_2+1}$$

By the construction of this recurrence formula, we have $W_{L,M} \leq Y_{L,M}$ for all L and M, that is, $W_L(X) \ll Y_L(X) = \sum_{M\geq 0} Y_{L,M} X^M$ for all L.

By an easy calculation, $Y(T,X) = \sum_{L\geq 2} Y_L(X)\, T^L$ satisfies the following equation.

$$Y(T,X) = |a|(X)T^2 + CXY(T,X) + 4CY(T,X) \cdot S(Y)(T,X)$$

with $Y(T,X) = O(T^2)$.

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with $Y(T,X) = O(T^2)$.

Here we put $Y(T,X) = TZ(T,X) \ \ (Z(T,X) = O(T))$, $Z(T,X)$ satisfies the equation

$$Z(T,X) = |a|(X)T + CXZ(T,X) + 4CZ(T,X) \cdot TS(Z)(T,X).$$

By an easy calculation, $Y(T,X)$ satisfies the following equation.

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Here we put $Y(T,X) = TZ(T,X)$ $(Z(T,X) = O(T))$, $Z(T,X)$ satisfies the equation

(5) $$Z(T,X) = |a|(X)T + CXZ(T,X) + 4CZ(T,X) \cdot TS(Z)(T,X).$$

Since $Z(T,X) \gg 0$ and by the definition of shift operator, $XS(Z)(T,X)$ is estimated by

$$XS(Z)(T,X) = Z(T,X) - Z(T,0) \ll Z(T,X).$$

We put $\varphi(\rho) = Z(\rho, \rho)$. In this case, $\rho S(Z)(\rho, \rho) \ll Z(\rho, \rho) = \varphi(\rho)$ holds.

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For the equation

$$Z(T, X) = |a|(X)T + CXZ(T, X) + 4CZ(T, X) \cdot TS(Z)(T, X) \quad \text{with } Z(0, X) \equiv 0\ ,$$

we set $T = \rho, X = \rho$.
The formal solution $\psi(\rho)$ of the equation

$$\psi(\rho) = |a|(\rho)\rho + C\rho\psi(\rho) + 4C\psi(\rho)^2 \ \text{ with } \ \psi(0) = 0$$

is convergent in a neighborhood of the origin by the implicit function theorem.
Moreover by using the above majorant relation, we obtain $Z(\rho, \rho) \ll \psi(\rho)$.
This implies that the formal solution $Z(T, X)$ of (3) is also convergent near the origin.

We put $\varphi(\rho) = Z(\rho,\rho)$. In this case, $\rho S(Z)(\rho,\rho) \ll Z(\rho,\rho) = \varphi(\rho)$ holds.

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$$Z(T,X) = |a|(X)T + CXZ(T,X) + 4CZ(T,X) \cdot TS(Z)(T,X) \quad \text{with } Z(0,X) \equiv 0\,,$$

we set $T = \rho, X = \rho$.
The formal solution $\psi(\rho)$ of the equation

$$\psi(\rho) = |a|(\rho)\rho + C\rho\psi(\rho) + 4C\psi(\rho)^2 \ \text{ with } \ \psi(0) = 0$$

is convergent in a neighborhood of the origin by the implicit function theorem.
Moreover by using the above majorant relation, we obtain $Z(\rho,\rho) \ll \psi(\rho)$.
This implies that the formal solution $Z(T,X)$ of (3) is also convergent near the origin.

This implies that

$$\mathbb{C}\{T,X\} \ni TZ(T,X) = Y(T,X) \gg W(T,X) \gg \mathfrak{B}^{(3,3)}(V)(T,X)$$

Namely, $V(T,X) = V(t_1 + t_2, x) \in \mathcal{G}^{(3,3,3)}$.

Moreover, $\mathfrak{B}^{\tilde{s}}(U)(t,x) = \mathfrak{B}^{(1,2,1)}(U)(t,x) \ll V(T,X) \in \mathcal{G}^{(3,3,3)}$, then the Gevrey order of $U(t,x)$ is $(\boldsymbol{s}', \boldsymbol{\sigma}') = (1,2,1) + (3,3,3) - (1,1,1) = (3,4,3)$,
this is the consequence of Proposition 2.