Borel sums of Voros coefficients of Gauss's hypergeometric differential equation with a large parameter

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Formal Analytic Solutions of Differential, Difference and Discrete Equations
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Introduce

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with $Q=Q_0+\eta^{-2}Q_1$,

$$Q_0=\frac{(\alpha-\beta)^2x^2+2(2\alpha\beta-\alpha\gamma-\beta\gamma)x+\gamma^2}{4x^2(x-1)^2},$$

$$Q_1=-\frac{x^2-x+1}{4x^2(x-1)^2}.$$

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$$a = \frac{1}{2} + \eta\alpha, b = \frac{1}{2} + \eta\beta, c = 1 + \eta\gamma$$

and eliminate the first-order term by

$$\psi = x^{\frac{1}{2}+\frac{\eta\gamma}{2}}(1-x)^{\frac{1}{2}+\frac{\eta(\alpha+\beta-\gamma)}{2}}w.$$

Our equation:

$$\Big(-\frac{d^2}{dx^2}+\eta^2 Q\Big)\psi = 0$$

where $Q = Q_0 + \eta^{-2} Q_1$ with

$$Q_0 = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta-\alpha\gamma-\beta\gamma)x+\gamma^2}{4x^2(x-1)^2},\ Q_1 = -\frac{x^2-x+1}{4x^2(x-1)^2}.$$

We denote by ι_j the following mappings.

Q : invariant under involutions ι_j ($j = 0, 1, 2$)

$$\begin{aligned}
\iota_0 : \quad & (\alpha,\beta,\gamma) \mapsto (-\alpha,-\beta,-\gamma)\\
\iota_1 : \quad & \mapsto (\gamma-\alpha,\gamma-\beta,\gamma)\\
\iota_2 : \quad & \mapsto (\beta,\alpha,\gamma)
\end{aligned}$$

We have to keep in mind that Q is invariant under these involutions.

WKB solutions

Our equation has the following formal solutions (WKB solutions) :

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a is a zero of $\sqrt{Q_0}dx$. (a is a turning point.)
a formal solution $S = S_{\mathrm{odd}} + S_{\mathrm{even}} = \sum_{j=-1}^{\infty} \eta^{-j} S_j$
to Riccati equation

$$\frac{dS}{dx} + S^2 = \eta^2 Q$$

$S_{-1} = \sqrt{Q_0}$.

Stokes graph

A **Stokes curve** is an integral curve of $\mathrm{Im}\sqrt{Q_0}dx = 0$ emanating from a turning point.
A **Stokes graph** of our equation is a collection of all Stokes curves, turning points $a_k (k = 0, 1)$ and singular points $b_0 = 0, b_1 = 1, b_2 = \infty$.

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Let us assume

$$\text{(i) } \alpha\beta\gamma(\alpha-\beta)(\alpha-\gamma)(\alpha+\beta-\gamma) \neq 0$$

$$\text{(ii) } \mathrm{Re}\,\alpha\,\mathrm{Re}\,\beta\,\mathrm{Re}\,(\gamma-\alpha)\,\mathrm{Re}\,(\gamma-\beta) \neq 0$$

$$\text{(iii) } \mathrm{Re}\,(\alpha-\beta)\,\mathrm{Re}\,(\alpha+\beta-\gamma)\,\mathrm{Re}\,\gamma \neq 0$$

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Assumption (ii) and (iii)

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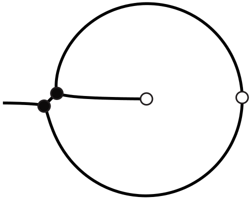
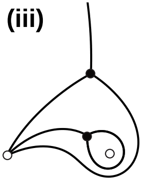
If the LHS of conditions (ii) or (iii) vanishes then the Stokes graph is degenerate.

(ii)



(iii)



We assume that (α, β, γ) are not contained in (i). Let n_0, n_1 and n_2 be numbers of Stokes curves that flow into 0, 1 and ∞, respectively. $\hat{n}$ will denote (n_0, n_1, n_2).

