Summability of Formal Solutions to the Cauchy Problem for Some Linear Partial Differential Equations

Hidetoshi TAHARA
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This is a joint work with Hiroshi YAMAZAWA

August 29, 2013, Poland (Będlewo)

In this talk, I will consider the following Cauchy problem for linear partial differential equation

$$\begin{cases} \partial_t^m u + \displaystyle\sum_{(j,\alpha)\in\Lambda} a_{j,\alpha}(t)\partial_t^j \partial_x^\alpha u = f(t,x), \\ \partial_t^i u\Big|_{t=0} = \varphi_i(x), \quad i = 0, 1, \ldots, m-1. \end{cases}$$

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The plan of the talk is as follows:

- 1. Equation and assumtions
- 2. Examples, motivation and problem
- 3. Main theorem
- 4. Idea of the proof

1. Equation and assumptions

1.1. Equation

We will consider the Cauchy problem:

$$(E) \qquad \begin{cases} \partial_t^m u + \sum\limits_{(j,\alpha)\in\Lambda} a_{j,\alpha}(t)\partial_t^j \partial_x^\alpha u = f(t,x), \\ \partial_t^i u\Big|_{t=0} = \varphi_i(x), \quad i = 0, 1, \ldots, m-1 \end{cases}$$

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where

- $(t,x) = (t, x_1, \ldots, x_N) \in \mathbb{C} \times \mathbb{C}^N$,
- $m \geq 1$ is an integer,
- Λ is a finite subset of $\mathbb{N} \times \mathbb{N}^N$,
- $a_{j,\alpha}(t)$, $f(t,x)$ and $\varphi_i(x)$ are holomorphic functions in a neighborhood of the origin.

1.2. Formal solution

We suppose:

$$(A_1) \quad \mathrm{ord}_t(a_{j,\alpha}) \geq \max\{0, j-m+1\}, \quad \forall (j,\alpha) \in \Lambda.$$

where $\mathrm{ord}_t(a_{j,\alpha})$ denotes the order of the zero of the function $a_{j,\alpha}(t)$ at $t=0$.

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Proposition 1. The Cauchy problem (E) has a unique formal solution $\hat{u}(t, x)$ of the form

$$\hat{u}(t, x) = \sum_{n=0}^{\infty} u_n(x) t^n \in \mathcal{O}_R[[t]]$$

for some $R > 0$, where $\mathcal{O}_R$ denotes the set of all holomorphic functions on $D_R = \{x \in \mathbb{C}^N ; |x| < R\}$.

1.3. Basic problem

(1) If

$$j + |\alpha| \leq m \quad \textbf{for all } (j, \alpha) \in \Lambda,$$

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Basic problem. How about the summability of the formal solution $\hat{u}(t,x)$ in the case:

$$(A_2) \qquad j+|\alpha|>m \quad \text{for some } (j,\alpha)\in\Lambda.$$

2. Examples, motivation and problem

2.1. Definition of Newton polygon

For $(a, b) \in \mathbb{R}^2$, we set
$C(a, b) = \{(x, y) \in \mathbb{R}^2 ; x \leq a, y \geq b\}$, and we make

$$t^p \partial_t^j \partial_x^\alpha \quad \text{correpond to} \quad C(j + |\alpha|, p - j).$$

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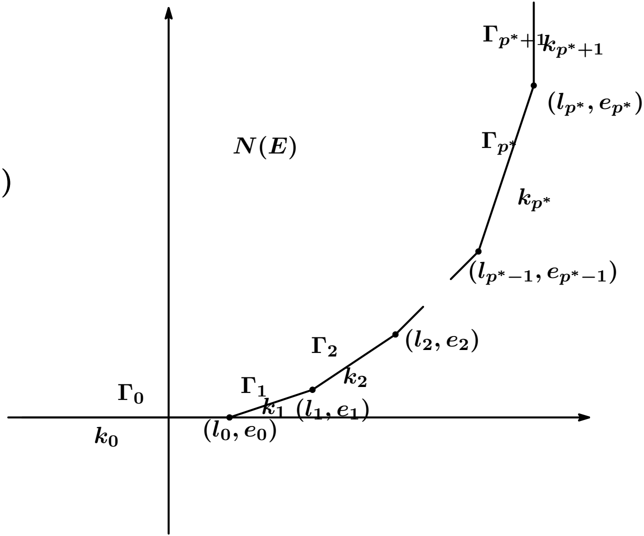
we define the Newton polygon $N(E)$ by

$$N(E) = \text{the convex hull of the set}$$
$$C(m, -m) \cup \bigcup_{(j,\alpha)\in\Lambda} C(j + |\alpha|, \mathrm{ord}_t(a_{j,\alpha}) - j).$$

