Meromorphic Solutions of some Second order linear q-Difference Equations

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1 Introduction

At first, we consider the relation of the following two q-difference equations. One of them is first order q-difference equation

$$\boldsymbol{f(z) = \bar{a}(z)f(qz) + \bar{b}(z), \quad (|q| < 1)} \tag{$*$}$$

and another equation is second order q-difference equation

$$\boldsymbol{f(c^2z) + \tilde{a}(z)f(cz) + \tilde{b}(z)f(z) = 0, \quad (|c| < 1)} \tag{$**$}$$

which have been considered in [3], where $\bar{a}$, $\bar{b}$, $\tilde{a}$, $\tilde{b}$ are polynomial functions.

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[3]: W. Bergweiler, K. Ishizaki and N. Yanagihara, *Meromorphic solutions of some functional equations*, **Methods Appl. Anal. Vol. 5(3) (1998), 248–258.** *Correction Methods*, **Appl. Anal. 6(4) (1999).**

We want to write the second order Eq. $(**)$ making use of the first order Eq. $(*)$. Because, we want to consider some problem for the equation

$$f(s^2 z) = a_1(z) f(sz) + a_2(z) f(z), \qquad (***)$$

which was proposed by the late prof. Niro Yanagihara.

Where $\quad s = |s| e^{-2\pi\lambda i}$, $|s| > 1$, $0 \leq \lambda < 1$, $\lambda \notin \mathbb{Q}$,

$a_j(z) \in \mathbb{C}[z], (j = 1, 2)$, and $a_2(z) \neq 0$.

However the second order Eq. $(**)$ could't been derived by the first order Eq. $(*)$ under the condition that $\tilde{a}$, $\tilde{b}$ are polynomial.

Then we must consider the following second order q-difference equation

$$f(q^2 z) + a^*(z) f(qz) + b^*(z) f(z) = 0, \quad (|q| < 1), \qquad (1.1)$$

where $a^*(z)$, $b^*(z)$ are rational functions.

The Previous Researches

R. D. Carmichael (1912), Raymond Adams (1928-1929):

They have studied nth linear q-difference equation, however they haven't considered meromorphic solutions.

J. P. Ramis (1992):

They have considered the existence of meromorphic solutions for nth linear q-difference equation under the condition that coefficients are constants.

D. C. Barnett, R. G. Halburd, W. Morgan, R. J. Korhonen (2007), W. Bergweiler, K. Ishizaki, N. Yanagihara (2002):

They have considered meromorphic solutions of nth linear of q-difference equation with polynomial coefficients, however the existence of meromorphic solutions haven't been considered.

Put $a^*(z) = \dfrac{b(z)}{c(z)}$, $b^*(z) = \dfrac{a(z)}{c(z)}$ by some polynomial functions $a(z)$, $b(z)$ and $c(z)$, then we can write Eq.(1.1) as

$$a(z)f(z) + b(z)f(qz) + c(z)f(q^2z) = 0, \quad (|q| < 1), \tag{1.2}$$

where

$$a(z) = \sum_{k=0}^{A} a_k z^k,\ b(z) = \sum_{k=0}^{B} b_k z^k,\ c(z) = \sum_{k=0}^{C} c_k z^k.$$

In this talk, we will consider the existence and behavior of meromorphic solutions of Eq.(1.2).

Our aim in this talk is to propose the following Theorem 1.

Theorem 1

We have as follows.

(i) If there exists no integer k satisfying $a_0 + b_0 q^k + c_0 q^{2k} = 0$, then the Eq. (1.2) possesses no non-trivial meromorphic solutions.

(ii) If $a_0 \neq 0$ and there exists an integer k such that $a_0 + b_0 q^k + c_0 q^{2k} = 0$, then (1.2) possesses a non-trivial meromorphic solution.

(iii) For the case $a_0 = 0$, we have the following tree cases.

($\sharp$) We suppose that $a(z) = a_A z^A$, $\max(B, C) \leq A$, $A \geq 1$. If (1.2) possesses a non-trivial meromorphic solution, then the solution is not transcendental, and the solution has no poles at any $z \neq 0$.