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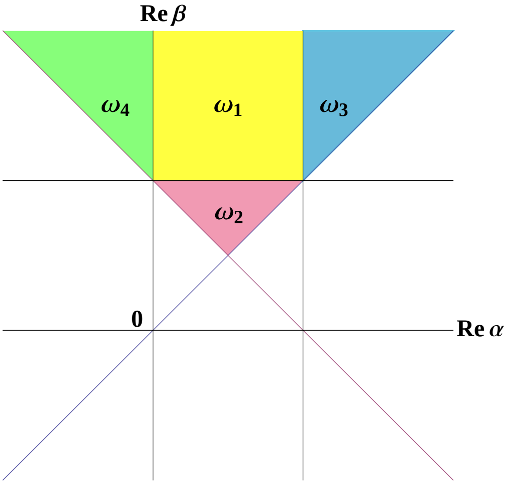
- **$\hat{n}$ characterizes topological configration of Stokes graphs.**
- **$\hat{n}$ is constant on a connected component of the set of all (α, β, γ) satisfying (ii) and (iii).**

We defined

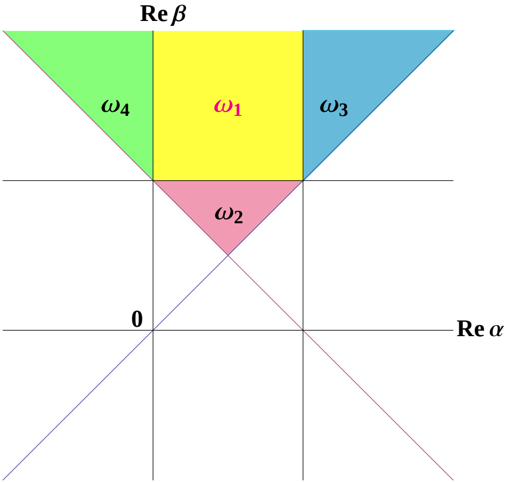
$$
\begin{aligned}
\omega_1 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\alpha < \mathrm{Re}\gamma < \mathrm{Re}\beta\}, \\
\omega_2 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\alpha < \mathrm{Re}\beta < \mathrm{Re}\gamma < \mathrm{Re}\alpha + \mathrm{Re}\beta\}, \\
\omega_3 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\gamma < \mathrm{Re}\alpha < \mathrm{Re}\beta\}, \\
\omega_4 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\gamma < \mathrm{Re}\alpha + \mathrm{Re}\beta < \mathrm{Re}\beta\}.
\end{aligned}
$$

If (α, β, γ) are contained in ω_h $(h = 1, 2, 3, 4)$ respectively, we give a characterization of the Stokes geometry of our equation.

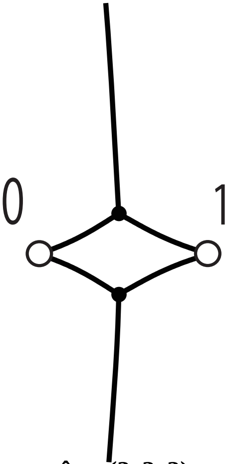
$\mathrm{Re}\,\gamma > 0$ fixed
$\mathrm{Re}\alpha$-$\mathrm{Re}\beta$ plane



