Long-time asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation

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Dzień dobry. Nazywam się
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$$Q = -2iz^2\sigma_3 + 2z \begin{bmatrix} 0 & y \\ \bar{y} & 0 \end{bmatrix} + \begin{bmatrix} -i|y|^2 & iy_x \\ -i\bar{y}_x & i|y|^2 \end{bmatrix}$$

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$\Leftrightarrow$ compatibility ($\psi_{xt} = \psi_{tx}$) of the Lax pair.

(Zakharov-Shabat, Ablowitz-Kaup-Newell-Segur)

- x-part: reflection coefficient r
- t-part: its time evolution

2. Reflection coefficient

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Eigenfunctions $\psi \sim {}^t[0,1]e^{izx}$, $\psi^* \sim {}^t[1,0]e^{-izx}$ as $x \to \infty$(right).

The *reflection coefficient* $r = r(z)$ is defined by:

$$\underbrace{r\psi}_{\text{reflection}} + \underbrace{\psi^*}_{\text{incidence}} \sim \underbrace{\text{const.}\,{}^t[1,0]e^{-izx}}_{\text{transmission}} \text{ as } x \to -\infty\text{(left)}.$$

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Summary $y(x,0) \underset{x}{\mapsto} r(z,0) \underset{t}{\mapsto} r(z,t)\ (t > 0)$

3. Our tool: Riemann-Hilbert problem

Γ: oriented contour (the left-hand is the $+$ side).

$m(z)$: unknown matrix, holomorphic in $\mathbb{C} \setminus \Gamma$

Examples:

1. $\Gamma = \mathbb{R}$, $m(z)$ in the upper/lower half planes
2. $\Gamma = \{|z| = 1\}$, $m(z)$: holo. in $|z| \neq 1$.

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m_+, m_-: boundary values on Γ from the $\pm$ sides

RHP: $m_+ = m_- J$ on Γ (J : **the jump matrix**)

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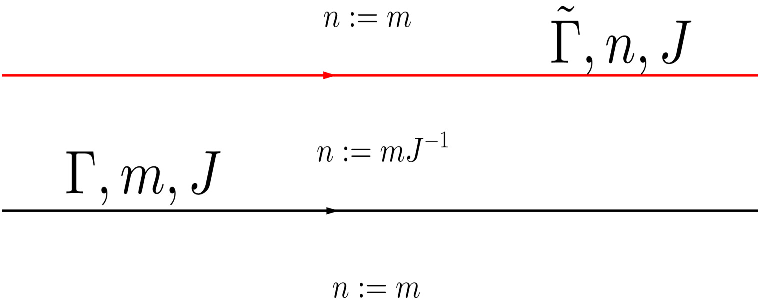
If $J = I$, then $m_+ = m_-$ and m is holomorphic near Γ

$\Rightarrow$*Neglect* Γ.

If $J \approx I$ on some parts of Γ, we *neglect* those parts up to a certain error (*asymptotic analysis*).

4. Contour Deformation

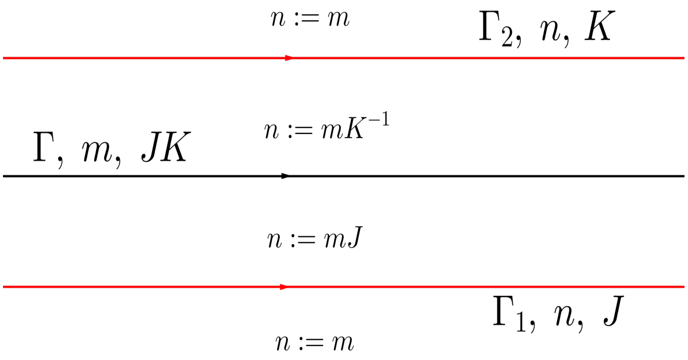
$m_+ = m_- J$ on Γ (black line) $\Leftrightarrow$ $n_+ = n_- J$ on $\tilde{\Gamma}$ (*red line*).
$(J = J(z))$



5. Factorization and contour deformation

$m_+ = m_- JK$ on Γ (black line).

$\Leftrightarrow \; n_+ = n_- J$ on Γ_1 and $n_+ = n_- K$ on Γ_2 (*red lines*).



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- Assume their jump matrices are close.
- Assume RHP No.1 is explicitly solvable.
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This observation, together with *contour deformation*, leads to the Riemann-Hilbert version of the classical method of steepest descent: *the Deift-Zhou method, nonlinear steepest descent*.

