On singularly perturbed ordinary differential equations with several Gevrey orders.

Catherine STENGER

University of La Rochelle - France.

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Let a system of singularly perturbed differential equations

$$(S) \quad \varepsilon^D \frac{dY}{dx} = A(x)Y + F(x)$$

- x is a complex variable
- ε is a small complex parameter
- $D = \mathrm{diag}(d_1, \ldots, d_n), \quad d_i \in \mathbb{N}^*$
- $A(x)$ is an analytic matrix function in a neighborhood of $x = 0$
- $F(x)$ is an analytic vector function in a neighborhood of $x = 0$.

Question

We want to study the existence of analytic functions $Y(x, \varepsilon)$ that solve (S) and remain bounded as $\varepsilon \to 0$, uniformly in x in a full neighborhood of $x = 0$

and this in the case where

$$D = \mathrm{diag}(d_1, \ldots, d_n) \neq d.\mathrm{Id}_n.$$

Tools

- formal solutions
- Gevrey character
- Gevrey asymptotic expansions

Motivations

We know by Canalis-Durand, Ramis, Schäfke and Sibuya (2000) that if

$$\det\left(A(0)\right) \neq 0$$

then

(i) (S) has a unique formal solution

$$\hat{Y}(x,\varepsilon) = \sum_{r=0}^{+\infty} Y_r(x)\varepsilon^r$$

where $Y_r(x)$ are analytic in a neighborhood of $x = 0$

(ii) $\hat{Y}(x,\varepsilon)$ is Gevrey of order $1/\min_i d_i$

(iii) the formal Borel transforms $\mathcal{B}\hat{Y}$ and the truncated Laplace transforms $\mathcal{L}\left(\mathcal{B}\hat{Y}\right)$ provide quasi-solutions $Y_q = \mathcal{L}\left(\mathcal{B}\hat{Y}\right)$ of (S)

$$\left|\varepsilon^D Y_q' - A(x)Y_q - F(x)\right| \leq K \exp\left(-\frac{T}{|\varepsilon|^{\min_i d_i}}\right)$$

(iv) the existence of solutions Y of (S) in full neighborhoods of $x = 0$ independent of ε follows by Gronwall's lemma

$$|Y - Y_q| \leq c \exp\left(-\frac{C}{|\varepsilon|^{\min_i d_i}}\right)$$

for $|x| \leq K|\varepsilon|^{\max_i d_i - \min_i d_i}$, if

$$D = \mathrm{diag}(d_1, \ldots, d_n) = d.\mathrm{Id}_n.$$

Now, we are interested at the the case

- $n = 2$
- $D = \mathrm{diag}(d_1, d_2), \quad d_i \in \mathbb{N}^*, \quad d_1 \neq d_2$

our idea is to find quasi-solutions $\tilde{Y}_q$ that satisfy

$$|\varepsilon^D Y_q' - A(x) Y_q - F(x)| \leq c \exp\left(-\frac{C}{|\varepsilon|^{\max_i d_i}}\right)$$

We write the system (S)

$$\varepsilon^{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}' = \begin{pmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix}$$

by using its associated second order differential equation

$$(E) \quad \varepsilon^{d_1+d_2} Y_2'' + a_1(x,\varepsilon) Y_2' + a_0(x,\varepsilon) Y_2 = f(x,\varepsilon)$$

where if $A_{21}(0) \neq 0$, the coefficients $a_i(x,\varepsilon), f(x,\varepsilon)$ are analytic in x in a neighborhood of $x = 0$ and polynomial in ε.

We choose to write the unique formal solution $\hat{Y}_2$ of (E) as a power series in the variable x

$$\hat{Y}_2(x,\varepsilon) = \sum_{m=0}^{+\infty} \hat{u}_m(\varepsilon) x^m.$$

Therefore, the formal coefficients $\hat{u}_m(\varepsilon)$ satisfy

$$(R) \quad (m+1)(m+2)\varepsilon^{d_1+d_2}\hat{u}_{m+2}(\varepsilon) + \sum_{n=-1}^{m} c_n(m,\varepsilon)\hat{u}_{m-n}(\varepsilon) = g(m,\varepsilon)$$

where the coefficients $c_n(m,\varepsilon)$, $g(m,\varepsilon)$ are polynomial in ε and m.
To study the behavior of $\hat{u}_m(\varepsilon)$, we suppose that

$$(R) \quad (m+1)(m+2)\varepsilon^{d_1+d_2}\hat{u}_{m+2}(\varepsilon) + c_p(m,\varepsilon)\hat{u}_{m-p}(\varepsilon) = g(m,\varepsilon)$$

where $p \in \{-1, 0, 1, \ldots, m\}$.