Re $\gamma > 0$ fixed
Reα-Reβ plane

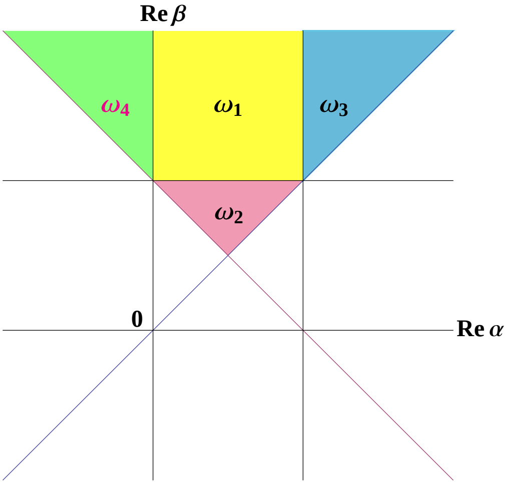


Example1. $(\alpha, \beta, \gamma) = (0.1, 2, 1)$ in ω_1

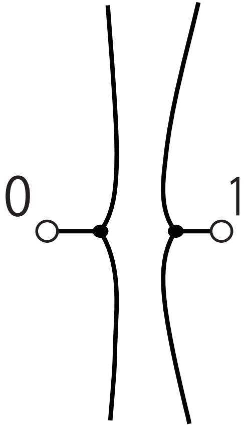


$\hat{n} = (2, 2, 2)$

Re $\gamma > 0$ fixed
Reα-Reβ plane

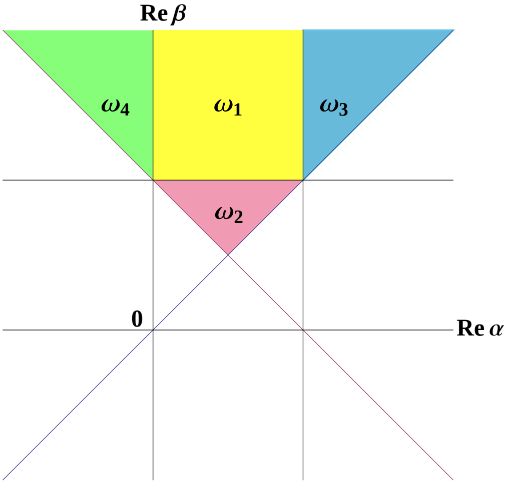


Example 2. $(\alpha, \beta, \gamma) = (-0.1, 2, 1)$ **in** ω_4



$\hat{n} = (1, 1, 4)$

Re $\gamma > 0$ fixed
Reα-Reβ plane



Each uncolored domain is covered by one of colored domains via involutions.

Q : invariant under involutions ι_j $(j = 0, 1, 2)$

$$\begin{aligned} \iota_0 : \quad & (\alpha, \beta, \gamma) \mapsto (-\alpha, -\beta, -\gamma) \\ \iota_1 : \quad & \mapsto (\gamma - \alpha, \gamma - \beta, \gamma) \\ \iota_2 : \quad & \mapsto (\beta, \alpha, \gamma) \end{aligned}$$

We set G=the group generated by ι_j $(j = 0, 1, 2)$

$$\Pi_h = \bigcup_{r \in G} r(\omega_h) \qquad (h = 1, 2, 3, 4).$$

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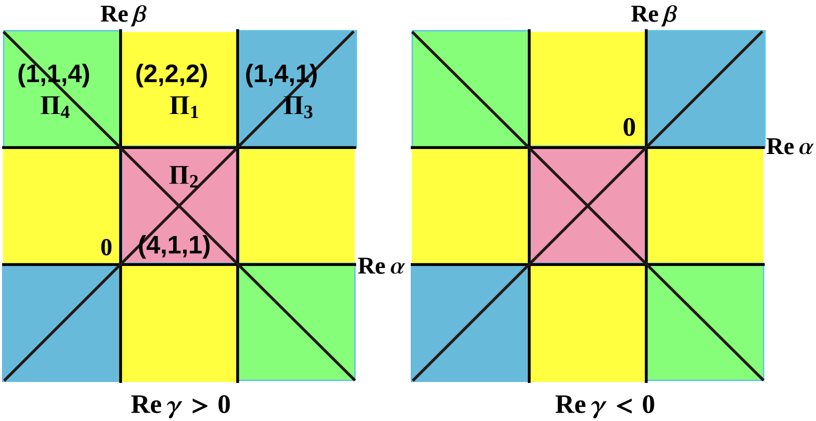
Theorem 1

(1) If $(\alpha, \beta, \gamma) \in \Pi_1$, then $\hat{n} = (2, 2, 2)$.
(2) If $(\alpha, \beta, \gamma) \in \Pi_2$, then $\hat{n} = (4, 1, 1)$.
(3) If $(\alpha, \beta, \gamma) \in \Pi_3$, then $\hat{n} = (1, 4, 1)$.
(4) If $(\alpha, \beta, \gamma) \in \Pi_4$, then $\hat{n} = (1, 1, 4)$.