NB: A *nonlinear* problem has been reduced to an RHP, a *linear* problem. Superposition is now possible.

7. Inverse scattering and RHP

Initial value problem of NLS can be solved by inverse scattering.

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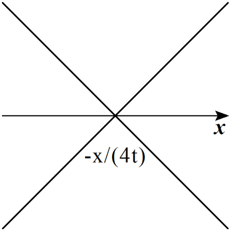
♯ $y(x,0) \mapsto r(z,0) \mapsto r(z,t) \mapsto m(x,t;z) \mapsto y(x,t)$

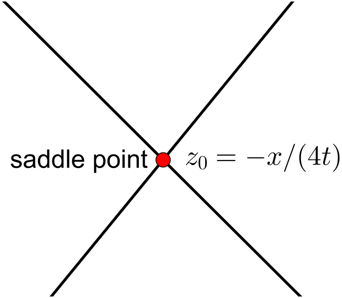
8. Long-time asymptotics of the defocusing NLS

1. Zakhalov-Manakov: formal calculation
2. Deift-Its-Zhou: proof by *nonlinear steepest descent*
   RHP with oscillatory coefficients (with a *phase function*)
   $\Rightarrow$ new contour, new unknown and new jump matrix
   $\Rightarrow$ new RHP, equivalent to the original one.



If $t \gg 0$, the jump matrix on $\mathbb{R}$ is almost I. Can be neglected.

9. saddle point and decaying oscillation



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$z_0 = -x/(4t)$ is the only saddle point of the phase function θ.

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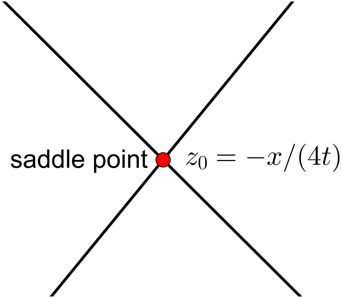

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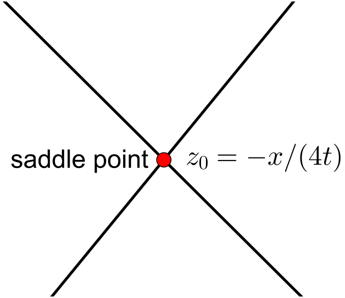

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$$y(x,t) \sim \alpha(z_0) t^{-1/2} \exp\left(4itz_0^2 - i\nu(z_0)\log 8t\right)$$

decaying oscillation

α, ν determined by the reflection coefficient.

10. Integrable Discrete NLS (IDNLS)

Ablowitz-Ladik introduced

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If $\sum_{n\in\mathbb{Z}} n^{10}|R_n(0)| < \infty$ *and* $\sup_{n\in\mathbb{Z}}|R_n(0)| < 1$,
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such that in $|n|<2t$ *("timelike"), we have as* $t\to\infty$,

$$R_n(t)=\sum_{j=1}^{2}\underbrace{C_j t^{-1/2}\exp\Big(-i(p_j t+q_j\log t)\Big)}_{\text{DECAYING OSCILLATION}}+O(t^{-1}\log t)$$

cf. Formal calc. by Novokshënov-Habibullin (1981),
focusing, without solitons.

Go to details.

11. IDNLS and its Lax pair

Lax pair (AKNS pair):

$$X_{n+1} = \begin{bmatrix} z & \overline{R}_n \\ R_n & z^{-1} \end{bmatrix} X_n$$

n-part, Ablowitz-Ladik scattering problem,

$$\frac{d}{dt} X_n = \begin{bmatrix} iR_{n-1}\overline{R}_n - \frac{i}{2}(z - z^{-1})^2 & -i(z\overline{R}_n - z^{-1}\overline{R}_{n-1}) \\ i(z^{-1}R_n - zR_{n-1}) & -iR_n\overline{R}_{n-1} + \frac{i}{2}(z - z^{-1})^2 \end{bmatrix} X_n$$

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(IDNLS) *is the compatiblity condition.*

$$i\frac{d}{dt}R_n + (R_{n+1} - 2R_n + R_{n-1}) - |R_n|^2(R_{n+1} + R_{n-1}) = 0 \cdots \text{(IDNLS)}$$

