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Singular perturbative analysis of Poincaré's theorem for a singular vector field

Masafumi Yoshino (Hiroshima University)

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1. Linearization problem of a singular vector field

Let $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$, $n \geq 2$ be the variable of $\mathbb{C}^n$, and consider the holomorphic vector field in some domain of $\mathbb{C}^n$ containing the origin

$$\mathcal{X} = \sum_{j=1}^{n} a_j(y) \frac{\partial}{\partial y_j}. \tag{1}$$

We assume that the number of singular points of $\mathcal{X}$ is finite, hence the singular points are isolated. Moreover we suppose $a_j(0) = 0$ for $j = 1, \ldots, n$.

Assume that the change of coordinates preserving the origin

$$y = u(x),\ u = (u_1, \ldots, u_n),\quad x = (x_1, \ldots, x_n),\ n \geq 1$$

transforms $\mathcal{X}$ to its linear part. This is equivalent to

$$A(u(x)) \left(\frac{\partial u}{\partial x}\right)^{-1} = x\Lambda, \tag{2}$$

where $A(y) = (a_1(t), \ldots, a_n(y))$ and $\frac{\partial u}{\partial x}$ is the Jacobian matrix, and $\Lambda = DA(0)$ is the linear part of $\mathcal{X}$ at the origin.

We define $v(x)$ by $u(x) = x + v(x)$, $v(x) = O(|x|^2)$ and we set

$$A(y) = y\Lambda + R(y),\ R(y) = (R_1(y), \ldots, R_n(y)) = O(|y|^2).$$

Then we have $A(u) = (x+v)\Lambda + R(x+v)$ and $\frac{\partial u}{\partial x} = I + \frac{\partial v}{\partial x}$. Hence by (2) we have

$$A(u(x)) = (x+v)\Lambda + R(x+v) = x\Lambda\left(I + \frac{\partial v}{\partial x}\right).$$

It follows that our linearization condition can be written in

$$v\Lambda + R(x+v) = x\Lambda\frac{\partial v}{\partial x}. \tag{3}$$

This is a system of semilinear first order partial differential equations for v. Let λ_j, $j = 1, \ldots, n$ be the eigenvalues of Λ. Poincaré's theorem asserts the existence of a local holomorphic solution provided the non resonance condition and the Poincaré condition, Re $\lambda_j > 0$ are satisfied. We note that the solution may not be globally defined because the nonlinear term $R(x+v)$ may cause the singularity.

We note that similar relation like (3) holds at every isolated singular point of $\mathcal{X}$.

Instead of solving (3) globally we introduce a parameter η in the equation and we want to construct an approximate global transformation. Namely, we approximate our equation with the following

$$v\Lambda + R(x+v) = \eta^{-1}x\Lambda\frac{\partial v}{\partial x}. \tag{4}$$

Clearly if $\eta = 1$, then we have the linearization relation (3).

In this talk I will show the global solvability of (4) by virtue of the Borel sum with respect to η of some formal series solution. Then we show that the solution of (4) is naturally related to the solution of the original equation (3).

2. Formal transformation

We assume that Λ is a diagonal matrix with eigenvalues $\lambda_j \neq 0$, $j = 1, 2, \ldots, n$. Define

$$\mathcal{L} = \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j}. \tag{5}$$

Then (4) is written in

$$\eta^{-1} \mathcal{L} v_j = \lambda_j v_j + R_j(x + v(x)), \quad j = 1, \ldots, n. \tag{6}$$

Definition 1 A singular perturbative solution (SP-solution in short) $v(x, \eta)$ of (6) is a formal power series in η^{-1} of the form

$$v(x, \eta) = \sum_{\nu=0}^{\infty} \eta^{-\nu} v_\nu(x) = v_0(x) + \eta^{-1} v_1(x) + \cdots, \tag{7}$$

where the coefficients $v_\nu(x)$ are holomorphic vector functions of x in some open set independent of ν.

We want to construct a SP-solution of (6) in the following form

$$v_j \equiv v_j(x,\eta) = \sum_{\nu=0}^{\infty} v_\nu^j(x)\eta^{-\nu}, \qquad v_\nu^j(x) = O(|x|^2),\ \ j=1,\ldots,n. \tag{8}$$

By substituting the expansion (8) into (6), and by comparing the coefficients of η, $\eta^0 = 1$ we obtain

$$\lambda_j v_0^j(x) + R_j(x_1 + v_0^1, \ldots, x_n + v_0^n) = 0. \tag{9}$$

One can determine $v_0 = (v_0^1, \ldots, v_0^n)$ from (9). In the rest of the talk we assume that v_0 is holomorphic in the domain $\Omega(v_0)$ which contains the origin. The other terms are determined inductively.

We define the set Σ_0 by

$$\Sigma_0 := \{x \in \mathbb{C}^n;\ \det\left(\Lambda + \nabla R(x + v_0(x))\right) = 0\}. \tag{10}$$

We assume

$$0 \notin \Sigma_0. \tag{11}$$

We note that (11) implies $\lambda_k \neq 0$ for every k. The next theorem gives the existence of a SP-solution.