We yield that necessary $p = 0$ and we prove that the formal solutions $\hat{u}_m(\varepsilon)$ of (R) are

$$\hat{u}_{2q+r}(\varepsilon) = (-1)^q \prod_{k=0}^{q-1} \frac{c_0(2k+r,\varepsilon)}{c_{-2}(2k+r,\varepsilon)}$$

$$\times \sum_{\ell=q}^{+\infty} (-1)^\ell \frac{g(2\ell+r,\varepsilon)}{c_0(2\ell+r,\varepsilon)} \prod_{k=0}^{\ell-1} \frac{c_{-2}(2k+r,\varepsilon)}{c_0(2k+r,\varepsilon)}$$

Localization of the singularities of $\hat{u}_m(\varepsilon)$

The coefficient $\hat{u}_m(\varepsilon)$ can have a singularity at $\varepsilon = \varepsilon(m)$ where $\varepsilon(m)$ is defined by

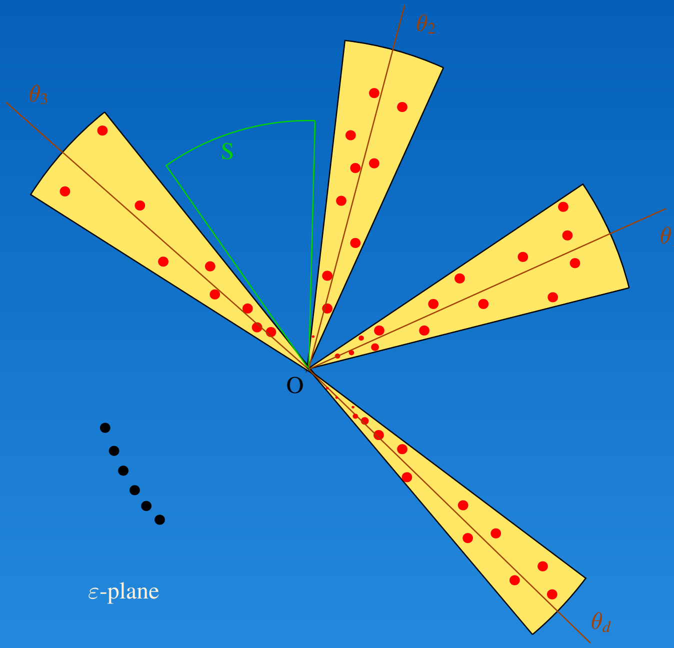
$$c_0(m, \varepsilon(m)) = 0.$$

We prove that

$$\begin{cases} \lim_{m\to 0} \varepsilon(m) = 0 \\ \lim_{m\to 0} \arg \varepsilon(m) = \theta_j \end{cases}$$

for $j \in J$, where J is a finite set dependent on d_1, d_2.

Now, we are able to consider a sector $\mathcal{S}$ of the ε-plane which doesn't contain singularities $\varepsilon(m)$.


θ_2
θ_3
S
θ_1
O
θ_d
ε-plane

Construction of our quasi-solution

- Let

$$f(t,x,\varepsilon) = \sum_{q=0}^{+\infty}\sum_{k=0}^{+\infty} a_{2q+r,k}(\varepsilon)t^k x^{2q+r}$$

  where the coefficients $a_{2q+r,k}(\varepsilon)$ are definied from the $\hat{u}_{2q+r}(\varepsilon)$.

  We prove that f is an analytic function for $|t|$ and $|x|$ small enough and for ε in our sector $\mathcal{S}$.

- We consider the truncated Laplace transform

$$F_\theta(u,x,\varepsilon) = \frac{1}{u}\int_0^{Te^{i\theta}} e^{-t/u} f(t,x,\varepsilon)dt$$

  where

$$F_\theta(u,x,\varepsilon) \sim \sum_{q=0}^{+\infty}\sum_{k=0}^{+\infty} a_{2q+r,k}k!(\varepsilon)t^k x^{2q+r}$$

  as $u \to 0$ in some sector $\mathcal{S}_u$, uniformly for x and ε.

- The function

$$Y_\theta(x,\varepsilon) = \frac{1}{\varepsilon^d} F_\theta(\varepsilon^{\tilde{d}}, x, \varepsilon)$$

(where d, $\tilde{d}$ depend on d_1, d_2)
is a quasi-solution of (E) such that

$$\left|\varepsilon^{d_1+d_2} Y_\theta'' + a_1(x,\varepsilon) Y_\theta' + a_0(x,\varepsilon) Y_\theta - f(x,\varepsilon)\right|$$

$$\leq K \exp\left(-\frac{T}{|\varepsilon|^{\max_i d_i}}\right)$$