q-Analogue of summability of formal solutions of linear q-difference-differential equations

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Schedule

Known facts (Equations)

Let $m \geq 1$ be an integer, let $(t, x) = (t, z_1, \ldots, z_d) \in \mathbb{C}_t \times \mathbb{C}_z^d$ be the complex variables. Let us consider the linear partial differential equation

$$\sum_{j+|\alpha| \leq m} a_{j,\alpha}(t, z)(t\partial_t)^j \partial_z^\alpha X = F(t, z), \tag{1.1}$$

with the unknown function $X = X(t, z)$, where $a_{j,\alpha}(t, x)$ $(j + |\alpha| \leq m)$ and $F(t, z)$ are holomorphic functions in a neighborhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_z^d$.

Known facts (Conditions)

We suppose: there is an $m_0 \in \mathbb{N}$ with $0 \le m_0 \le m$ such that

$$\begin{cases} ord_t(a_{j,0}) \ge \max\{0, j - m_0\}, & \text{if } |\alpha| = 0, \\ ord_t(a_{j,\alpha}) \ge \max\{1, j + |\alpha| - m_0 + 1\}, & \text{if } |\alpha| > 0, \end{cases} \tag{1.2}$$

where $ord_t(a)$ denotes the order of the zeros of the function $a(t,z)$ at $t = 0$.

For $r > 0$ we write $D_r = \{t \in \mathbb{C} \ ; \ |t| < r\}$; for $R > 0$ we write $D_R = \{z \in \mathbb{C}^d; \ |z_i| < R \ (i = 1, \dots, d)\}$ and we denote by $\mathcal{O}_R[[t]]$ the set of all formal power series in t with coefficients in $\mathcal{O}_R$.

Known result 1

For the equation (1.1) we have

Theorem 1.1

(Baouendi-Goulauic[1]). Suppose the conditions (1.2), $m_0 = m$ and

$$a_{m,0}(0,0) \neq 0.$$

If the equation (1.1) has a formal solution $\hat{X}(t,z) = \sum_{n\geq 0} X_n(z)t^n$ $\in \mathcal{O}_R[[t]]$ (with $R > 0$), then it is convergent in a neighborhood of the origin $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$.

Known result 2

Theorem 1.2

(Ōuchi[7]). Suppose the conditions (1.2), $0 < m_0 < m$ *and*

$$a_{m_0,0}(0,0) \neq 0, \ \ \textit{and} \ \ \left.\frac{a_{m,0}(t,0)}{t^{m-m_0}}\right|_{t=0} \neq 0.$$

If the equation (1.1) has a formal solution $\hat{X}(t,z) = \sum_{n\geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$ *(with* $R > 0$*), then it is Borel summable in* t *in a suitable direction.*

Definition of q-difference

Let $q > 1$: for a function $f(t,x)$ we define the $q-$difference operator D_q by

$$(D_q f)(t,z) = \frac{f(qt,z) - f(t,z)}{qt - t}.$$

In this talk, we will try to $q-$discrete the equation (1.1) to the following $q-$difference-differential equation

$$\sum_{j+|\alpha|\leq m} a_{j,\alpha}(t,z)(tD_q)^j \partial_z^\alpha X = F(t,z), \tag{2.1}$$

and we will consider the following problem.

Problems

Problem 2.1

(1) ($q-$Analogue of [1]). Under what condition can we get the result that every formal solution $\hat{X} = \sum_{n \geq 0} X_n(z) t^n \in \mathcal{O}_R[[t]]$ *is convergent in a neighborhood of the origin* $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$*?*

(2) ($q-$Analogue of [7]). Under what condition can we get the result that every formal solution $\hat{X} = \sum_{n \geq 0} X_n(z) t^n \in \mathcal{O}_R[[t]]$ *is* $q-$*summable in* t *in a suitable direction* λ *(in the sense of Definition 2.2 given below)?*

Definition of q-summable

For $\lambda \in \mathbb{C} \setminus \{0\}$ and $\epsilon > 0$ we set

$$\mathcal{S}_\lambda = \{-\lambda q^m;\ m \in \mathbb{Z}\},$$
$$\mathcal{S}_{\lambda,\epsilon} = \bigcup_{m \in \mathbb{Z}} \{t \in \mathbb{C} \setminus \{0\};\ |1 + \lambda q^m / t| \le \epsilon\}.$$

It is easy to see that if $\epsilon > 0$ is sufficiently small the set $\mathcal{S}_{\lambda,\epsilon}$ is a disjoint union of closed disks. The following definition is due to Ramis-Zhang [8] (though, they did not use the word "$q-$summable")