12. AL scattering problem and eigenfunctions

$$X_{n+1} = \begin{bmatrix} z & \overline{R}_n \\ R_n & z^{-1} \end{bmatrix} X_n, \qquad \text{n-part, AL scattering problem}$$

$\phi_n(z,t), \psi_n(z,t)$: holo. sol. in $|z| > 1$, continuous in $|z| \geq 1$

$\psi_n^*(z,t)$: holo. sol. in $|z| < 1$, continuous in $|z| \leq 1$

$$\phi_n(z,t) \sim z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{as } n \to -\infty \text{(LEFT)},$$

$$\psi_n(z,t) \sim z^{-n} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \psi_n^*(z,t) \sim z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{as } n \to \infty \text{(RIGHT)}.$$

13. Reflection coefficient

On $C\colon |z| = 1$, for some $a(z,t) \neq 0$ and $b(z,t)$,

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The *time evolution* is
$r(z,t) = r(z)\exp\left(it(z - z^{-1})^2\right)$, where $r(z) = r(z,0)$.

14. Oscillatory RHP

$m_+(z) = m_-(z)v(z)$ on $C\colon |z| = 1$(clockwise),

$m(z) \to I$ as $z \to \infty$,

$$v(z) = \begin{bmatrix} 1 - |r(z)|^2 & -e^{-2\varphi}\bar{r}(z) \\ e^{2\varphi}r(z) & 1 \end{bmatrix}$$ *oscillatory* jump matrix

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Potential reconstruction $R_n(t) = -\left.\frac{d}{dz}m(z)_{21}\right|_{z=0}$

RHP gives $\{R_n\}$. Ref. book by Ablowitz-Prinari-Trubatch

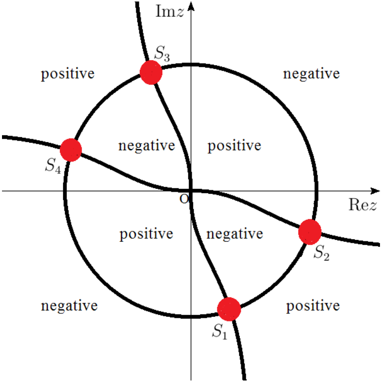
15. Jump matrix and saddle points

$$v(z) = \begin{bmatrix} 1 - |r(z)|^2 & -e^{-2\varphi}\bar{r}(z) \\ e^{2\varphi} r(z) & 1 \end{bmatrix}$$ oscillatory jump matrix

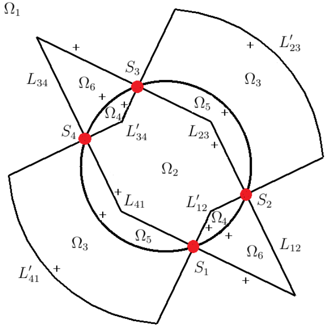
$\varphi = \frac{1}{2} it(z - z^{-1})^2 - n \log z$ (*phase function*)

$S_1, \ldots, S_4$: saddle points of φ.

$\{\mathrm{Re}\,\varphi = 0\}$ and the signature of $\mathrm{Re}\,\varphi$ is shown in the figure.



16. Rewriting into an equivalent RHP



A new unknown function and a new contour.

Crosses near S_j: the direction of the steepest descent of $\pm\varphi$.

The new jump matrix has components coming from $e^{\pm 2\varphi}$.

17. New jump matrix

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Decompose r. Extract terms that can be continued analytically to the inside/outside of the circle.

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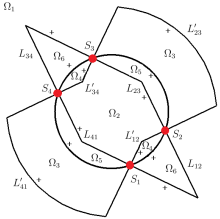
Decompose r*. Extract terms that can be continued analytically to the inside/outside of the circle.*

$r =$Taylor polynomial and remainder (Fourier integral).

Divide Fourier integral — analytic continuation

$$f(x) = \underbrace{\int_{-\infty}^{0} e^{ix\xi}\hat{f}(\xi)\,d\xi}_{\text{holo in } \operatorname{Im} x<0} + \underbrace{\int_{0}^{\infty} e^{ix\xi}\hat{f}(\xi)\,d\xi}_{\text{holo in } \operatorname{Im} x>0}$$

The remaining term decreases on the circle rapidly as $t \to \infty$.