2.2. Picture of Newton polygon

$$(2.1)$$



2.3. Ouchi's theorem

In the picture (2.1) we denoted:

$$(l_0, e_0), \;\; \cdots, \;\; (l_{p^*}, e_{p^*}): \;\text{ the vertices of } N(\mathrm{E}),$$
$$\Gamma_0, \ldots, \Gamma_{p^*+1}: \;\text{ the boundary of } N(\mathrm{E}),$$
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Theorem (Ouchi (J. Diff. Equations, 2002)). If

(C) $$(j, \alpha) \in \Lambda \text{ and } |\alpha| > 0$$
$$\Longrightarrow (j + |\alpha|, \mathrm{ord}_t(a_{j,\alpha}) - j) \in (N(\mathrm{E}))^\circ,$$

the formal solution $\hat{u}(t, x)$ is $(k_{p^*}, \ldots, k_1)$-multisummable (in a suitable direction).

2.4. Example

Let us consider

$$(e)\qquad \begin{cases} \partial_t u = \partial_x^2 u + c(t)(t\partial_t)^3 u, \\ u\big|_{t=0} = \varphi(x). \end{cases}$$

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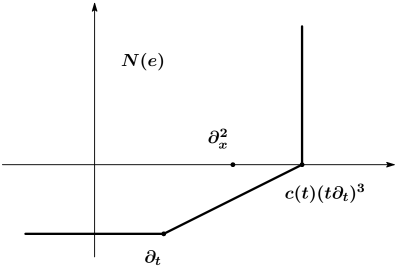

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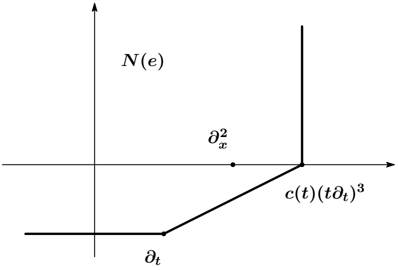
Therefore, by Ouchi's result we see:

the formal solution is $1/2$-summable in the direction d ($\neq 0, \pi$).



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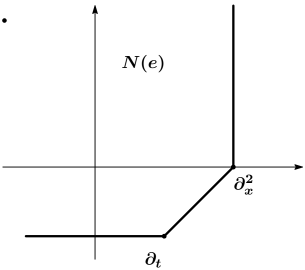

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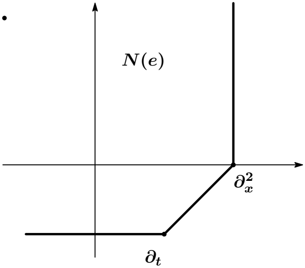

Theorem (Lutz-Miyake-Schäfke, (Nagoya J. Math., 1999)).
The formal solution $\hat{u}(t,x)$ is 1-summable in the direction d, if and only if $\varphi(x)$ can be analytically continued to infinity in directions $d/2$ and $\pi + d/2$, and is of exponential order at most 2 when x is going to infinity in these directions.

2.6. In the case $c(t) = t\gamma(t)$

In the case $c(t) = t\gamma(t)$ with $\gamma(0) \neq 0$, our equation is

$$(e) \qquad \partial_t u = \partial_x^2 u + \gamma(t)t(t\partial_t)^3 u, \quad u\big|_{t=0} = \varphi(x).$$

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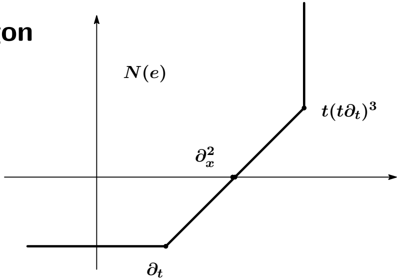

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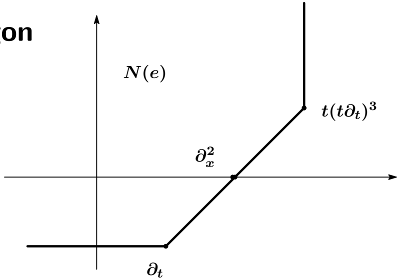

Theorem (Yamazawa, 2012). If $\varphi(x)$ is an entire function of order 2, the formal solution $\hat{u}(t,x)$ is 1-summable in the direction d ($\neq 0, \pi$).

