The fourth order PI equation and coalescing phenomena of nonlinear turning points

(work in progress)

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Hierarchy of higher order PI equations

(cf. Kudryashev, *Phys. Lett. A* **(1997))**

$(P_{\mathrm{I}})_m$ **: $(2m)$-th order PI equation with a large parameter**

For example,

$$(P_{\mathrm{I}})_1 : \quad \eta^{-1}\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \quad \eta^{-1}\frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda}$$

where $H = H(\lambda, \mu, t) = \mu^2 - (\lambda^3 + t\lambda)$

and $\eta\ (>0)$ **: large parameter**

$$\Longleftrightarrow \quad \eta^{-1}\frac{d\lambda}{dt} = 2\mu, \quad \eta^{-1}\frac{d\mu}{dt} = 3\lambda^2 + t$$

$$\Longleftrightarrow \quad \eta^{-2}\frac{d\lambda^2}{dt^2} = 6\lambda^2 + 2t$$

$$(P_{\mathrm{I}})_2: \qquad \eta^{-1}\frac{d\lambda_j}{dt} = \frac{\partial H}{\partial \mu_j}, \quad \eta^{-1}\frac{d\mu_j}{dt} = -\frac{\partial H}{\partial \lambda_j} \quad (j = 1, 2)$$

where

$$\begin{aligned} H &= H(\lambda_1, \lambda_2, \mu_1, \mu_2, t, c) \\ &= \frac{\mu_1^2 - \mu_2^2}{\lambda_1 - \lambda_2} - \frac{(\lambda_1^5 - \lambda_2^5) + c(\lambda_1^3 - \lambda_2^3) + t(\lambda_1^2 - \lambda_2^2)}{\lambda_1 - \lambda_2} \end{aligned}$$

In terms of the symmetric variables

$$u_1 = \lambda_1 + \lambda_2, \quad u_2 = -\lambda_1\lambda_2,$$

$$v_1 = \frac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2}, \quad v_2 = \frac{\lambda_1\mu_2 - \lambda_2\mu_1}{\lambda_1 - \lambda_2},$$

it is equivalent to

$$\eta^{-4}\frac{d^4u_1}{dt^4}-10\eta^{-2}\left[2u_1\frac{d^2u_1}{dt^2}+\left(\frac{du_1}{dt}\right)^2\right]+40u_1^3+8cu_1-8t=0$$

Purpose of this talk

To show the importance of the fourth order PI equation $(P_{\mathrm{I}})_2$.

Motivation

Dubrovin's result (*Comm. Math. Phys.* **(2006)**)