($\sharp\sharp$) We suppose that $a_K \neq 0$, $1 \leq K < A$ and

$$a(z) = a_A z^K (z-\mu_1)\cdots(z-\mu_{A-K}),\ (\mu_j \neq 0,\ j = 1,\cdots,A-K). \tag{1.3}$$

Put

$$g(z) = \prod_{n=0}^{\infty}\left(\prod_{j=1}^{A-K}\Big(1-\frac{q^n z}{\mu_j}\Big)\right) f(z), \tag{1.4}$$

$$U(z) = \prod_{n=0}^{\infty}\left(\prod_{j=1}^{A-K}\Big(1-\frac{q^n z}{\mu_j}\Big)\right),\ V(z) = \prod_{j=1}^{A-K}\Big(1-\frac{z}{\mu_j}\Big), \tag{1.5}$$

and

$$C_0 = (-1)^{A-K} a_K \prod_{j=1}^{A-K} \mu_j. \tag{1.6}$$

If (1.2) possesses a rational solution $f(z)$, then

$$1 \leq K < \min\left(A, \max\left(B, \frac{A+C}{2}\right)\right), \tag{1.7}$$

and the following second order q-difference equation

$$C_0 z^K g(z) + b(z)g(qz) + c(z)V(qz)g(q^2 z) = 0, \tag{1.8}$$

possesses a transcendental meromorphic solution. Further the Eq. (1.8) is reducible.

($\sharp\sharp\sharp$) We suppose

$$\max\left(B, \frac{A+C}{2}\right) \leq K < A. \tag{1.9}$$

If (1.2) possesses a non-trivial meromorphic solution, then the solution is transcendental.

The late professor Yanagihara talked about "irreducibility" of functional equations.

There may be several definitions.

I) If any solution of a second order equation satisfies a first order equation, we call the second order equation reducible, otherwise irreducible.

II) If at least one solution of a second order equation satisfies a first order equation, we call reducible, otherwise irreducible.

In this talk, we call "reducible equation" by the latter definition.

We will prove Theorem 1 in following 7 Steps.

1) We determine a formal solution.

2) The convergence radius of the formal solution is positive in the case $a_0 \neq 0$.

3) The local solution will be has an analytic continuation.

4) We change parameters and estimate them.

5) Under the condition $a_0 = 0$, if we have a non-trivial meromorphic solution of (1.2), then the solution is not transcendental with the conditions of ($\sharp$).

6) Further under the conditions of ($\sharp$ $\sharp$), if we have a non-trivial meromorphic solution of (1.2), then the second order q-difference equation (1.8) possesses a transcendental meromorphic solution and it is reducible.

7) At last, under the conditions of ($\sharp$ $\sharp$ $\sharp$), if we have a non-trivial meromorphic solution of (1.2), then the solution is transcendental.

2 Proof of Theorem 1 (i) and (ii)

2.1 A formal solution (step 1)

We consider a formal solution of (1.2) which is given by a power series at the origin. Let p be an integer (negative may be possible). Set

$$f(z) = \sum_{k=p}^{\infty} \alpha_k z^k, \quad (\alpha_p \neq 0). \tag{2.1}$$

We may avoid to consider the case all of $a(z)$, $b(z)$ and $c(z)$ are constants. In above [3], they have considered meromorphic solutions of (1.2) in which the case $c(z) \equiv 1$, however some parts of the proof in it, need corrections. Hence in this talk we avoid the case $c(z) \equiv 1$.

Then we assume that $M := \max(A, B, C) > 0$.