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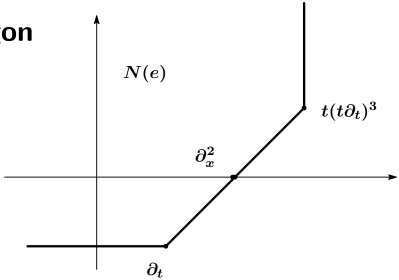

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Later, "of order 2" is improved to "of finite order".

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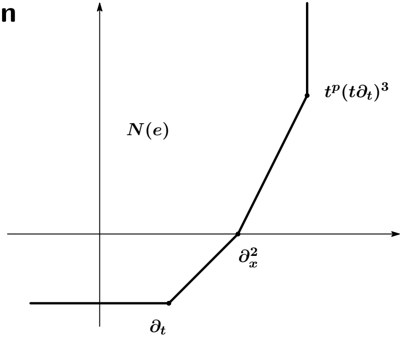

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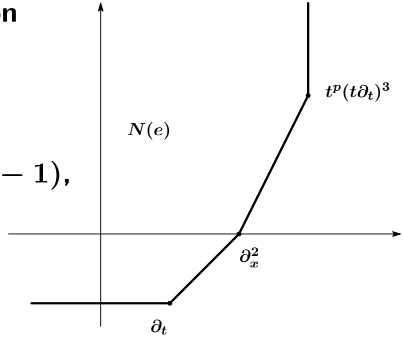

By our calculation.
If $\varphi(x)$ is an entire function of order less than $2(p+1)/(p-1)$, the formal solution $\hat{u}(t,x)$ is $(p+1)/2$-summable in the direction d ($\neq 0, \pi$).

Thus, on the equation

$$(e) \qquad \partial_t u = \partial_x^2 u + c(t)(t\partial_t)^3 u, \quad u\Big|_{t=0} = \varphi(x).$$

we have seen that the admissible exponential order at $x = \infty$ is as follows (in our calculation):

$$\begin{aligned}
&\text{case 1)}: \quad c(t) \equiv 0 \Longrightarrow \text{exponential order} \leq 2,\\
&\text{case 2)}: \quad c(t) = t\gamma(t) \text{ and } \gamma(0) \neq 0\\
&\qquad\qquad \Longrightarrow \text{exponential order} < \infty,\\
&\text{case 3)}: \quad c(t) = t^p\gamma(t), p \geq 2 \text{ and } \gamma(0) \neq 0\\
&\qquad\qquad \Longrightarrow \text{exponential order} < \frac{2(p+1)}{p-1}.
\end{aligned}$$

2.9. Problem

By looking at these examples, we have come to be interested in the following problem:

Problem. What determine the bound of the admissible exponential order

$$\leq 2 \text{ ?}, \quad < \infty \text{ ?} \quad \text{or} \quad < \frac{2(p+1)}{p-1} \text{ ?}$$

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Since to study this problem in the general case is very difficult, from now we will study this problem under the assumption that

the data are entire functions in the x-variable.

3. Main Theorem
- A sufficient condition for summability -

3.1. A class of entire functions

Let $\gamma > 0$. We say that a function $\varphi(x)$ is an entire function of order γ, if it is holomorphic on $\mathbb{C}^N$ and satisfies

$$|\varphi(x)| \leq A \exp\bigl(a|x|^\gamma\bigr) \quad \text{on } \mathbb{C}^N$$

for some $A > 0$ and $a > 0$. We denote by $\mathrm{Exp}^{\{\gamma\}}(\mathbb{C}^N)$ the set of all entire functions of order γ.

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Similarly, we denote by $\mathrm{Exp}^{\{\gamma\}}(D_r \times \mathbb{C}^N)$ the set of all holomorphic functions $f(t,x)$ on $D_r \times \mathbb{C}^N$ having the estimate

$$|f(t,x)| \leq B \exp\bigl(b|x|^{\gamma}\bigr) \quad \text{on } D_r \times \mathbb{C}^N$$

for some $B > 0$ and $b > 0$.