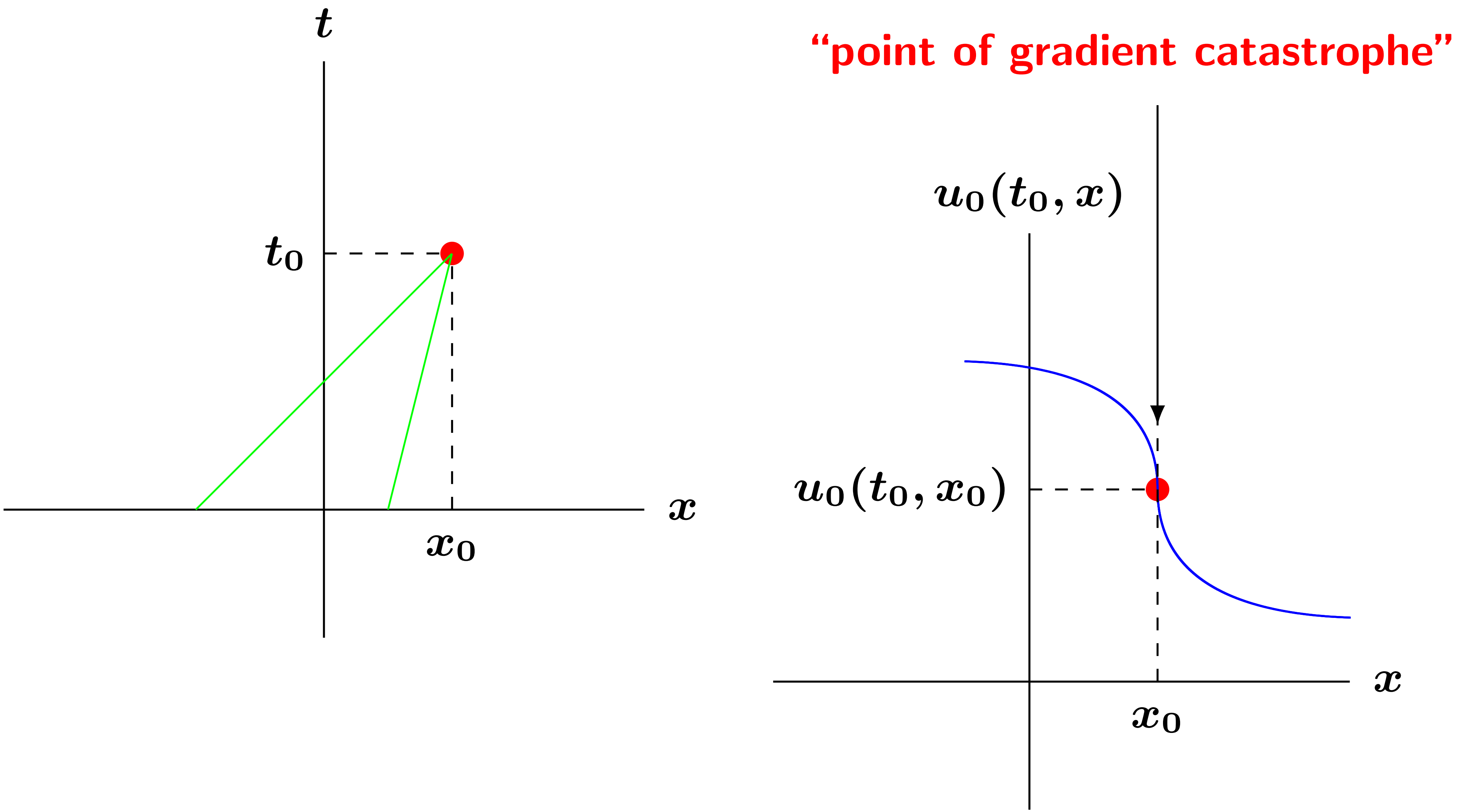
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + c\epsilon^2\frac{\partial^3 u}{\partial x^3} = 0 \quad : \quad \textbf{KdV equation}$$

($\epsilon > 0$: **small parameter,** c : **constant**)

$$u = u_0 + \epsilon^2 u_2 + \cdots \quad : \quad \textbf{perturbative solution}$$

where

$$\frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} = 0$$



t
t0
x
x0
"point of gradient catastrophe"
u0(t0, x)
u0(t0, x0)
x
x0

Claim

Under some genericity condition near a point of gradient catastrophe the behavior of the perturbative solution u of the KdV equation is described by a (special) solution of $(P_{\mathrm{I}})_2$.

Remark (Correspondence of variables)

KdV		$(P_{\mathrm{I}})_2$
t	$\longleftrightarrow$	c
x	$\longleftrightarrow$	t

Remark

The above claim holds universally for any Hamiltonian perturbations of the equation $\dfrac{\partial u}{\partial t} + a(u)\dfrac{\partial u}{\partial x} = 0.$

Question

- Why does $(P_{\mathrm{I}})_2$ appear in the description of the behavior of solutions of the KdV equation near a point of gradient catastrophe ?
- What is the characteristic feature of $(P_{\mathrm{I}})_2$ related to this problem ?

Key facts in our approach

(1) Relation between $(P_{\mathrm{I}})_2$ and Garnier systems.

(2) Stokes geometry (especially, turning points) of $(P_{\mathrm{I}})_2$.

Relation between $(P_{\mathrm{I}})_2$ and Garnier systems

Theorem (**Koike,** *RIMS Kôkyûroku Bessatsu* **(2007, 2008))**

$(P_{\mathrm{I}})_2$ is the restriction of the most degenerate Garnier system $G(9/2;2)$ of two variables

$$\eta^{-1}\frac{\partial \lambda_j}{\partial t_k} = \frac{\partial h_k}{\partial \mu_j}, \quad \eta^{-1}\frac{\partial \mu_j}{\partial t_k} = -\frac{\partial h_k}{\partial \lambda_j} \quad (j,k=1,2)$$

with

$$h_1 = H(\lambda_1,\lambda_2,\mu_1,\mu_2,t_1,t_2)$$

$$\begin{aligned} h_2 &= \frac{\lambda_1\mu_2^2-\lambda_2\mu_1^2}{\lambda_1-\lambda_2} - \eta^{-1}\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2} \\ &\quad + \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\{(\lambda_1^4-\lambda_2^4)+t_2(\lambda_1^2-\lambda_2^2)+t_1(\lambda_1-\lambda_2)\} \end{aligned}$$

onto the complex line $\{\, t_2 = c \,\}$.