Comparing the coefficients α_l of z^l, we have

$$(a_0 + b_0 q^k + c_0 q^{2k})\alpha_k = -\sum_{h=1}^{k-p} (a_h + b_h q^{k-h} + c_h q^{2(k-h)})\alpha_{k-h}. \quad (2.2)$$

If (1.2) has a non-trivial meromorphic solution, then there exists an integer k_0 such that $k = k_0$ is a solution of

$$a_0 + b_0 q^k + c_0 q^{2k} = 0. \quad (2.3)$$

For the integer solutions k_1 and k_2, we define

$$k_0 = \max(k_1, k_2).$$

Since $|q| < 1$, we see

$$a_0 + b_0 q^k + c_0 q^{2k} \neq 0, \ \text{for } (k > k_0,).$$

Letting $p = k_0$, we can determine the coefficients α_k such that

$$\begin{cases} \alpha_k = 0, \quad (k \leq k_0 - 1) \\ \alpha_{k_0} = \text{any value} \\ \alpha_k = \dfrac{\sum_{h=1}^{k-k_0} \left(a_h \alpha_{k-h} + b_h \alpha_{k-h} q^{k-h} + c_h \alpha_{k-h} q^{2(k-h)} \right)}{a_0 + b_0 q^k + c_0 q^{2k}}, \ (k \geq k_0 + 1), \end{cases} \tag{2.4}$$

where from definition of functions a, b, c, we assume

$$a_h = 0, \ (h > A), \ b_i = 0, \ (i > B), \ c_j = 0, \ (j > C).$$

Therefore we can determine a formal solution

$$f(z) = \sum_{k=k_0}^{\infty} \alpha_k z^k, \quad (\alpha_{k_0} \neq 0), \tag{2.5}$$

of (1.2). Here we have proved (i) of Theorem 1.

2.2 Existence of meromorphic solution (step 2)

Hereafter in this section, we suppose that $\boxed{a_0 \neq 0}$, and we will prove (ii) of Theorem 1.

2.2.1 A positive convergence radius of the formal solution

In the case of $\boxed{a_0 \neq 0}$, we will prove the formal solution (2.5) of (1.2), has no positive convergence radius by Cauchy's root test. Therefore the Eq. (1.2) possesses a local meromorphic solution $f(z)$ which has a power series such that

$$f(z) = \sum_{k=k_0}^{\infty} \alpha_k z^k, \quad (\alpha_{k_0} \neq 0), \tag{2.6}$$

in a neighborhood of the origin.

2.2.2 Global solutions by the analytic continuation (Step 3)

We define a domain

$$D_m = \{|z| < \frac{R}{|q|^m}\}, \tag{2.7}$$

and we assume that

$$\text{"the Eq. (1.2) possesses a local meromorphic solution } f(z) \text{ in } D_0\text{"}. \tag{2.8}$$

Since $0 < |q| < 1$, $D_m \subset D_{m'}$ as $m < m'$ and we see $\cup_{m=0}^{\infty} D_m = \mathbb{C}$.
We can write (1.2) as

$$f(z) = -\frac{b(z)}{a(z)} f(qz) - \frac{c(z)}{a(z)} f(q^2 z) \quad (|q| < 1). \tag{2.9}$$

For a $z \in D_1/D_0$, the $f(qz)$, $f(q^2 z)$, $\frac{b(z)}{a(z)}$, and $\frac{c(z)}{a(z)}$ are also meromorphic in D_1. Thus the $f(z)$ satisfying (2.9) is meromorphic in D_1.

Repeating this process we construct a global solution.

3 Proof of Theorem 1 (iii), the case $a_0 = 0$

When we assume that Eq. (1.2) possesses a non-trivial meromorphic solution, we will investigate the solution under the condition $\boxed{a_0 = 0}$.

3.1. Some preparations (previous Step 4)

3.1.1. A constant K

At first we define K to be the smallest integer such that

$$(a_0, a_1, \cdots, a_{K-1}) = (0, 0, \cdots, 0), \ a_K \neq 0, \quad K \geq 1. \tag{3.1}$$

For the K, we set $\gamma = \frac{1}{2K}$.

3.1.2. Conditions of the constant b_0 and c_0

When we have a non-trivial solution of (1.2), then we can assume that $c_0 \neq 0$, $b_0 \neq 0$ without lost generality.