18. Four small crosses

The jump matrix $\to I$ as $t \to \infty$ on parts of the contour.
Those parts can be **neglected**.
On the crosses, *only small neighborhoods of the saddle points matter.*

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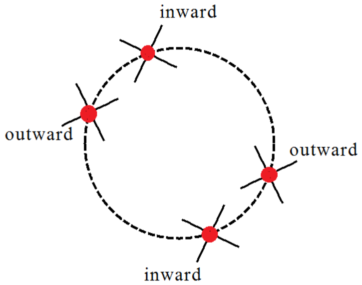

Our RHP can be approximated by a concrete, calculable one.
Its solution is given by Deift-Its-Zhou (or Deift-Zhou on MKdV.)

20. Result (details)

Go to outline

$r(z) := r(z,0)$ (*initial reflection coefficient*)

$$S_1 = e^{-\pi i/4}A,\ S_2 = e^{-\pi i/4}\bar{A},\ S_3 = -S_1,\ S_4 = -S_2,:$$ *saddle points*,

$$A = 2^{-1}\left(\sqrt{2+n/t} - i\sqrt{2-n/t}\,\right),$$

$$\delta(0) = \exp\left(\frac{-1}{\pi i}\int_{S_1}^{S_2} \log(1-|r(\tau)|^2)\frac{d\tau}{\tau}\right) \geq 1,$$

$$\beta_1 = \frac{-e^{\pi i/4}A}{2(4t^2-n^2)^{1/4}},\ \beta_2 = \frac{e^{\pi i/4}\bar{A}}{2(4t^2-n^2)^{1/4}}$$ ← Decay, $O(t^{-1/2})$

$$D_1 = \frac{-iA}{2(4t^2-n^2)^{1/4}(A-1)},\ D_2 = \frac{i\bar{A}}{2(4t^2-n^2)^{1/4}(\bar{A}-1)}.$$

For $j = 1, 2$,

$$\chi_j(S_j) = \frac{1}{2\pi i}\int_{\exp(-\pi i/4)}^{S_j} \log\frac{1-|r(\tau)|^2}{1-|r(S_j)|^2}\frac{d\tau}{\tau - S_j},$$

$$\nu_j = -\frac{1}{2\pi}\log(1-|r(S_j)|^2) \geq 0,$$

$$\widehat{\delta}_j(S_j) = \exp\left(\frac{1}{2\pi}\left[(-1)^j\int_{e^{-\pi i/4}}^{S_{3-j}} - \int_{-S_1}^{-S_2}\right]\frac{\log(1-|r(\tau)|^2)}{\tau - S_j}\,d\tau\right),$$

$$\delta_j^0 = S_j^n e^{-it(S_j - S_j^{-1})^2/2} D_j^{(-1)^{j-1} i\nu_j} e^{(-1)^{j-1}\chi_j(S_j)}\widehat{\delta}_j(S_j)$$

Oscillation

Theorem (Y; two terms, decaying oscillation)

Assume $\sum n^{10}|R_n(0)| < \infty$ and $\sup|R_n(0)| < 1$.
Then on $|n| < 2t$ ("timelike"), as $t \to \infty$,

$$R_n(t) = -\frac{\delta(0)}{\pi i}\sum_{j=1}^{2} \underbrace{\beta_j}_{\text{Decay}} \underbrace{(\delta_j^0)^{-2}}_{\text{Oscill.}} S_j^{-2} M_j + O(t^{-1}\log t).$$

Here

$$M_j = \begin{cases} \dfrac{\sqrt{2\pi}\exp\left((-1)^j 3\pi i/4 - \pi\nu_j/2\right)}{\bar{r}(S_j)\Gamma((-1)^{j-1}i\nu_j)} & \text{if} \quad r(S_j) \neq 0, \\ 0 & \text{if} \quad r(S_j) = 0. \end{cases}$$

Four crosses can be dealt with separately.
(RHP being linear, superposition is possible).
Two pairs of antipodals⇒two terms.

Go to outline

23. Work in progress and an open problem

WORK in PROGRESS (to be announced in FASDE 4?):
We studied the asymptotic behavior in $|n| < 2t$.
What are the behaviors in $|n| \approx 2t$ and in $|n| > 2t$?
It seems that Painlevé II appears.
Similar phenomena have been observed in the cases of MKdV (Deift-Zhou) and Toda (Kamvissis).

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OPEN PROBLEM
focusing case, a sum of solitons asymptotically.
cf. Toda (Krüger-Teschl), NLS (Fokas-Its-Sung)

Thank you very much!

Dziękuję! Wymowa polska jest trundna.