Thus we should consider the following system:

$$\begin{cases} \eta^{-1}\dfrac{\partial u_1}{\partial t_1} = 2v_1, \\ \eta^{-1}\dfrac{\partial u_2}{\partial t_1} = 2v_2, \\ \eta^{-1}\dfrac{\partial v_1}{\partial t_1} = 3u_1^2 + 2u_2 + t_2, \\ \eta^{-1}\dfrac{\partial v_2}{\partial t_1} = u_1^3 + 4u_1u_2 - v_1^2 + t_2u_1 + t_1. \end{cases}$$

$$\begin{cases} \eta^{-1}\dfrac{\partial u_1}{\partial t_2} = \frac{2}{3}v_2, \\ \eta^{-1}\dfrac{\partial u_2}{\partial t_2} = \frac{2}{3}(v_1u_2 - u_1v_2) - \frac{1}{3}\eta^{-1}, \\ \eta^{-1}\dfrac{\partial v_1}{\partial t_2} = \frac{1}{3}(u_1^3 + 4u_1u_2 - v_1^2 + t_2u_1 + t_1), \\ \eta^{-1}\dfrac{\partial v_2}{\partial t_2} = -\frac{1}{3}(u_1^4 + u_2u_1^2 - 2u_2^2 - u_1v_1^2 + t_2(u_1^2 - u_2) + t_1u_1). \end{cases}$$

Stokes geometry of $(P_{\mathrm{I}})_2$

We denote the above system as follows:

$$(P_{\mathrm{I}})_2 \qquad \begin{cases} \eta^{-1}\dfrac{\partial u_j}{\partial t_k} = F_{jk}(u, v, t, \eta) \\ \eta^{-1}\dfrac{\partial v_j}{\partial t_k} = G_{jk}(u, v, t, \eta) \end{cases} \qquad (j, k = 1, 2)$$

Proposition 1

There exists a formal power series solution (w.r.t. η^{-1}) of $(P_{\mathrm{I}})_2$:

$$\begin{cases} u_j^{(0)} = \hat{u}_j(t) + \eta^{-1}\hat{u}_{j,1}(t) + \cdots \\ v_j^{(0)} = \hat{v}_j(t) + \eta^{-1}\hat{v}_{j,1}(t) + \cdots \end{cases}$$

where $(\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2)$ satisfies a system of algebraic equations

$$5\hat{u}_1^3 + t_2\hat{u}_1 - t_1 = 0, \quad 3\hat{u}_1^2 + 2\hat{u}_2 + t_2 = 0,$$
$$\hat{v}_1 = \hat{v}_2 = 0$$

and higher order terms $(\hat{u}_{1,l}, \hat{u}_{2,l}, \hat{v}_{1,l}, \hat{v}_{2,l})$ $(l \geq 1)$ are recursively determined.

To define the Stokes geometry of $(P_{\mathrm{I}})_2$, we consider the Fréchet derivative of $(P_{\mathrm{I}})_2$ at the formal power series solution $(u_j^{(0)}, v_j^{(0)})$.

Definition (Fréchet derivative of $(P_{\mathrm{I}})_2$ at $(u_j^{(0)}, v_j^{(0)})$)

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial t_k}(\Delta u_j) = \displaystyle\sum_p \Big[\frac{\partial F_{jk}}{\partial u_p}(u_j^{(0)}, v_j^{(0)}, t, \eta)\Delta u_p \\ \qquad\qquad\qquad\qquad + \dfrac{\partial F_{jk}}{\partial v_p}(u_j^{(0)}, v_j^{(0)}, t, \eta)\Delta v_p\Big] \\ \eta^{-1}\dfrac{\partial}{\partial t_k}(\Delta v_j) = \displaystyle\sum_p \Big[\frac{\partial G_{jk}}{\partial u_p}(u_j^{(0)}, v_j^{(0)}, t, \eta)\Delta u_p \\ \qquad\qquad\qquad\qquad + \dfrac{\partial G_{jk}}{\partial v_p}(u_j^{(0)}, v_j^{(0)}, t, \eta)\Delta v_p\Big] \end{cases}$$

$(j, k = 1, 2)$.