3.1.3. Change parameter α_k to δ_k

For α_k, we have

$$\sum_{j=0}^{M}(a_j + b_j q^{k-j} + c_j q^{2(k-j)})\alpha_{k-j} = 0, \tag{3.2}$$

where $a_h = 0$ $(A < h)$, $b_h = 0$ $(B < h)$, $c_h = 0$ $(C < h)$.

Next we define a new parameter β_k, δ_k and a constant t as in

$$\alpha_k = \beta_k q^{-\gamma k^2}, \ \ (k = k_0, k_0 + 1, \cdots), \tag{3.3}$$

$$\beta_k = t^k \delta_k \text{ for } k \in \mathbb{Z}, \ \ (\delta_k = 0 \text{ for } k < k_0), \ \text{ and } \ -\frac{a_K}{b_0} q^{-\frac{K}{2}} = t^K. \tag{3.4}$$

Then we write (3.2) as in the following form

$$\boxed{\begin{aligned}\left(1 + \frac{c_0 q^k}{b_0}\right)\delta_k = \delta_{k-K} - q^{2\gamma k}(r_1\delta_{k-1} + r_2\delta_{k-2} \\ + \cdots + r_K\delta_{k-K} + \cdots + r_M\delta_{k-M}).\end{aligned}} \tag{3.5}$$

where

$$\eta_j = (k-j) - \gamma(k-j)^2 - 2\gamma k + \gamma(k-K)^2 \quad (1 \le j \le K), \tag{3.6}$$

$$\zeta_j = -\gamma(k-j)^2 - 2\gamma k + \gamma(k-K)^2 \quad (K+1 \le j \le M), \tag{3.7}$$

$$r_j = \begin{cases} \big(b_j + c_j q^{k-j}\big)\big(\dfrac{1}{b_0 t^j} q^{\eta_j - K/2}\big), & (j = 1, 2, \cdots, K), \\ \big(a_j + b_j q^{k-j} + c_j q^{2(k-j)}\big)\Big(\dfrac{1}{b_0 t^j} q^{\zeta_j - K/2}\Big), & (j = K+1, \cdots, M). \end{cases} \tag{3.8}$$

3.1.4. An existence of a constant ρ

As the last preparation, we have a constant ρ such that

$$|r_1| + \cdots + |r_K| + |r_{K+1}| + \cdots + |r_M| \leq \rho, \tag{3.9}$$

where

$$\rho = \sum_{j=1}^{K} \big(|b_j| + |c_j|\big) \frac{1}{|b_0||t^j|} |q|^{\left(\gamma(K^2-j^2)-j\right)-K/2} + \sum_{j=K+1}^{M} \big(|a_j| + |b_j| + |c_j|\big) \left| \frac{1}{b_0 t^j} q^{\gamma(K^2-j^2)-K/2} \right|. \tag{3.10}$$

3.2. Estimations of $|\delta_k|$ (Step 4)

Here we have following lemmas for estimations of $|\delta_k|$.

Lemma 3.1. *For any $p > 0$, there exists a constant $T > 0$ such that for any k*

$$|\delta_k| \leq (1+p)^k T, \tag{3.11}$$

where δ_k are defined in (3.4).

Lemma 3.2.

$$\lim_{k\to\infty} \delta_k = 0 \tag{3.12}$$

Lemma 3.3. *Putting $q_1 = |q|^{2\gamma}(1+p)$ and $L = \left(\left|\frac{c_0}{b_0}\right| + \frac{\rho}{q_1^M}\right)\left(\frac{1}{1-q_1^K}\right)$,*

$$|\delta_k| < q_1^k T L, \tag{3.13}$$

for any $k \in \mathbb{Z}$, where $\delta_k = 0$ for $k < k_0$.