3.2. Equation and Problem

As before, we consider the Cauchy problem

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under the consitions (A_1), (A_2), $a_{j,\alpha}(t)$ is holomorphic on $D_r = \{|t| < r\}$ and the following:

$$(A_3)\qquad \begin{cases} f(t,x) \in \mathrm{Exp}^{\{\gamma\}}(D_r \times \mathbb{C}^N), \\ \varphi_i(x) \in \mathrm{Exp}^{\{\gamma\}}(\mathbb{C}^N),\ \ i = 0,1,\ldots,m-1. \end{cases}$$

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Problem. Under what condition on γ, can we get the summability of the formal solution $\hat{u}(t,x)$?

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Recall that usual Newton polygon $N(E)$ was defined by

$$N(E) = \text{the convex hull of the set}$$
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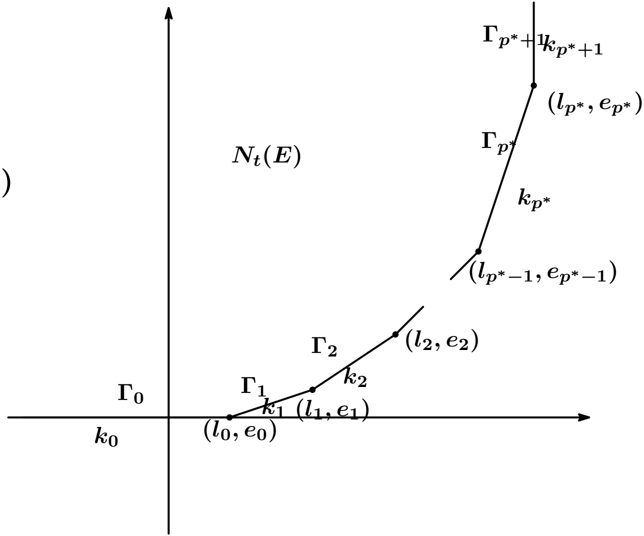
3.4. Picture of t-Newton polygon

(3.1)



3.5. Important data in (3.1)

In the picture (3.1) we denoted:

$$(l_0, e_0), \ \cdots, \ (l_{p^*}, e_{p^*}) : \ \text{the vertices of } N_t(\mathbf{E}),$$
$$\Gamma_0, \ldots, \Gamma_{p^*+1} : \ \text{the boundary of } N_t(\mathbf{E}),$$
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Then we have

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Then we have

$$k_0 = 0 < k_1 < k_2 < \cdots < k_{p^*} < k_{p^*+1} = \infty.$$

We denote by $(N_t(\mathrm{E}))^\circ$ the interior of the set $N_t(\mathrm{E})$, and we suppose:

$$(A_4) \qquad (j, \alpha) \in \Lambda \text{ and } |\alpha| > 0 \\ \Longrightarrow (j, \mathrm{ord}_t(a_{j,\alpha}) - j) \in (N_t(\mathrm{E}))^\circ.$$

3.6. Set of admissible exponents $\mathscr{C}$

Next, let us define the set of admissible exponets $\mathscr{C}$. We set

$$\Lambda^* = \{(j, \alpha) \in \Lambda\,;\ (j + |\alpha|, \mathrm{ord}_t(a_{j,\alpha}) - j) \notin N_t(E)\}.$$

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(2) If $\Lambda^* \neq \emptyset$, we define the interval $\mathscr{C}$ by

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and $\mathscr{C}_{j,\alpha}$ are defined as follows:

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and $\mathscr{C}_{j,\alpha}$ are defined as follows:

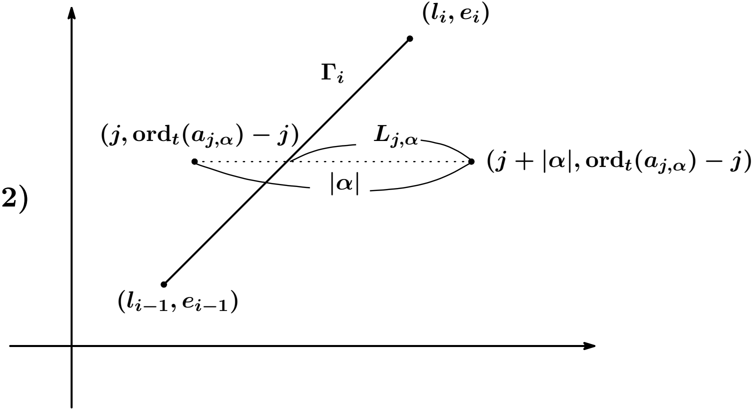
If $(j,\alpha) \in \Lambda^*$, we have $|\alpha| > 0$,