The Fréchet derivative of $(P_{\mathrm{I}})_2$ at $(u_j^{(0)}, v_j^{(0)})$, denoted by $(\Delta P_{\mathrm{I}})_2$ in what follows, is a system of linear differential equations for $(\Delta u_j, \Delta v_j)$.

Note that, in the case of a linear differential equation (with a large parameter), a turning point should be defined as a point where two characteristic roots of the differential equation merge, that is, a zero of the descriminant of the characteristic equation.

Proposition 2

The characteristic equation of $(\Delta P_{\mathrm{I}})_2$ is given as follows:

- (in the t_1 direction)

$$\nu_1^4 - 20\hat{u}_1\nu_1^2 + 16(6\hat{u}_1^2 - \hat{u}_2) = 0$$

Hence zeros of the discriminant are given by

- $6\hat{u}_1^2 - \hat{u}_2 = 0$
- $(10\hat{u}_1)^2 - 16(6\hat{u}_1^2 - \hat{u}_2) = 4(\hat{u}_1^2 + 4\hat{u}_2) = 0$

- (in the t_2 direction)

$$\nu_1^4 - \frac{4}{9}\hat{u}_1(2\hat{u}_1^2 + 3\hat{u}_2)\nu_1^2 + \frac{16}{81}\hat{u}_2^2(6\hat{u}_1^2 - \hat{u}_2) = 0$$

Hence zeros of the discriminant are given by

- $\hat{u}_2^2(6\hat{u}_1^2 - \hat{u}_2) = 0$
- $\frac{4}{81}(2\hat{u}_1^2 - \hat{u}_2)^2(\hat{u}_1^2 + 4\hat{u}_2) = 0$