$\boxed{\text{Lemma 3.4.}}$ *Let $p > 0$, n, m_1, m_2 be constants satisfying the following conditions:*

$$q_1 := |q|^{2\gamma}(1+p) < 1, \tag{3.14}$$

$$|q|^{\frac{n}{2}}\Big(\Big|\frac{c_0}{b_0}\Big| + \frac{\rho}{q_1^M}\Big)\Big(\frac{1}{1-q_1^K}\Big) < 1, \tag{3.15}$$

$$n \geq 2M, \tag{3.16}$$

$$\frac{1+\rho(|q|^{2\gamma})^m}{1-(|c_0|/|b_0|)|q|^m} < 1+p, \text{ for any } \ m > m_1 > 0, \tag{3.17}$$

$$|q|^{2\gamma k}L = |q|^{\frac{k}{K}}\Big(\Big|\frac{c_0}{b_0}\Big| + \frac{\rho}{q_1^M}\Big)\Big(\frac{1}{1-q_1^K}\Big) < |q|^{n\gamma M}, \text{ for any } k > m_2 > 0. \tag{3.18}$$

$$N_0 = \max(nM-K,\, m_1,\, m_2). \tag{3.19}$$

If $K = M$, then for any $\nu (\in \mathbb{N})$, we have

$$|\delta_k| \leq q_1^k T(|q|^{n\gamma M}L)^\nu, \tag{3.20}$$

provided

$$k \geq N_0. \tag{3.21}$$

We note when $K < M$, Then we have

$$|\delta_k| \leq q_1^k T(|q|^{n\gamma M} L)^\nu, \tag{3.20}$$

holds for $\nu = i + 2$ for $k \geq N_0 + 2(M - K)$. However, when $\nu \to \infty$, (3.20) holds for $k \to \infty$, i.e., we don't have this proof.

We note that we haven't been able to derive the inequality [3.20] in proof of Theorem 3.1 (iii) in [3]. Hence we replace the [3.20] in [3] to (3.20) in this paper, with the additional condition $K = A = M$.

3.3. The proof of iii) ($\sharp$) of Theorem 1 (Step 5)

Proof At first we assume that there exist a meromorphic solution

$$f(z) = \sum_{k=k_0}^{\infty} \alpha_k z^k \tag{2.6}$$

of (1.2). In the case $K = M$ of ($\sharp$), from Lemma 3.4. and $|q|^{n\gamma M} L < 1$, we have

$$|\delta_k| \to 0 \text{ as } \nu \to \infty.$$

From definition of α, β and t we get

$$\alpha_k = \beta_k q^{-\gamma k^2} = t^k \delta_k q^{-\gamma k^2} = 0, \quad (k \geq nM - K = (n-1)M).$$

Thus we see that the solution $f(z)$ is a non-transcendental solution of Eq. (1.2). If $k_0 < 0$, $f(z)$ admits a pole at $z = 0$. However it has no poles at any $z = z_0 \neq 0$ in this case.

3.4. The proof of iii) (♯♯) of Theorem 1 (Step 6)

In this case we suppose $1 \leq K < A \leq M$. From definitions we have

$$a(z) = a_A z^A + \cdots + a_K z^K$$
$$= \left((-1)^{A-K} a_K \prod_{j=1}^{A-K} \mu_j\right) z^K \prod_{j=1}^{A-K} \left(1 - \frac{z}{\mu_j}\right) = C_0 z^K V(z), \quad (3.22)$$

$$g(z) = U(z) f(z), \quad \left(U(z) = \prod_{n=0}^{\infty}\left(\prod_{j=1}^{A-K}\left(1 - \frac{q^n z}{\mu_j}\right)\right),\right) \quad (3.23)$$

Thus we write (1.2) as

$$C_0 z^K g(z) + b(z) g(qz) + c(z) V(qz) g(q^2 z) = 0. \quad (1.8)$$

Where the coefficient function $c(z)V(qz)$ is a polynomial in which $C + A - K$ degree. If (1.2) possesses a rational solution $f(z) = R(z)$, then $g(z)$ is a transcendental meromorphic solution of Eq. (1.8).

On the other hand, from Theorem 1 (iii) ($\sharp$),
if $1 \leq K = \max(K, B, C + A - K) = M'$, then the Eq. (1.8) don' t possess a transcendental meromorphic solution which has a power series of (2.6).