Reα-Reβ planes



Voros coefficients

$$\sqrt{Q_0} \sim -\frac{\gamma}{2x} \quad \text{at } x = 0,$$
$$\sqrt{Q_0} \sim \frac{\alpha+\beta-\gamma}{2(x-1)} \quad \text{at } x = 1,$$
$$\sqrt{Q_0} \sim \frac{\beta-\alpha}{2x} \quad \text{at } x = \infty,$$

V_j for (b_j, a) $(j = 0, 1, 2)$ has following form: the Voros coefficient

$$V_0 = V_0(\alpha,\beta,\gamma) := \int_0^a (S_{\text{odd}} - \eta S_{-1})dx,$$
$$V_1 = V_1(\alpha,\beta,\gamma) := \int_1^a (S_{\text{odd}} - \eta S_{-1})dx,$$
$$V_2 = V_2(\alpha,\beta,\gamma) := \int_\infty^a (S_{\text{odd}} - \eta S_{-1})dx$$

Since residues of S_{odd} and ηS_{-1} as the singular points coincide, V_j are well defined and we have a formal power series V_j in η^{-1}.

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Since residues of S_{odd} and ηS_{-1} as the singular points coincide, V_j are well defined and we have a formal power series V_j in η^{-1}.

Voros coefficient $V_j(\alpha,\beta,\gamma)$ describes the discrepancy between WKB solutions normalized at a and those normalized at b_j:

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_a^x S_{\mathrm{odd}} dx\right)$$

$$\psi_{\pm}^{(b_j)} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_{b_j}^x (S_{\mathrm{odd}} - \eta S_{-1}) dx \pm \eta \int_a^x S_{-1} dx\right)$$

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$$\Longrightarrow \psi_{\pm}^{(b_j)} = \exp(\pm V_j)\psi_{\pm}$$

Theorem 2

V_j for (j, a) $(j = 0, 1, 2)$ has following forms:

$$V_0 = -\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)}\left\{(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}+\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}\right)+\frac{2}{\gamma^{n-1}}\right\},$$

$$V_1 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)}\left\{(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}-\frac{1}{(\gamma-\alpha)^{n-1}}-\frac{1}{(\gamma-\beta)^{n-1}}\right)+\frac{2}{(\alpha+\beta-\gamma)^{n-1}}\right\},$$

$$V_2 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)}\Big\{(1-2^{1-n})\Big(\frac{1}{\alpha^{n-1}}-\frac{1}{\beta^{n-1}}-\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}\Big)-\frac{2}{(\beta-\alpha)^{n-1}}\Big\}.$$