Definition

(i) A first kind turning point of $(P_{\mathrm{I}})_2$

$$\iff \quad 6\hat{u}_1^2 - \hat{u}_2 = 0$$

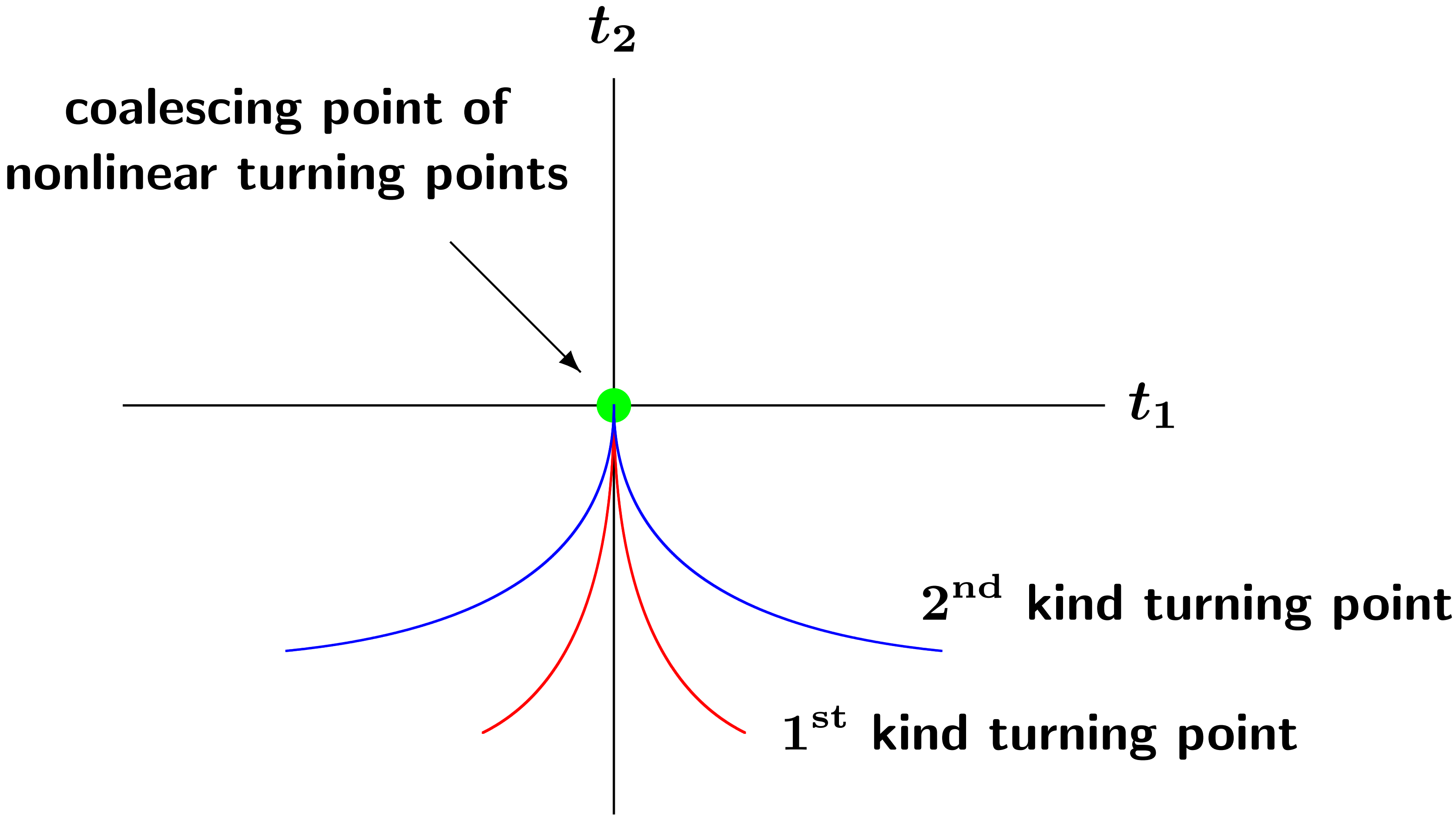
$$\iff \quad 135 t_1^2 + 4 t_2^3 = 0$$

(ii) A second kind turning point of $(P_{\mathrm{I}})_2$

$$\iff \quad \hat{u}_1^2 + 4\hat{u}_2 = 0$$

$$\iff \quad 5 t_1^2 + 2 t_2^3 = 0$$

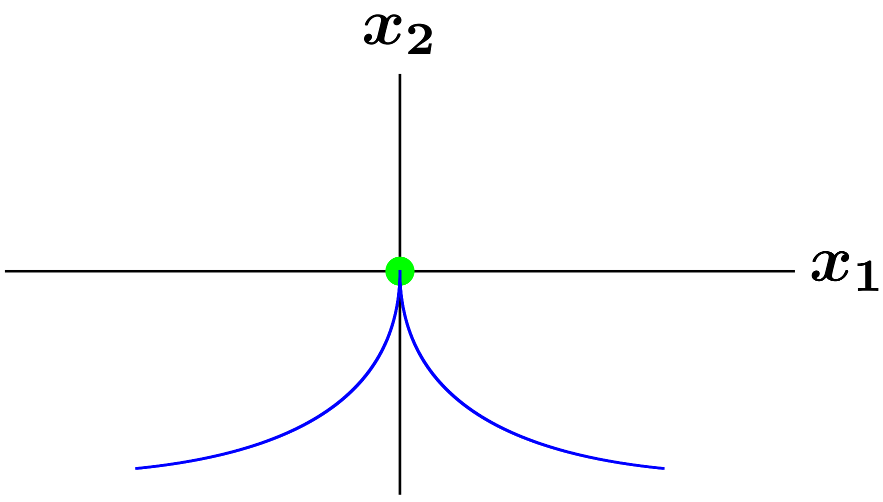
Coalescence of nonlinear turning points



Pearcey system

$$\begin{cases} \left(\dfrac{\partial^3}{\partial x_1^3} + \dfrac{x_2}{2}\eta^2 \dfrac{\partial}{\partial x_1} + \dfrac{x_1}{4}\eta^3\right)\psi = 0, \\ \left(\eta \dfrac{\partial}{\partial x_1} - \dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0. \end{cases}$$

Turning points : $\{\, 27x_1^2 + 8x_2^3 = 0 \,\}$



Theorem (Hirose, to appear in *Publ. RIMS*)

In the case of a system of linear differential equations of two variables, the Pearcey system gives a normal form near a point of coalescence of turning points.