Therefore, in the case (iii) ($\sharp\sharp$), if (1.2) possess a rational solution and (1.8) possess a transcendental meromorphic solution, then the condition $K < \max(K, B, C + (A - K)) = M'$ is satisfied.
Further since $1 \leq K < A$, we have

$$1 \leq K < \min\Big(A, \max\Big(B, \frac{A+C}{2}\Big)\Big). \tag{1.7}$$

Further if (1.2) possess a rational solution $R(z)$, then the solution $g(z) = U(z)R(z)$ of (1.8) satisfies the following first order equation

$$g(z) - \frac{V(z)R(z)}{R(qz)} g(qz) = 0, \tag{3.24}$$

in which the form

$$a^{**}(z)g(z) + b^{**}(z)g(qz) = 0, \tag{3.25}$$

where $a^{**}(z)$ and $b^{**}(z)$ are polynomials.

Now (1.2) equivalent to (1.8) which has a solution the $g(z)$. Therefore the Eq. (1.2) is reducible.

3.5. The proof of iii) ($\sharp\sharp\sharp$) of Theorem 1 (Step 7)

When $K < A$, we see

$$\boxed{K = \max(K, B, C + (A - K)) \Longleftrightarrow \max(B, \frac{A+C}{2}) \leq K < A.}$$

From step 6, if $\max(B, \frac{A+C}{2}) \leq K < A$, Eq. (1.8) don' t possess a transcendental meromorphic solution. Thus if (1.2) possess a non-trivial rational solution $f(z)$, the existence of a transcendental solution $g(z)$ of Eq. (1.8) is contradiction.

We have completed the proof of Theorem 1.

4 Example

In this section we will propose same examples Eq. (1.2) possesses polynomial solution, rational solution, or transcendental meromorphic solution, in the case $a_0 \neq 0$ and $a_0 = 0$.

4.1. The case of $a_0 \neq 0$

Example 4.1.: A monomial solution.

In the Eq. (1.2), We suppose that

$$b(z) = \sum_{k=0}^{B} b_k z^k, \quad c(z) = \sum_{k=0}^{C} c_k z^k.$$

Further we suppose that

$$a(z) = -q^m b(z) - q^{2m} c(z), \text{ and } a_0 = -q^m b_0 - q^{2m} c_0 \neq 0, \ m \in \mathbb{N},$$

We write Eq. (1.2) as

$$\boxed{\big(- q^m b(z) - q^{2m} c(z) \big) f(z) + b(z) f(qz) + c(z) f(q^2 z) = 0, \quad (4.1)}$$

and $f(z) = z^m$ is a monomial solution of Eq. (4.1).

Example 4.2.: A polynomial solution.

We suppose that $a(z) = z^2 + 3z + \frac{131}{11}$, $b(z) = -\frac{101}{88}z^2 + z + 1$, $c(z) = -\frac{1}{22}(251z^2 + 290z + 284)$, and $q = \frac{1}{2}$. We write the Eq.(1.2) as

$$\boxed{\begin{aligned}&\Big(z^2 + 3z + \frac{131}{11}\Big) f(z) + \Big(-\frac{101}{88}z^2 + z + 1\Big) f(\frac{z}{2}) \\ &\qquad + \Big(-\frac{1}{22}(251z^2 + 290z + 284)\Big) f(\frac{z}{4}) = 0, \qquad (4.2)\end{aligned}}$$

and we have a polynomial solution $f(z) = z^2 + z + 1$ of (4.2).

Example 4.3.: A rational solution which has a pole at $z = 0$.

We suppose that

$$a(z) = 2z^2 - \frac{1}{3}, \quad b(z) = -\frac{9}{2}z^2 + 1, \quad c(z) = z^2 - \frac{2}{3},$$

and $q = \frac{1}{2}$. We write the Eq. (1.2) as

$$\boxed{\left(z^2 - \frac{2}{3}\right) f(z) + \left(-\frac{9}{2}z^2 + 1\right) f\left(\frac{z}{2}\right) + \left(2z^2 - \frac{1}{3}\right) f\left(\frac{z}{4}\right) = 0, \qquad (4.3)}$$

and have a rational solution $f(z) = \frac{1}{z} - \frac{7}{24} + z^2$ of (4.3) which has only one pole at $z = 0$.

