Constructing measures of orthogonality with applications

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Formal and Analytic Solutions of Differential, Difference and Discrete Equations

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Outline

Definition

Let us define $\mathfrak{F} : \operatorname{Dom}\mathfrak{F} \to \mathbb{C}$ by relation

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty}(-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1},$$

where

$$\operatorname{Dom}\mathfrak{F} = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables let me identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$.

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- However, function $\mathfrak{F}$ is also related with *continued fractions*, *bilateral second order difference equations*, as well as *orthogonal polynomials*.
- In this talk we focus on usage of $\mathfrak{F}$ for description of the *measure of orthogonality* of orthogonal polynomials.

1. Put $x_k = z/(\nu + k)$, then

$$\mathfrak{F}(x) = \Gamma(\nu + 1) z^{-\nu} J_\nu(2z),$$

for $z \in \mathbb{C}$ and $-\nu \notin \mathbb{N}$, where J_ν is the *Bessel function* of the first kind.

Some examples

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2. Put $x_k = z^{1/2} q^{(2k-1)/4}$, then

$$\mathfrak{F}(x) = A_q(z) := {}_0\phi_1(;0;q,-qz),$$

for $z \in \mathbb{C}$ and $q \in (0,1)$, where A_q is *Ramanujan function* (or *q-Airy function*).

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3. Put

$$x_k = \frac{q^{\frac{1}{2}(\alpha+\gamma+k)-\frac{3}{4}} (q^{\gamma-\alpha+k}; q^2)_\infty \, z^{\frac{1}{2}}}{(q^{\gamma-\alpha+k+1}; q^2)_\infty \left(1-(1-z)q^{\gamma+k-1}\right)},$$

then

$$\mathfrak{F}(x) = \frac{(q^\gamma; q)_\infty}{((1-z)q^\gamma; q)_\infty} \, {}_1\phi_1(q^\alpha; q^\gamma; q, -q^\gamma z),$$

for $z, \alpha, \gamma \in \mathbb{C}$, $(1-z)q^\gamma \notin q^{-\mathbb{Z}_+}$ and $q \in (0,1)$, where ${}_1\phi_1$ is *q-confluent hypergeometric function* (proof in [F. Š., P. Šťovíček, LAA, 2013]).

Fundamental property of $\mathfrak{F}$

- For all $x \in \operatorname{Dom}\mathfrak{F}$ and $k = 1, 2, \ldots$ one has

Recursive relation

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \ldots, x_k)\,\mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x)$$

where T denotes the left shift operator defined on the space of all sequences:

$$T(\{x_k\}_{k=1}^{\infty}) = \{x_{k+1}\}_{k=1}^{\infty}.$$

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Typical example: For $x_k = z/(\nu + k - 1)$, the simple recurrence relation for $\mathfrak{F}$ yields the well known formula for Bessel functions:

$$J_{\nu-1}(2z) = \frac{\nu}{z} J_\nu(2z) - J_{\nu+1}(2z).$$

Function $\mathfrak{F}$ and orthogonal polynomials

- By the *Favard's theorem*, the couple of polynomial sequences $(\{F_n\}_{n=0}^{\infty}, \{G_n\}_{n=0}^{\infty})$ defined recursively by equation

$$u_{n+1} = (x - \lambda_n)u_n - w_{n-1}^2 u_{n-1}, \quad n = 1, 2, \dots,$$

where $\lambda_n \in \mathbb{R}$ and $w_n > 0$, and with initial conditions

$$\begin{aligned} F_0(x) &= 1, & F_1(x) &= x - \lambda_0, \\ G_0(x) &= 0, & G_1(x) &= 1, \end{aligned}$$

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- As one easily verifies by induction, polynomials F_n and G_n can be expressed in terms of $\mathfrak{F}$,

$$F_n(x) = \prod_{k=0}^{n-1} (x - \lambda_k)\, \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - x} \right\}_{l=0}^{n-1} \right), \quad n = 0, 1 \ldots,$$

and

$$G_n(x) = \prod_{k=1}^{n-1} (x - \lambda_k)\, \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - x} \right\}_{l=1}^{n-1} \right), \quad n = 0, 1 \ldots,$$

where the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is defined recursively by $\gamma_0 = 1$, $\gamma_{k+1} = w_k/\gamma_k$.