Here, B_n are Bernoulli numbers defined by

$$\frac{te^t}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n.$$

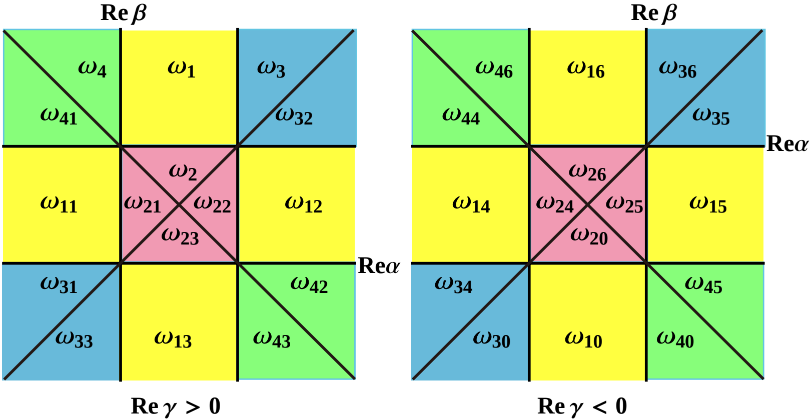
Borel sums of Voros coefficients

Q : :invariant under involutions ι_j $(j = 0, 1, \cdots, 6)$

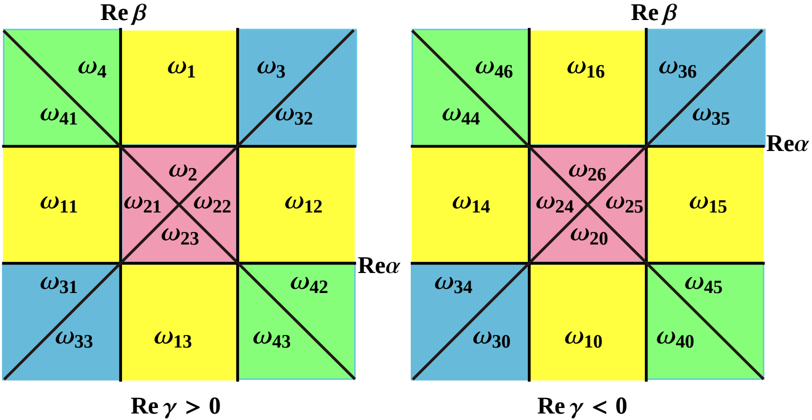
$$
\begin{aligned}
\iota_0 \quad &: (\alpha, \beta, \gamma) \mapsto (-\alpha, -\beta, -\gamma) \\
\iota_1 \quad &: \quad \mapsto (\gamma - \beta, \gamma - \alpha, \gamma) \\
\iota_2 \quad &: \quad \mapsto (\beta, \alpha, \gamma) \\
\iota_3 = \iota_1\iota_2 \quad &: \quad \mapsto (\gamma - \alpha, \gamma - \beta, \gamma) \\
\iota_4 = \iota_0\iota_2 \quad &: \quad \mapsto (-\beta, -\alpha, -\gamma) \\
\iota_5 = \iota_0\iota_1 \quad &: \quad \mapsto (\beta - \gamma, \alpha - \gamma, -\gamma) \\
\iota_6 = \iota_0\iota_1\iota_2 \quad &: \quad \mapsto (\alpha - \gamma, \beta - \gamma, -\gamma)
\end{aligned}
$$

$\omega_{hm} = \iota_m(\omega_h)$: Images in ω_h by ι_m.
Here, $h = 1, 2, 3, 4$, $m = 0, 1, \cdots, 6$.

Reα-Reβ planes



Reα-Reβ planes

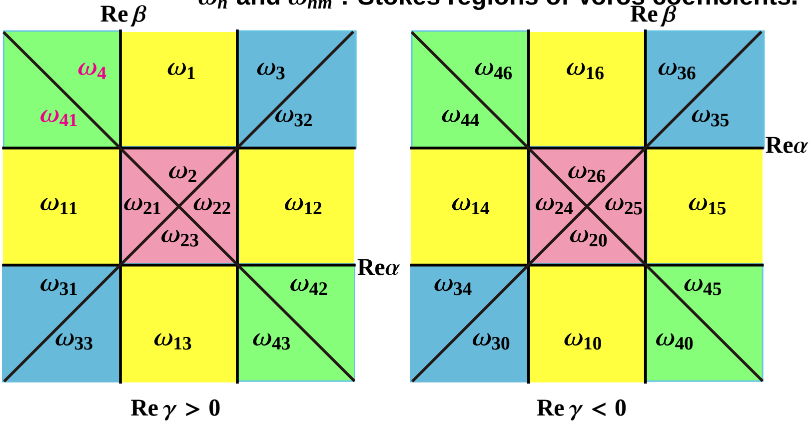


V_j are Borel summable in ω_h and $\omega_{hm}(h = 1, 2, 3, 4;\ m = 0, 1, \cdots, 6)$.

V_j^h or V_j^{hm} : The Borel sums of V_j in ω_h or $\omega_{hm}(h = 1, 2, 3, 4)$.