To be more specific, let us consider

$$(1)\quad \begin{cases} \eta^{-1}\dfrac{\partial}{\partial \tilde{x}_1}\tilde{\Psi} = \tilde{P}(\tilde{x})\tilde{\Psi} \\ \eta^{-1}\dfrac{\partial}{\partial \tilde{x}_2}\tilde{\Psi} = \tilde{Q}(\tilde{x})\tilde{\Psi} \end{cases} \qquad (\tilde{P}(\tilde{x}), \tilde{Q}(\tilde{x}) : 3 \times 3 \text{ matrices})$$

Assume that coalescence of turning points occurs at $\tilde{x} = (0, 0)$.
Then, under some genericity condition, (1) can be transformed to
(2)

$$\begin{cases} \eta^{-1}\dfrac{\partial}{\partial x_1}\Psi = P(x)\Psi, \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -x_1/4 & -x_2/2 & 0 \end{pmatrix} \\ \eta^{-1}\dfrac{\partial}{\partial x_2}\Psi = Q(x)\Psi, \quad Q = P^2 + \dfrac{x_2}{3} - \dfrac{\eta^{-1}}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{cases}$$

That is, there exist

$$x(\tilde{x}) = (x_1(\tilde{x}_1, \tilde{x}_2), x_2(\tilde{x}_1, \tilde{x}_2)) \ : \ \text{coordinate transform},$$

$$T(\tilde{x}, \eta) = \sum_{n=0}^{\infty} \eta^{-n} T_n(\tilde{x}) \ : \ \text{formal Gauge transform of } 3 \times 3 \text{ matrices}$$

so that the following holds

$$\tilde{\Psi}(\tilde{x}, \eta) = T(\tilde{x}, \eta)\Psi(x(\tilde{x}), \eta).$$

Remark

The completely integrable system (2) is equivalent to the Pearcey system.

Main claim (conjecture)

Claim (Conjecture)

In the case of a higher order Painlevé equation, $(P_{\mathrm{I}})_2$ gives a normal form near a point of coalescence of nonlinear turning points.

To state the main claim in a more specific manner, let us consider, for example, the fourth order PII equation $(P_{\mathrm{II}})_2$:

$$\eta^{-1}\frac{\partial\tilde{\lambda}_j}{\partial\tilde{t}_k}=\frac{\partial\tilde{h}_k}{\partial\tilde{\mu}_j},\quad \eta^{-1}\frac{\partial\tilde{\mu}_j}{\partial\tilde{t}_k}=-\frac{\partial\tilde{h}_k}{\partial\tilde{\lambda}_j}\quad (j,k=1,2)$$

with

$$\begin{aligned}\tilde{h}_1=&\frac{1}{2}\frac{\tilde{\mu}_1^2-\tilde{\mu}_2^2}{\tilde{\lambda}_1-\tilde{\lambda}_2}-\frac{\tilde{\lambda}_1^3\tilde{\mu}_1-\tilde{\lambda}_2^3\tilde{\mu}_2}{\tilde{\lambda}_1-\tilde{\lambda}_2}-\tilde{t}_2\frac{\tilde{\lambda}_1\tilde{\mu}_1-\tilde{\lambda}_2\tilde{\mu}_2}{\tilde{\lambda}_1-\tilde{\lambda}_2}\\ &-\frac{\tilde{t}_1}{2}\frac{\tilde{\mu}_1-\tilde{\mu}_2}{\tilde{\lambda}_1-\tilde{\lambda}_2}-\alpha(\tilde{\lambda}_1+\tilde{\lambda}_2)+\frac{1}{2}\tilde{t}_1\tilde{t}_2,\end{aligned}$$

$$\begin{aligned}\tilde{h}_2=&\frac{1}{2}\frac{\tilde{\lambda}_1\tilde{\mu}_2^2-\tilde{\lambda}_2\tilde{\mu}_1^2}{\tilde{\lambda}_1-\tilde{\lambda}_2}+\tilde{\lambda}_1\tilde{\lambda}_2\frac{\tilde{\lambda}_1^2\tilde{\mu}_1-\tilde{\lambda}_2^2\tilde{\mu}_2}{\tilde{\lambda}_1-\tilde{\lambda}_2}+\tilde{t}_2\tilde{\lambda}_1\tilde{\lambda}_2\frac{\tilde{\mu}_1-\tilde{\mu}_2}{\tilde{\lambda}_1-\tilde{\lambda}_2}\\ &-\frac{\tilde{t}_1}{2}\frac{\tilde{\lambda}_1\tilde{\mu}_2-\tilde{\lambda}_2\tilde{\mu}_1}{\tilde{\lambda}_1-\tilde{\lambda}_2}-\frac{\eta^{-1}}{2}\frac{\tilde{\mu}_1-\tilde{\mu}_2}{\tilde{\lambda}_1-\tilde{\lambda}_2}-\alpha\tilde{\lambda}_1\tilde{\lambda}_2+\frac{1}{8}\tilde{t}_1^2+\frac{1}{2}\tilde{t}_2.\end{aligned}$$