Example 4.4.: A rational solution which has a pole at $z \neq 0$.

We suppose that $a(z) = z^2 - \frac{59}{15}z + \frac{44}{15}$, $b(z) = -\frac{1}{2}z^2 + z$, $c(z) = -\frac{2}{15}(z-4)(105z-22)$, and $q = \frac{1}{2}$. We write the Eq. (1.2) as

$$\Big(z^2 - \frac{59}{15}z + \frac{44}{15}\Big)f(z) + \Big(-\frac{1}{2}z^2 + z\Big)f\big(\frac{z}{2}\big) + \Big(-\frac{2}{15}(z-4)(105z-22)\Big)f\big(\frac{z}{4}\big) = 0. \qquad (4.4)$$

and have a rational solution $f(z) = \dfrac{135z^3 - 135z^2 + 22z}{135(z-1)}$ which has a pole at $z = 1$

The case of $a_0 = 0$

Example 4.5.: A rational solution has a pole at $z = 0$, (iii) ($\sharp$)

We suppose that $b(z) = b_2 z^2 + b_1 z + 4$, $c(z) = c_2 z^2 + c_1 z - 1$, $b_2 = \dfrac{142 \pm \sqrt{566857}}{441}$, $c_2 \neq 0$, $a(z) = z^2$, and $q = \dfrac{1}{2}$. We write the Eq. (1.2) as

$$\boxed{z^2 f(z) + \Big(b_2 z^2 + b_1 z + 4\Big) f\big(\frac{z}{2}\big) + \Big(c_2 z^2 + c_1 z - 1\Big) f\big(\frac{z}{4}\big) = 0. \quad (4.5)}$$

and have a rational solution $f(z) = \dfrac{1}{z^2} - \dfrac{1}{z} - 1 + \Big(-\dfrac{108}{7} b_2 + \dfrac{220}{7}\Big) z$. Put $h(z) = z^2 f(z)$, $h(z)$ satisfies the following equation

$$\boxed{z^2 h(z) + 4\Big(b_2 z^2 + b_1 z + 4\Big) h\Big(\frac{z}{2}\Big) + 16\Big(c_2 z^2 + c_1 z - 1\Big) h\Big(\frac{z}{4}\Big) = 0. \quad (4.6)}$$

Here the solution f has a pole at $z = 0$ order 2, and h has no pole.

Example 4.6: A reducible equation in the case (iii) ($\sharp\sharp$).

We suppose that $a(z) = z^2 + 3z$, $b(z) = -\frac{21}{16}z^2 + z + 1$, $c(z) = -\frac{1}{4}(43z^2 + 58z + 8)$, and $q = \frac{1}{2}$. Then we write the Eq. (1.2) as

$$\boxed{\left(z^2+3z\right)f(z)+\left(-\frac{21}{16}z^2+z+1\right)f\left(\frac{z}{2}\right)-\frac{1}{4}(43z^2+58z+8)f\left(\frac{z}{4}\right)=0. \qquad (4.7)}$$

and have a solution $f(z) = z^2 + z$,

Put $g(z) = (z^2 + z)\prod_{n=0}^{\infty}\left(1 + \dfrac{z}{3\cdot 2^n}\right)$, $C_0 = 3$, and $V(z) = 1 + \dfrac{z}{3}$.

The $g(z)$ is a transcendental meromorphic function and satisfies

$$\boxed{3zg(z)+\left(-\frac{21}{16}z^2+z+1\right)g\left(\frac{z}{2}\right)-\frac{1}{4}(43z^2+58z+8)\left(1+\frac{z}{6}\right)g\left(\frac{z}{4}\right).} \tag{4.8}$$

We also see that $g(z)$ satisfies the first order q-difference equation

$$\boxed{\left(\frac{z^2}{2^2}+\frac{z}{2}\right)g(z)-\left(1+\frac{z}{3}\right)(z^2+z)g\left(\frac{z}{2}\right)=0.}$$

The second order linear q-difference equation (4.8) equivalent to (4.7), and possesses a transcendental meromorphic solution and it is reducible.