Proposition

If $\sum_{k\geq 0} \left|\frac{w_k^2}{(x-\lambda_k)(x-\lambda_{k+1})}\right| < \infty$, for some $x \in \mathbb{C}$, then the limit relation

$$\lim_{n\to\infty} \prod_{k=0}^{n-1} (x-\lambda_k)^{-1} F_n(x) = \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - x}\right\}_{k=0}^{\infty}\right)$$

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Typical example: By setting $\lambda_k = 0$ and $w_k = [4(k+\nu)(k+\nu+1)]^{-1/2}$, polynomials

$$F_n(x) = x^n \, \mathfrak{F}\left(\left\{ \frac{1}{2x(\nu+k)} \right\}_{k=0}^{n-1}\right), \quad n = 0, 1, 2 \ldots,$$

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$$R_{n,\nu}(x) = 2^n\frac{\Gamma(\nu+n)}{\Gamma(\nu)}F_n(x^{-1}).$$

Asymptotic behavior of $F_n(x)$ as $n \to \infty$

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The above limit relation yields the Hurwitz's asymptotic formula for Lommel polynomials

$$\lim_{n\to\infty} \frac{x^n}{2^n\Gamma(\nu+n)} R_{n,\nu}(x) = \left(\frac{x}{2}\right)^{-\nu+1} J_{\nu-1}(x).$$

Regularization

- The asymptotic behavior of F_n, as $n \to \infty$, is expressed in terms of function

$$\Phi(\lambda, w; z) = \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=0}^{\infty}\right)$$

under the assumption that ensures the function to be well defined. This function is meromorphic on $\mathbb{C} \setminus \mathrm{der}(\lambda)$ with poles at $z = \lambda_k$ such that $\lambda_k \notin \mathrm{der}(\lambda)$.

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- Taking into account later application, we restrict sequences λ and w such that $\lambda \in \ell^1(\mathbb{Z}_+)$ and $w \in \ell^2(\mathbb{Z}_+)$. Then function
$$\psi_\lambda(z) = \prod_{n=0}^{\infty}(1 - z\lambda_n)$$
is well defined entire function and $\psi_\lambda^{(-1)}(\{0\}) = \{\lambda_n^{-1} : \lambda_n \neq 0, n \in \mathbb{Z}_+\}$.

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- Let us define function

$$G(\lambda, w; z) = \begin{cases} \psi_\lambda(z)\Phi(\lambda, w; z^{-1}) & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

Assuming $\lambda \in \ell^1(\mathbb{Z}_+)$ and $w \in \ell^2(\mathbb{Z}_+)$, function $G(\lambda, w; .)$ is entire.

- For the limit of the ratio $G_n(z^{-1})/F_n(z^{-1})$, now we have

$$\lim_{n\to\infty}\frac{G_n(z^{-1})}{F_n(z^{-1})}=z\,\frac{G(T\lambda,Tw;z)}{G(\lambda,w;z)},$$

  for all $z\neq 0$ not being zeros of function $G(\lambda,w;.)$.

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Theorem (Markov)

Let λ be real and w positive sequence and, moreover, both bounded. Then polynomials $\{F_n\}_{n=0}^{\infty}$ are orthogonal with respect to measure μ, for which, it holds

$$\int_{\mathbb{R}} \frac{d\mu(x)}{z-x} = \lim_{n\to\infty} \frac{G_n(z)}{F_n(z)},$$

and the convergence is uniform on any compact subset of $\mathbb{C} \setminus \mathbb{R}$.

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- Thus, by the Markov theorem, one finds

$$\int_{\mathbb{R}}\frac{d\mu(x)}{1-xz} = \frac{G(T\lambda, Tw; z)}{G(\lambda, w; z)}.$$

Proposition

Let $\lambda \in \ell^1(\mathbb{Z}_+)$ be real and $w \in \ell^2(\mathbb{Z}_+)$ be positive sequence. Then all zeros of functions $G(\lambda, w; .)$ and $G(T\lambda, Tw; .)$ are real, simple, and there are infinitely many of them (for each function). Moreover, these two functions have no zero in common.