Reα-Reβ planes

ω_h **and** ω_{hm} **: Stokes regions of Voros coefficients.**



V_j **are Borel summable in** ω_h **and** $\omega_{hm}(h = 1, 2, 3, 4;\ m = 0, 1, \cdots, 6)$.

V_j^h **or** V_j^{hm} **: The Borel sums of** V_j **in** ω_h **or** $\omega_{hm}(h = 1, 2, 3, 4)$.

We compute V_j^h **and** V_j^{hm} **in all region. We have 96 case.**

Let show you V_j^4 **and** V_j^{41} **and the relation** V_j^h **and** V_j^{hm}**.**

Theorem 3

(i) The Borel sums V_j^4 of V_j in ω_4 have following forms:

$$V_0^4 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta},$$

$$V_1^4 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma^2((\alpha+\beta-\gamma)\eta)\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\eta}{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)(-\alpha)^{-\alpha\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\alpha+\beta-\gamma)^{2(\alpha+\beta-\gamma)\eta-1}2\pi},$$

$$V_2^4 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma^2((\beta-\alpha)\eta)(-\alpha)^{-\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}2\pi\eta}.$$

(ii) The Borel sums V_j^{41} of V_j in ω_{41} have following forms:

$$V_0^{41} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}.$$

$$V_1^{41} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}(\gamma-\alpha-\beta)^{2(\gamma-\alpha-\beta)\eta-1}2\pi}{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2((\gamma-\alpha-\beta)\eta)(-\alpha)^{-\alpha\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta},$$

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In the same way, we can compute the Borel sums of V_j in the other Stokes regions of Voros coefficients.

G the group generated by $\iota_m (m = 0, 1, \cdots, 6)$.
τ : An element of G of the form:

$$\tau = \iota_0^{\epsilon_0} \iota_1^{\epsilon_1} \iota_2^{\epsilon_2} = \iota_m$$

$(\epsilon_n = 0, 1; m = 0, 1, \cdots, 6)$
The unify the notation, we denote V_j^{hm} by $V_j^{h\tau}$ for $\tau = \iota_m$
We define the action of $\tau \in G$ on $V_j^h(\alpha, \beta, \gamma)$ by

$$\tau_* V_j^h(\alpha, \beta, \gamma) = V_j^h(\tau(\alpha, \beta, \gamma))$$

Theorem 5

Let $\mathrm{sgn}(\iota, j)$ denote the function defined by

$$\begin{aligned} \mathrm{sgn}(\tau, 0) &= (-1)^{\epsilon_0}, \\ \mathrm{sgn}(\tau, j) &= (-1)^{\epsilon_0 + \epsilon_j} \ \ (j = 1, 2). \end{aligned}$$

The Borel resummed Voros coefficients $V_j^{h\tau}$ in $\tau(\omega_h)$ are related to $\tau_* V_j^h$ by

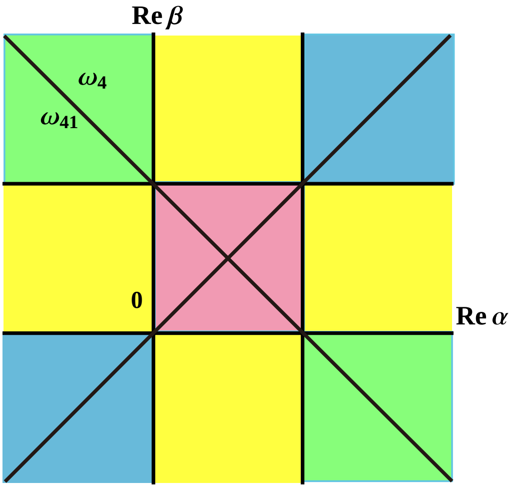
$$V_j^{h\tau} = \mathrm{sgn}(\tau; j)\, \tau_* V_j^h.$$

Proof

$\tau = \iota_1$

$\mathrm{Re}\,\gamma > 0$ fixed

$\mathrm{Re}\alpha$-$\mathrm{Re}\beta$ plane



We compare V_0^{41} with $\tau_* V_0^4$.