Assume $\alpha \neq 0$. We then find that coalescence of nonlinear turning points for $(P_{\mathrm{II}})_2$ occurs at

$$9\tilde{t}_2^2 + 10\alpha = 0, \quad 135\tilde{t}_1^2 + 512\tilde{t}_2^3 = 0.$$

Our claim is that, near such a point of coalescence of nonlinear turning points, there exist

$$t_1(\tilde{t}_1, \tilde{t}_2, \eta) = \sum_{n=0}^{\infty} \eta^{-n} t_{1,n}(\tilde{t}_1, \tilde{t}_2),$$

$$t_2(\tilde{t}_1, \tilde{t}_2, \eta) = \sum_{n=0}^{\infty} \eta^{-n} t_{2,n}(\tilde{t}_1, \tilde{t}_2),$$

and

$$x(\tilde{x}, \tilde{t}_1, \tilde{t}_2, \eta) = \sum_{n=0}^{\infty} \eta^{-n} x_n(\tilde{x}, \tilde{t}_1, \tilde{t}_2),$$

such that the following holds:

$$\lambda_j^{(0)}(t_1(\tilde{t}_1,\tilde{t}_2,\eta),t_2(\tilde{t}_1,\tilde{t}_2,\eta),\eta) = x(\tilde{\lambda}_j^{(0)}(\tilde{t}_1,\tilde{t}_2,\eta),\tilde{t}_1,\tilde{t}_2,\eta),$$

where

$\lambda_j^{(0)}(t_1,t_2,\eta)$: formal power series solution of $(P_{\mathrm{I}})_2$,

$\tilde{\lambda}_j^{(0)}(\tilde{t}_1,\tilde{t}_2,\eta)$: formal power series solution of $(P_{\mathrm{II}})_2$.

Remark

We guess that a coalescing phenomenon of nonlinear turning points is occurring at a point of gradient catastrophe of the KdV equation, and consequently Dubrovin's result can be deduced from our main claim. But this is still just a guess.

Toward the proof of the main claim

We make full use of the isomonodromic deformation theory associated to $(P_{\mathrm{I}})_2$, that is, we consider the Lax pair associated to $(P_{\mathrm{I}})_2$:

$$(LP_{\mathrm{I}})_2 \qquad \begin{cases} \eta^{-1}\dfrac{\partial}{\partial x}\Psi = A\Psi \\ \eta^{-1}\dfrac{\partial}{\partial t_1}\Psi = B_1\Psi \\ \eta^{-1}\dfrac{\partial}{\partial t_2}\Psi = B_2\Psi \end{cases}$$

where $A = A(x, t, \lambda, \mu, \eta)$ and $B_k = B_k(x, t, \lambda, \mu, \eta)$ $(k = 1, 2)$ are 2×2 matrices. Note that

$$\text{compatibility condition of } (LP_{\mathrm{I}})_2 \iff (P_{\mathrm{I}})_2$$

Substitute the formal power series solution $(\lambda_j^{(0)}, \mu_j^{(0)})$ of $(P_{\mathrm{I}})_2$ into the coefficients A and B_k of the Lax pair $(LP_{\mathrm{I}})_2$, then we find the following:

- $\eta^{-1}\dfrac{\partial}{\partial x}\Psi = A\Psi$ has double turning points at $x = \hat{\lambda}_j$ $(j = 1, 2)$ and one simple turning point at $x = -2\hat{\lambda}_j =: \hat{a}$.

- At the coalescing point $(t_1, t_2) = (0, 0)$ of nonlinear turning points of $(P_{\mathrm{I}})_2$, these three turning points $x = \hat{\lambda}_1$, $x = \hat{\lambda}_2$ and $x = \hat{a}$ merge to one point.

- Let

$\alpha^{\pm}$: characteristic root of A,
$\beta_k^{\pm}$: characteristic root of B_k $(k = 1, 2)$,
$\nu_{k,j}^{\pm}$: characteristic root of the Fréchet derivative of $(P_{\mathrm{I}})_2$ in the t_k direction $(j, k = 1, 2)$.