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Typical example: If we put $\lambda_k = 0$ and $w_k = [4(\nu + k)(\nu + k + 1)]^{-1/2}$, with $\nu > 0$, then the statement is about zeros of Bessel functions $z^{-\nu+1}J_{\nu-1}(z)$ and $z^{-\nu}J_{\nu}(z)$.

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- Recall we know

$$\int_{\mathbb{R}} \frac{d\mu(x)}{1 - xz} = \frac{G(T\lambda, Tw; z)}{G(\lambda, w; z)}$$

  where μ is the measure of orthogonality for polynomials $\{F_n\}_{n=0}^{\infty}$.

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where μ is the measure of orthogonality for polynomials $\{F_n\}_{n=0}^{\infty}$.
- It can be shown from this formula measure μ is supported by reciprocal values of points, where the RHS has poles, and the origin, i.e.,
$$\mathrm{supp}(\mu) = \{0\} \cup \{z^{-1} : G(\lambda, w, z) = 0\}.$$

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- It can be shown from this formula measure μ is supported by reciprocal values of points, where the RHS has poles, and the origin, i.e.,
$$\operatorname{supp}(\mu) = \{0\} \cup \{z^{-1} : G(\lambda, w, z) = 0\}.$$
- Furthermore, denoting by μ_k, $k \in \mathbb{N}$, zeros of $G(\lambda, w; .)$, we have the Mittag-Leffler expansion
$$\Lambda_0 + \sum_{k=1}^{\infty} \frac{\Lambda_k}{1 - \mu_k^{-1} z} = \frac{G(T\lambda, Tw; z)}{G(\lambda, w; z)}$$
where the convergence of the sum is local uniform in $z \notin \{\mu_k : k \in \mathbb{N}\}$.

Towards orthogonality

- Numbers Λ_k represents jumps of distribution function $F_\mu(x) := \mu((-\infty, x])$ at $x = \mu_k^{-1}$ and Λ_0 jump at $x = 0$. We can express these jumps as

$$\Lambda_k = \lim_{z \to \mu_k} (1 - \mu_k^{-1} z) \frac{G(T\lambda, Tw; z)}{G(\lambda, w; z)} = -\mu_k^{-1} \frac{G(T\lambda, Tw; \mu_k)}{(\partial_z G)(\lambda, w; \mu_k)}.$$

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$$\Lambda_k = \lim_{z\to\mu_k}(1-\mu_k^{-1}z)\frac{G(T\lambda, Tw; z)}{G(\lambda, w; z)} = -\mu_k^{-1}\frac{G(T\lambda, Tw; \mu_k)}{(\partial_z G)(\lambda, w; \mu_k)}.$$

- Finally, the orthogonality relation for polynomials $\{F_n\}_{n=0}^{\infty}$ reads

$$\int_{\mathbb{R}} F_m(x)F_n(x)d\mu(x) = \left(\prod_{k=0}^{n-1} w_k^2\right)\delta_{mn}, \quad m, n \in \mathbb{Z}_+.$$

Theorem

For $\lambda \in \ell^1(\mathbb{Z}_+)$ be real and $w \in \ell^2(\mathbb{Z}_+)$ positive sequence we introduce function

$$G(\lambda, w; z) = \prod_{n=0}^{\infty}(1 - z\lambda_n)\, \mathfrak{F}\left(\left\{\frac{z\gamma_k^2}{1 - z\lambda_k}\right\}_{k=0}^{\infty}\right),$$

Then the measure of orthogonality μ of corresponding orthogonal polynomials $\{F_n\}_{n=0}^{\infty}$ is supported by a real sequence with 0, the only cluster point. Moreover, we have

$$\operatorname{supp}(\mu) = \{0\} \cup \{z^{-1} : G(\lambda, w; z) = 0\}.$$

The orthogonality relation reads

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and, for $x \in \operatorname{supp}(\mu) \setminus \{0\}$, distribution function $F_\mu(x) := \mu((-\infty, x])$ has jumps

$$F_\mu(x) - F_\mu(x - 0) = -x\frac{G(T\lambda, Tw; x^{-1})}{(\partial_z G)(\lambda, w; x^{-1})}.$$