$$V_0^{41} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}.$$

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$$\iota_1 \ : \ (\alpha,\beta,\gamma) \mapsto (\gamma-\beta,\gamma-\alpha,\gamma)$$

$$(\gamma-\alpha,\beta-\gamma,\gamma-\alpha-\beta,\beta-\alpha) \mapsto (\beta,-\alpha,\alpha+\beta-\gamma,\beta-\alpha)$$

$$V_0^{4} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta},$$

$$\iota_{1*}V_0^{4} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}$$

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$$\iota_{1*}V_0^{4} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}$$

We compare V_0^{41} with $\tau_* V_0^4$.

$$V_0^{41} = \frac{1}{2}\log \frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}.$$

$$\iota_1 \ : \ (\alpha,\beta,\gamma) \mapsto (\gamma-\beta,\gamma-\alpha,\gamma)$$

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$$\iota_{1*}V_0^{4} = \frac{1}{2}\log \frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}$$

We compare V_0^{41} with $\tau_* V_0^4$.

$$V_0^{41} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}.$$

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$$\iota_{1*}V_0^{4} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}$$

$$V_0^{4\tau} = \tau_* V_0^4$$

We compare V_1^{41} with $\tau_* V_1^4$ and V_2^{41} with $\tau_* V_2^4$.

$$V_1^{41} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}(\gamma-\alpha-\beta)^{2(\gamma-\alpha-\beta)\eta-1}2\pi}{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2((\gamma-\alpha-\beta)\eta)(-\alpha)^{-\alpha\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta},$$

$$V_2^{41} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma^2((\beta-\alpha)\eta)(-\alpha)^{-\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}2\pi\eta}.$$

$$\iota_1 \ : \ (\alpha,\beta,\gamma) \mapsto (\gamma-\beta,\gamma-\alpha,\gamma)$$

$$(\gamma-\alpha,\beta-\gamma,\gamma-\alpha-\beta,\beta-\alpha) \mapsto (\beta,-\alpha,\alpha+\beta-\gamma,\beta-\alpha)$$

$$\iota_{1*}V_1^{4} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2((\gamma-\alpha-\beta)\eta)(-\alpha)^{-\alpha\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}(\gamma-\alpha-\beta)^{2(\gamma-\alpha-\beta)\eta-1}2\pi}$$

$$\iota_{1*}V_2^{4} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma^2((\beta-\alpha)\eta)(-\alpha)^{-\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}2\pi\eta}$$

We compare V_1^{41} with $\tau_* V_1^4$ and V_2^{41} with $\tau_* V_2^4$.

$$V_1^{41} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}(\gamma-\alpha-\beta)^{2(\gamma-\alpha-\beta)\eta-1}2\pi}{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2((\gamma-\alpha-\beta)\eta)(-\alpha)^{-\alpha\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta},$$

$$V_2^{41} = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma^2((\beta-\alpha)\eta)(-\alpha)^{-\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}2\pi\eta}.$$

$$\iota_1 \ : \ (\alpha,\beta,\gamma) \mapsto (\gamma-\beta,\gamma-\alpha,\gamma)$$

$$(\gamma-\alpha,\beta-\gamma,\gamma-\alpha-\beta,\beta-\alpha) \mapsto (\beta,-\alpha,\alpha+\beta-\gamma,\beta-\alpha)$$

$$\iota_{1*}V_1^4 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2((\gamma-\alpha-\beta)\eta)(-\alpha)^{-\alpha\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}(\gamma-\alpha-\beta)^{2(\gamma-\alpha-\beta)\eta-1}2\pi}$$

$$\iota_{1*}V_2^4 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma^2((\beta-\alpha)\eta)(-\alpha)^{-\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}2\pi\eta}$$

$$V_1^{4\tau} = -\tau_* V_1^4, \quad V_2^{\tau 4} = \tau_* V_2^4$$

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END

Thank you for your attention.