Then we have

$$\nu_{k,j}^{\pm} = 2\beta_k^{\pm}\Big|_{x=\hat{\lambda}_j}$$

- The following relation holds:

$$\int_{(0,0)}^{(t_1,t_2)} \left(\nu_{1,j}^{\pm}\, dt_1 + \nu_{2,j}^{\pm}\, dt_2\right) = 2\int_{\hat{a}}^{\hat{\lambda}_j} \alpha^{\pm}\, dx$$

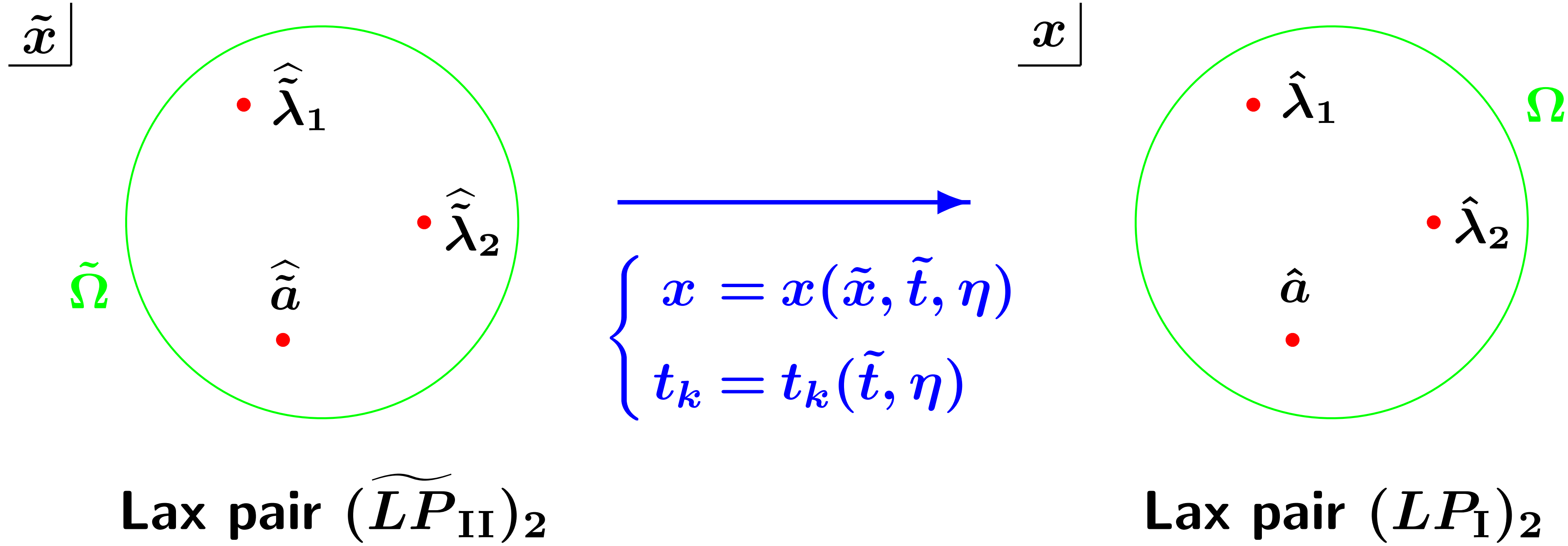
Remark

Similar results also hold for $(P_{\mathrm{II}})_2$.

We then construct a formal transformation

$$(3) \qquad \begin{cases} x = x(\tilde{x}, \tilde{t}_1, \tilde{t}_2, \eta) = \sum_{n=0}^{\infty} \eta^{-n} x_n(\tilde{x}, \tilde{t}_1, \tilde{t}_2), \\ t_1 = t_1(\tilde{t}_1, \tilde{t}_2, \eta) = \sum_{n=0}^{\infty} \eta^{-n} t_{1,n}(\tilde{t}_1, \tilde{t}_2), \\ t_2 = t_2(\tilde{t}_1, \tilde{t}_2, \eta) = \sum_{n=0}^{\infty} \eta^{-n} t_{2,n}(\tilde{t}_1, \tilde{t}_2) \end{cases}$$

that transforms $(\widetilde{LP}_{\mathrm{II}})_2$, the Lax pair associated to $(P_{\mathrm{II}})_2$, to $(LP_{\mathrm{I}})_2$ in an open set $\tilde{\Omega}$ containing three turning points $\widehat{\tilde{\lambda}}_1$, $\widehat{\tilde{\lambda}}_2$ and $\widehat{\tilde{a}}$.



The transformation (3) is expected to give a transformation from $(P_{\mathrm{II}})_2$ to $(P_{\mathrm{I}})_2$. This is our strategy to prove the main claim.