Example 4.7: A reducible equation in the case (iii) (♯♯).

We suppose that $a(z) = z^2 + z$, $b(z) = -\dfrac{112}{71}z^2 - \dfrac{619}{284}z + 1$, $c(z) = -\dfrac{60}{71}z^2 + \dfrac{203}{284}z - \dfrac{1}{2}$, $b_2 c_2 \neq 0$, $q = \dfrac{1}{2}$. **Then we write the Eq. (1.2) as**

$$\begin{aligned}\Big(z^2 + z\Big)f(z) + \Big(-\frac{112}{71}z^2 - \frac{619}{284}z + 1\Big)f\big(\frac{z}{2}\big) \\ + \Big(-\frac{60}{71}z^2 + \frac{203}{284}z - \frac{1}{2}\Big)f\big(\frac{z}{4}\big) = 0,\end{aligned} \tag{4.9}$$

and we have a rational solution

$f(z) = \dfrac{1}{z} + 1 + 16z = \dfrac{1 + z + 16z^2}{z}$, **where the solution has a pole at** $z = 0$.

Put $g(z) = \dfrac{1+z+16z^2}{z} \prod_{n=0}^{\infty}\left(1+\dfrac{z}{2^n}\right)$, $C_0 = 1$, $V(z) = 1 + z$.

The $g(z)$ is a transcendental meromorphic function and satisfies

$$\boxed{\begin{aligned} &zg(z)+\Big(-\frac{112}{71}z^2-\frac{619}{284}z+1\Big)g\Big(\frac{z}{2}\Big)\\ &\qquad +\Big(-\frac{60}{71}z^2+\frac{203}{284}z-\frac{1}{2}\Big)\Big(1+\frac{z}{2}\Big)g\Big(\frac{z^2}{2^2}\Big)=0. \qquad (4.10)\end{aligned}}$$

We also see that $g(z)$ satisfies the following equation,

$$\boxed{(2+z+8z^2)g(z)-(1+z)(1+z+16z^2)g\Big(\frac{z}{2}\Big)=0. \qquad (4.11)}$$

Thus the second order linear q-difference equation (4.10) is equivalent to (4.9) possesses a transcendental meromorphic solution and it is reducible.

Example 4.8: A solution in case (iii) ($\sharp\sharp\sharp$) .

I am sorry, we have't non-trivial meromorphic solution in this case.

I will propose that this is open problem.

5 Conclusion

$$\sum_{k=0}^{A} a_k z^k f(z) + \sum_{k=0}^{B} b_k z^k f(qz) + \sum_{k=0}^{C} c_k z^k f(q^2 z) = 0, \ \ (|q| < 1). \quad (1.2)$$

1) If $a_0 + b_0 q^k + c_0 q^{2k} = 0$ has a integer solution k and $a_0 \neq 0$, then (1.2) has a non-trivial meromorphic solution.

2) When $a_0 = 0$,

i) if $\sum_{k=1}^{A} a_k z^k = a_A z^A \neq 0$, Eq. (1.2) has no transcendental meromorphic solution,

ii) if $\sum_{k=1}^{A} a_k z^k = a_K z^K + \cdots + a_A z^A \neq 0$, $(a_K \neq 0)$ and Eq. (1.2) has a rational solution, then $1 \leq K < \min\left(A, \max\left(B, \dfrac{A+C}{2}\right)\right)$ and the equation is reducible,

iii) we suppose that Eq. (1.2) has a non-trivial meromorphic solution, $\sum_{k=0}^{A} a_k z^k = a_K z^K + \cdots + a_A z^A \neq 0$, $(a_K \neq 0)$ and $\max\left(B, \dfrac{A+C}{2}\right) \leq K < A$, then the solution is transcendental.

Thank you very much

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