Definition of q-summable

Let $\hat{X}(t,z)=\sum_{n\geq 0} X_n(z)t^n \in \mathcal{O}_{R_0}[[t]]$: we say that the series $\hat{X}(t,z)$ is $q-$summable in t in the direction λ if there are $r>0$, $R>0$, $M>0$, $H>0$ and a holomorphic function $W(t,z)$ on $(D_r \backslash \mathcal{S}_\lambda)\times D_R$ such that

$$\left| W(t,z)-\sum_{n=0}^{N-1} X_n(z)t^n \right| \leq \frac{MH^N}{\epsilon} q^{N(N-1)/2}|t|^N \tag{2.2}$$

holds on $(D_r \setminus \mathcal{S}_{\lambda,\epsilon})\times D_R$ for any sufficiently small $\epsilon>0$ and any $N=0,1,2,\ldots$.

Reference of q-analogue

To solve Problem 2.1 we will use the framework of $q-$Borel and $q-$Laplace transformations via Jacobi theta function, developped by Ramis-Zhang [8] and Zhang [11].

In the case of q-difference equations, $q-$analogues of summbility of formal solutions have been studied quite well by Zhang [10], Marotte-Zhang [4] and Ramis-Sauloy-Zhang [9].

In the case of $q-$difference-differential equations, we have some references, Malek [5][6], Lastra-Malek [2] and Lastra-Malek-Sanz [3]: but their problem is different from ours.

Our Equation

Throughout this paper, we let $q > 1$ be a fixed real number, $m \geq 1$ be an integer, and $\delta > 0$ be a real number. As a generalization of (2.2), we will treat the following equation

$$\sum_{j+\delta|\alpha| \leq m} a_{j,\alpha}(t,z)(tD_q)^j \partial_z^\alpha X = F(t,z), \tag{E}$$

with the unknown function $X = X(t,z)$, where $a_{j,\alpha}(t,z)$ $(j + \delta|\alpha| \leq m)$ and $F(t,z)$ are holomorphic functions in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$.

Condition

Instead of (1.2), we suppose: there is an integer m_0 with $0 \le m_0 \le m$ such that

$$\begin{cases} ord_t(a_{j,0}) \ge \max\{0, j - m_0\}, & \text{if } |\alpha| = 0, \\ ord_t(a_{j,\alpha}) \ge \max\{1, j - m_0 + 1\}, & \text{if } |\alpha| > 0. \end{cases} \tag{A}$$

$$\begin{cases} ord_t(a_{j,0}) \ge \max\{0, j - m_0\}, & \text{if } |\alpha| = 0, \\ ord_t(a_{j,\alpha}) \ge \max\{1, j + |\alpha| - m_0 + 1\}, & \text{if } |\alpha| > 0, \end{cases} \tag{1.2}$$

A q-analogue version of [1]

Theorem 3.1

Suppose the conditions (A), $m_0 = m$ and

$$a_{m,0}(0,0) \neq 0. \tag{3.1}$$

Then, if the equation (E) has a formal solution $\hat{X}(t,z) = \sum_{n\geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$ (with $R > 0$), it is convergent in a neighborhood of the origin $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$.

A q-analogue version of [7]

Theorem 3.2 (Formal solution)

Suppose the conditions (A), $0 \leq m_0 < m$ and

$$a_{m_0,0}(0,0) \neq 0. \tag{3.2}$$

Then, $\hat{X}(t,z) = \sum_{n \geq 0} X_n(z) t^n \in \mathcal{O}_R[[t]]$ (with $R > 0$) is a formal solution of (E), there are $A > 0$, $h > 0$ and $0 < R_1 < R$ such that

$$|X_n(z)| \leq A h^n q^{n(n-1)/2} \quad \text{on } D_{R_1}, \quad n = 0, 1, 2, \ldots. \tag{3.3}$$

A q-analogue version of [7]

Theorem 3.3 (Summability)

Suppose that the conditions of Theorem 3.2 hold. In addition, if the conditions

$$\frac{a_{m,0}(t,0)}{t^{m-m_0}}\Big|_{t=0} \neq 0, \tag{3.4}$$

$$ord_t(a_{j,\alpha}) \geq j - m_0 + 2, \ \ if\ |\alpha| > 0 \ and\ m_0 \leq j < m \tag{3.5}$$

are satisfied, the formal solution $\hat{X}(t,z)$ is $q-$summable in any direction $\lambda \in \mathbb{C} \setminus (\{0\} \cup S)$, where S is the set of singular directions defined below.