- $(j, \mathrm{ord}_t(a_{j,\alpha}) - j) \in (N_t(\mathrm{E}))^\circ$ by (A_4),
- $(j+|\alpha|, \mathrm{ord}_t(a_{j,\alpha}) - j) \notin N_t(\mathrm{E})$.

3.7. Definition of $\mathscr{C}_{j,\alpha}$ (1)

Case 1.

(3.2)



3.7. Definition of $\mathscr{C}_{j,\alpha}$ (1)

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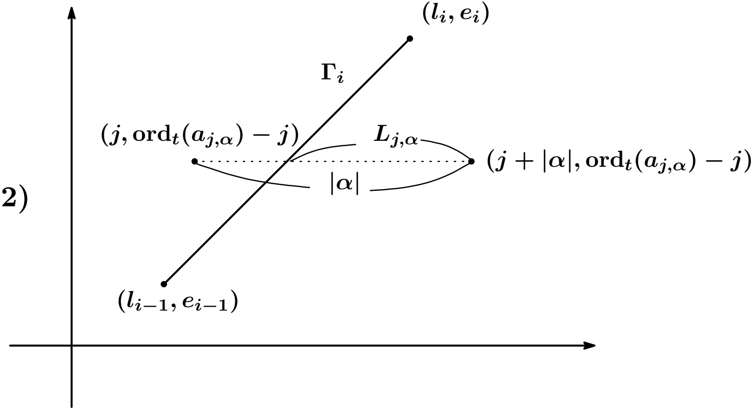


(3.2)

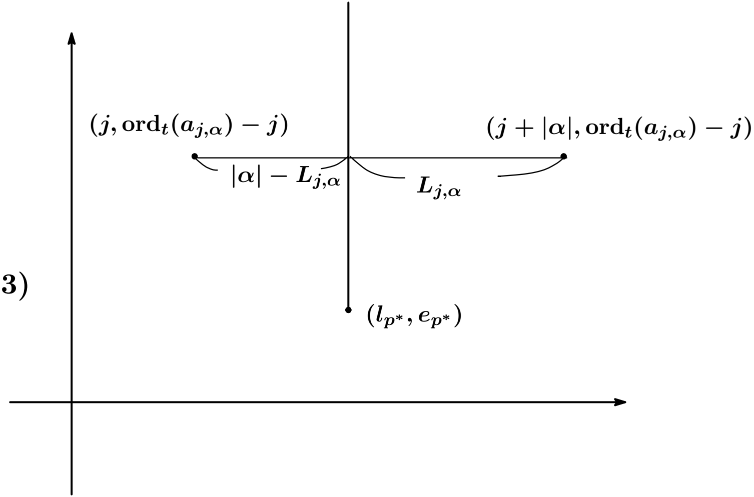
We set

$$\mathscr{C}_{j,\alpha} = \left(0, \frac{|\alpha|}{L_{j,\alpha}}\right)$$

3.8. Definition of $\mathscr{C}_{j,\alpha}$ (2)

Case (2).