- The example with q-confluent hypergeometric function introduced at the beginning, slightly reparametrized, yields

$$\mathfrak{F}\left(\left\{\frac{q^{\frac{1}{2}(\delta+k)-\frac{1}{4}}\,(q^{k+1-\delta};q^2)_\infty\,\sqrt{-a}}{(q^{k+2-\delta};q^2)_\infty\,\left((a+1)q^k-x\right)}\right\}_{k=0}^{\infty}\right)=\frac{(x^{-1};q)_\infty}{(x^{-1}(a+1);q)_\infty}\,{}_1\phi_1\left(x^{-1}q^{\delta};x^{-1};q,ax^{-1}\right)$$

where $x\notin(a+1)q^{\mathbb{Z}_+}\cup\{0\}$.

Application: Orthogonal polynomial arising from example with ${}_1\phi_1$

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- This identity correspond with the polynomial sequence $U_n(a,\delta;q,x)$, $n\in\mathbb{Z}_+$, which is generated by recursion

$$v_{n+1}=\left(x-(a+1)q^n\right)v_n+aq^{n+\delta-1}(1-q^{n-\delta})v_{n-1},\quad n\in\mathbb{N},$$

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- Thus, in this case, sequences $\{\lambda_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are as follows:

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- For $\delta=0$, polynomials $U_n(a,0;q,x)$ are known as Al-Salam-Carlitz I and are listed in the q-Askey scheme. They can be expressed as

$$U_n(a,0;q,x)=(-a)^n q^{\binom{n}{2}}\,{}_2\phi_1\left(q^{-n},x^{-1};0;q,a^{-1}qx\right).$$

- In this case, one deduces

$$G(T^k\lambda, T^k w; x) = {}_1\tilde{\phi}_1(xq^{\delta}; q^k x; q, aq^k x), \quad \text{for } k = 0, 1, 2, \ldots,$$

where ${}_1\tilde{\phi}_1$ denoted regularized q-confluent hypergeometric function defined by

$${}_1\tilde{\phi}_1(a; b; q, z) := (b; q)_\infty {}_1\phi_1(a; b; q, z).$$

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$$\operatorname{supp}(\mu) = \{x^{-1} \in \mathbb{C} : {}_1\tilde{\phi}_1\left(xq^{\delta}; x; q, ax\right) = 0\} \cup \{0\},$$

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The orthogonality of generalized Al-Salam-Carlitz I polynomials

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- If $\delta = 0$ these results yields orthogonality for Al-Salam-Carlitz I polynomials, which can be described fully explicitly.

Suppose $x \neq 0$ then the generating function for $U_n(a, \delta; q, x)$ reads:

i) if $\delta < 0$,

$$\sum_{n=0}^{\infty} \frac{U_n(a,\delta;q,x)}{(q^{-\delta};q)_{n+1}} t^n = \sum_{k=0}^{\infty} \frac{(aq^{\delta}t;q)_k (q^{\delta}t;q)_k}{(xt;q)_{k+1}} q^{-k\delta}, \quad |xt| < 1,$$

Other properties - Generating function

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iii) if $\delta > 0$,

-unknown-

- For $n \in \mathbb{Z}_+$, it holds

$$\mathcal{D}_q U_n(a,\delta;q,x) = \frac{1-q^{n-\delta}}{1-q} U_{n-1}(a,\delta;q,x) - q^n \frac{1-q^{-\delta}}{1-q} U_{n-1}(a,\delta-1;q,x).$$

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- This formula is a generalization of the *forward shift* formula for Al-Salam-Carlitz I polynomials,

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- On the other hand, it seems there is no simple formula which would generalize the *backward shift* for Al-Salam-Carlitz I polynomials, which reads

$$(a-x)(1-x)U_n(a,0;q,q^{-1}x) - aU_n(a,0;q,x) = xq^{-n}U_{n+1}(a,0;q,x).$$

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- Consequently, we do not know if there is a second order q-difference equation for polynomials $U_n(a, \delta; q, x)$.

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Thank you!