Singular direction S

By the assumption (A) we have the expression

$$a_{j,0}(t,z) = t^{j-m_0} b_{j,0}(t,z) \text{ for } m_0 < j \leq m$$

for some holomorphic functions $b_{j,0}(t,z)$ $(m_0 < j \leq m)$ in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$. By (3.4) we have $b_{m,0}(0,0) \neq 0$. We set

$$P(\xi, z) = \sum_{m_0 < j \leq m} \frac{b_{j,0}(0,z)}{q^{j(j-1)/2}} \xi^{j-m_0} + \frac{a_{m_0,0}(0,z)}{q^{m_0(m_0-1)/2}}.$$

By the assumptions (3.4) and (3.5) we see that $P(\xi, 0)$ is a polynomial of degree $m - m_0$ and it has $m - m_0$ non-zero roots $\tau_1, \ldots, \tau_{m-m_0}$. Then the set of singular directions is defined by

$$S = \{\tau \in \mathbb{C};\ \tau = t\tau_i \text{ for some } t > 0 \text{ and } 1 \leq i \leq m - m_0\}.$$

Sketch of proof

Let give a proof for a simple example.
By

$$tD_q f(t,z) = \frac{f(qt,z) - f(t,z)}{q-1} \tag{4.1}$$

we study the equation

$$\sum_{j+\delta|\alpha|\le m} b_{j,\alpha}(t,z)\sigma_q^j \partial_z^\alpha X = G(t,z), \tag{4.2}$$

where $\sigma_q f(t,z) = f(qt,z)$.
In this talk we study the following simple example:

$$X(t,z) + t\sigma_q X(t,z) - t^2 \partial_z^\alpha X(t,z) = a(z). \tag{4.3}$$

At first let us give an important proposition.

Proposition 4.1

For a series $\hat{X}(t,z)=\sum_{k=0}^{\infty} a_k(z)t^k$ the formal q-Borel transform $(\hat{B}_q\hat{X})(\tau,z)$ is defined by

$$(\hat{B}_q\hat{X})(\tau,z) := \sum_{k=0}^{\infty} a_k(z)\frac{\tau^k}{q^{k(k-1)/2}}. \tag{4.4}$$

Set $U(\tau,z)=(\hat{B}_q\hat{X})(\tau,z)$. Suppose that $U(\tau,z)$ satisfies

$$|U(\lambda q^l,z)| \le AB^l q^{l^2/2} \quad \text{for } l=0,1,\dots \tag{4.5}$$

for some $A,\ B>0$. Then $\hat{X}(t,z)$ is q-summable in t in the direction λ.

Let us give a proof for the example. By operating the formal q-Borel transform to the example (4.3) we get

$$(1+\tau)U(\tau,z) = a(z) + \frac{\tau^2}{q}\sigma_q^{-2}\partial_z^{\alpha}U(\tau,z). \tag{4.6}$$

We construct $U(t,z) = \sum_{k=0}^{\infty} U_k(\tau,z)$ with

$$\begin{aligned} (1+\tau)U_0(\tau,z) &= a(z) \\ (1+\tau)U_k(\tau,z) &= \frac{\tau^2}{q}\sigma_q^{-2}\partial_z^{\alpha}U_{k-1}(\tau,z). \end{aligned} \tag{4.7}$$

Then we have

Proposition 4.2

$$|U_k(\tau,z)| \le \frac{1}{q^k}\frac{|\tau|^k}{q^{k(k-1)}}|\partial_z^{k\alpha}a(z)| \tag{4.8}$$

Then for $z \in D_R$ we get

$$|U_k(\tau,z)| \le AB^k\frac{1}{q^k}\frac{|\tau|^k}{q^{(1-\epsilon)k(k-1)}}. \tag{4.9}$$

Hence we have

$$\begin{aligned} |U(\lambda q^l,z)| &\le \sum_{k=0}^{\infty} AB^k\frac{|\lambda|^k q^{kl}}{q^k q^{(1-\epsilon)k(k-1)}} \\ &\le q^{l^2/2}\sum_{k=0}^{\infty} AB^k\frac{|\lambda|^k q^{k^2/2}}{q^k q^{(1-\epsilon)k(k-1)}} \end{aligned} \tag{4.10}$$

by $kl \le k^2/2 + l^2/2$. Q.E.D.

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End

Thank you!