$\mathbf{(3.3)}$



3.8. Definition of $\mathscr{C}_{j,\alpha}$ (2)

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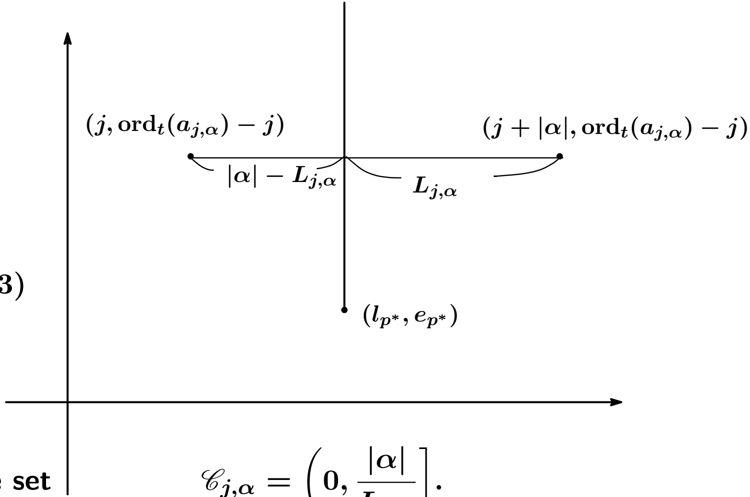
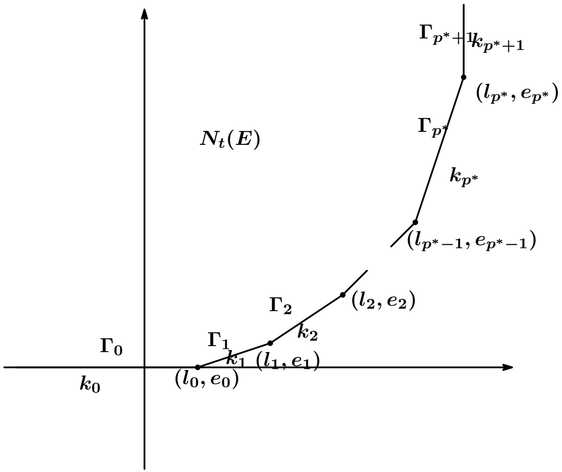


We set

$$\mathscr{C}_{j,\alpha} = \left(0, \frac{|\alpha|}{L_{j,\alpha}}\right].$$

Recall

Recall: $\mathscr{C} = \bigcap_{(j,\alpha)\in\Lambda^*} \mathscr{C}_{j,\alpha}$ **and the t-Newton polygon:**



3.9. Characteristic polynomial on Γ_i

In the case $p^* \geq 1$, we define the characteristic polynomial $P_i(\lambda)$ on Γ_i as follows.

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For $(j,0) \in I_1 \cup I_2 \cup \cdots \cup I_{p^*}$ we set $q_{j,0} = \mathrm{ord}_t(a_{j,0})$; then we have

$$a_{j,0}(t) = t^{q_{j,0}} a^0_{j,0}(t) \ \text{ with } \ a^0_{j,0}(0) \neq 0$$

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for some holomorphic function $a^0_{j,0}(t)$. We set

$$P_i(\lambda) = \begin{cases} \displaystyle\sum_{(j,0)\in I_1} a^0_{j,0}(0)\lambda^{j-m} - 1, & \text{if } i = 1, \\ \displaystyle\sum_{(j,0)\in I_i} a^0_{j,0}(0)\lambda^{j-l_{i-1}}, & \text{if } 2 \leq i \leq p^*. \end{cases}$$

3.10. Singular directions

In the case $p^* \geq 1$ we denote by

$$\lambda_{i,1} \ , \dots, \ \lambda_{i,\, l_i - l_{i-1}}$$

the roots of $P_i(\lambda) = 0$ that are called the characteristic roots on Γ_i.

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Definition (1) We define the set Ξ of singular directions by

$$\Xi = \bigcup_{i=1}^{p^*} \bigcup_{d=1}^{l_i - l_{i-1}} \Big\{ \frac{\arg \lambda_{i,d} + 2\pi j}{k_i}\ ;\ j = 0, \pm 1, \pm 2, \ldots \Big\}.$$

(2) We take $\mathscr{Z} \subset (-\pi, \pi]$ so that $\mathscr{Z} \equiv \Xi$ (mod. 2π). We note that $\mathscr{Z}$ is a finite set.

3.11. Main Theorem

Let us present our main theorem. Let

- **$\hat{u}(t,x)$ be the formal solution of (E),**
- **$\mathscr{C}$ be the set of admissible exponents,**
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Main Theorem. Suppose the conditions $(A_1)-(A_4)$ and $\gamma \in \mathscr{C}$. Then, we have:

(1) If $p^* = 0$, $\hat{u}(t,x)$ is convergent on $D_\delta \times \mathbb{C}^N$.
(2) If $p^* \geq 1$, $\hat{u}(t,x)$ is $(k_{p^*},\ldots,k_1)$-multisummable in any direction $d \in (-\pi,\pi] \setminus \mathscr{Z}$.

3.12. Consequence

Corollary. If $p^* \geq 1$, for any $d \in (-\pi, \pi] \setminus \mathscr{Z}$ we can find

- $S = \{t \in \mathcal{R}(\mathbb{C}_t \setminus \{0\})\,;\, 0 < |t| < \delta,$
  $|\arg t - d| < \pi/2k_{p^*} + \varepsilon\}$
  for some $\varepsilon > 0$ and $\delta > 0$, and
- a holomorphic solution $u(t,x)$ of (E) on $S \times \mathbb{C}^N$

such that the following asymptotic relation holds:

$$\left| u(t,x) - \sum_{n=0}^{N-1} u_n(x) t^n \right| \leq A H^N N!^{1/k_1} |t|^N \exp(b|x|^\gamma)$$

on $S \times \mathbb{C}^N$ for any $N = 0, 1, 2, \ldots$

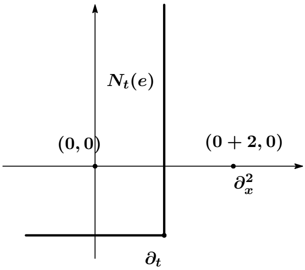
for some $A > 0$, $H > 0$ and $b > 0$.

3.13. Example (1)

In the case

$$(e) \qquad \partial_t u = \partial_x^2 u, \quad u\big|_{t=0} = \varphi(x).$$

In t-Newton polygon,
∂_x^2 corresponds to $(0,0)$
and so (A_4) is satisfied.
Note: in usual Newton polygon,
∂_x^2 corresponds to $(2,0)$.



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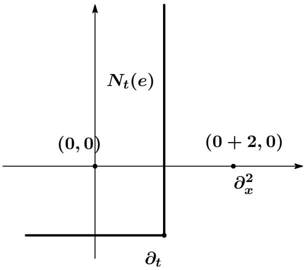

Therefore, we have $|\alpha| = 2$, $L = 1$, and so $\mathscr{C} = (0,2]$.

3.14. Example (2)

In the case

$$(e) \qquad \partial_t u = \partial_x^2 u + \gamma(t) t (t\partial_t)^3 u, \quad u\big|_{t=0} = \varphi(x).$$

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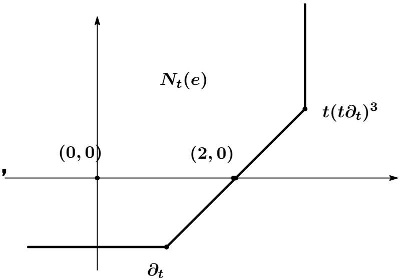

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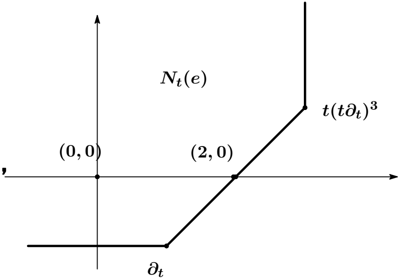
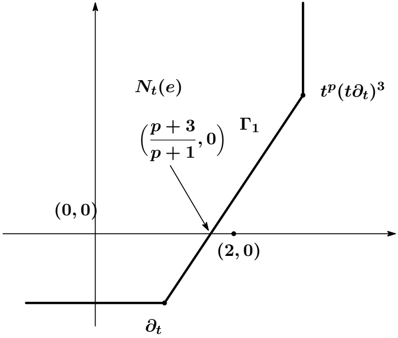


In this case, $(2,0)$ is on the boundary of $N_t(e)$ and so $\Lambda^* = \emptyset$: this shows $\mathscr{C} = (0,\infty)$. We can regard as $|\alpha| = 2$ and $L = 0$.

3.15. Example (3)

In the case

$$(e) \quad \partial_t u = \partial_x^2 u + \gamma(t) t^p (t\partial_t)^3 u, \ \ u\big|_{t=0} = \varphi(x) \ \ (p \geq 2).$$



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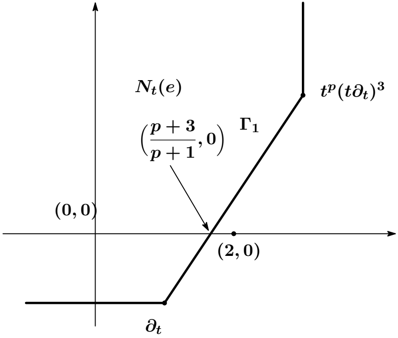
We have $|\alpha| = 2$ and

$L = 2 - \frac{p+3}{p+1} = \frac{p-1}{p+1}$,

$\mathscr{C} = \left(0, \frac{|\alpha|}{L}\right) = \left(0, \frac{2(p+1)}{(p-1)}\right)$.

We note that the slope of Γ_1 is $\frac{p+1}{2}$. Our conclusion is:

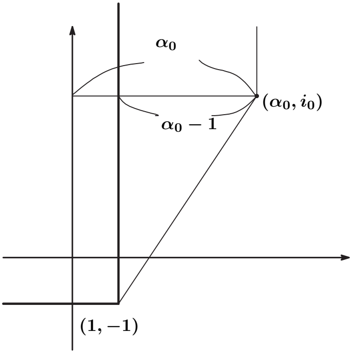
the formal solution $\hat{u}(t,x)$ is $\frac{p+1}{2}$-summable.



3.16. Ichinobe's result

In the lecture of Ichinobe, he has treated the following equation:

$$\partial_t u = \sum_{(i,\alpha)\in\Lambda} a_{i,\alpha} t^i \partial_x^\alpha u, \quad u\Big|_{t=0} = \varphi(x).$$



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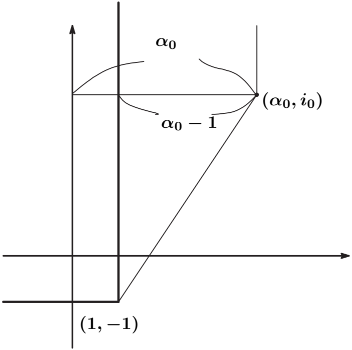
$$\partial_t u = \sum_{(i,\alpha)\in\Lambda} a_{i,\alpha} t^i \partial_x^\alpha u, \quad u\big|_{t=0} = \varphi(x).$$

In this case, we have $|\alpha| = \alpha_0$ and $L = \alpha_0 - 1$ and so

$$\mathscr{C} = \Big(0, \frac{\alpha_0}{\alpha_0 - 1}\Big].$$

Thus, the admissible exponential order $\leq \dfrac{\alpha_0}{\alpha_0 - 1}$.



4. Idea of the proof

In the proof, the following lemma is very important:

Lemma 1. Let $\gamma > 0$ and let $f(x)$ be an entire function. Set $\sigma = 1 - 1/\gamma$. The following two conditions are equivalent:
(1) $f(x) \in \mathrm{Exp}^{\{\gamma\}}(\mathbb{C}^N)$.
(2) For any compact subset K of $\mathbb{C}^N$ there are $A > 0$ and $h > 0$ such that the following estimates hold:

$$\max_K |\partial_x^\alpha f(x)| \leq A h^{|\alpha|} |\alpha|!^\sigma \quad \text{for any } \alpha \in \mathbb{N}^N.$$

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Hence, we can regard entire functions of order γ as functions of the Gevrey class of order σ. We note that $\gamma > 1$ is equivalent to $0 < \sigma < 1$.

4.2. Meaning of the condition $\gamma \in \mathscr{C}$

We note that by the definition of $\mathscr{C}$ we can see that

$$\mathscr{C} \supset (0, 1+\delta) \quad \text{for some } \delta > 0.$$

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Lemma 2. Under the relation $\sigma = 1 - 1/\gamma$, the following two conditions are equivalent:
(1) $\gamma \in \mathscr{C}$ and $\gamma > 1$.
(2) For any $(j, \alpha) \in \Lambda$ with $|\alpha| > 0$ we have

$$\begin{aligned}&(j + \sigma|\alpha|, \mathrm{ord}_t(a_{j,\alpha}) - j) \\ &\qquad \in (N_t(E))^{\circ} \cup (\Gamma_{p^*+1} \setminus \{(l_{p^*}, e_{p^*})\}).\end{aligned}$$

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This shows that if we regard entire functions as functions in the Gevrey lcass of order σ, then our equation is considered as a perturbation of ordinary differential equations.

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In the paper of Ouchi (J. Diff. Equations, 2002), he studied the summability of formal solutions to linear partial differential equations which can be considered as a perturbation of ordinary differential equations.

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For details, please see the following paper:

H. Tahara and H. Yamazawa, Multisummability of formal solutions to the Cauchy problem for some linear partial differential equations,
Journal of Differential Equations 255 (2013) pp. 3592-3637

Thank you very much.