Proposition 1 Assume (11). Then every coefficient of the SP-solution (8) is uniquely determined as a holomorphic function in a neighborhood of the origin $x = 0$ independent of ν.

Remark. Let $\widetilde{\mathbb{C}^n \setminus \Sigma_0}$ be the universal covering space of $\mathbb{C}^n \setminus \Sigma_0$. We can make analytic continuation of the formal SP-solution in Proposition 1 from the origin to $\widetilde{\mathbb{C}^n \setminus \Sigma_0} \cap \Omega(v_0)$, assuming that $R(x)$ is an entire function on $x \in \mathbb{C}^n$. Indeed, every coefficient of SP-solution (8) is analytically continued to $\widetilde{\mathbb{C}^n \setminus \Sigma_0} \cap \Omega(v_0)$ because it is calculated inductively through differentiations and algebraic calculations.

3. Definition of Borel sum

Let $v(x,\eta) = \sum_{\nu=0}^{\infty} v_\nu(x)\eta^{-\nu}$ be the SP-solution of (6). Then the formal Borel transform of $v(x,\eta)$ is defined by

$$B(v)(x,\zeta) := \sum_{\nu=0}^{\infty} v_\nu(x) \frac{\zeta^\nu}{\Gamma(\nu+1)}, \tag{12}$$

where $\Gamma(z)$ is the Gamma function.

For an opening $\theta > 0$ and the direction ξ we define the sector $S_{\theta,\xi}$ in the direction ξ by

$$S_{\theta,\xi} = \left\{ z \in \mathbb{C};\ |\mathrm{arg}\, z - \xi| < \frac{\theta}{2} \right\}. \tag{13}$$

We say that $v(x,\eta)$ is Borel summable in the direction ξ with respect to η if $B(v)(x,\zeta)$ converges in some neighborhood of the origin of (x,ζ), and there exist a neighborhood U of the origin $x = 0$ and a $\theta > 0$ such that $B(v)(x,\zeta)$ can be analytically continued to $(x,\zeta) \in U \times S_{\theta,\xi}$ and of exponential growth of order 1 with respect to ζ in $S_{\theta,\xi}$. For the sake of simplicity we denote the analytic continuation with the same notation $B(v)(x,\zeta)$. The Borel sum $V(x,\eta)$ of $v(x,\eta)$ is, then, given by the Laplace transform

$$V(x,\eta) := \int_{L_\xi} \zeta^{-1} e^{-\zeta\eta} B(v)(x,\zeta) d\zeta \tag{14}$$

where the integral is taken on the ray starting from the origin to the infinity in the direction ξ.

4. Convergence of the formal Borel transform

Theorem 2 Assume that $R(x)$ is an entire function on $x \in \mathbb{C}^n$. Let v be the SP-solution given by (8). Let K be the compact set in $\mathbb{C}^n \setminus \Sigma_0 \cap \Omega(v_0)$. Suppose that every $v_\nu(x)$ in v is analytic in some neighborhood of K independent of ν. Then there exist a neighborhood U of K and a neighborhood W of the origin $\zeta = 0$ in $\mathbb{C}$ such that the formal Borel transform $B(v)(x,\zeta)$ converges in $U \times W$.

Remark. If K is a neighborhood of the origin $x = 0$, then we only need to assume that $R(x)$ is analytic in some neighborhood of the origin $x \in \mathbb{C}^n$. Note that $0 \notin \Sigma_0$ by (11) and every $v_\nu(x)$ in v is analytic in some neighborhood of the origin independent of ν.

5. Summability at the origin.

Define C_0 as the smallest convex closed cone with vertex at the origin containing λ_j $(j = 1, 2, \ldots, n)$. Then we have

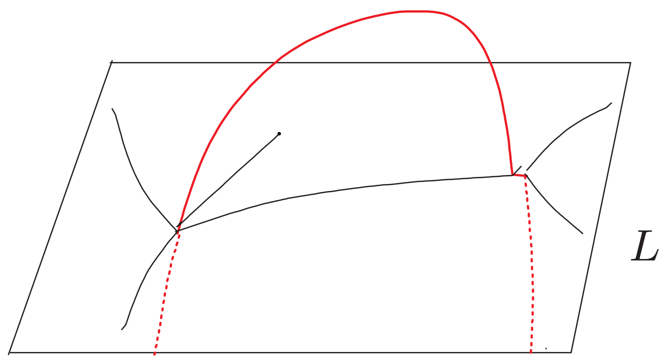
Theorem 3 Suppose (11). Assume that $\nabla R(x+v_0)$ is a diagonal matrix. Assume that, for some direction ξ

$$|\arg \lambda_j - \xi| < \pi/4 \quad \text{for } j = 1, 2, \ldots, n. \tag{15}$$

Then there exists a neighborhood U of the origin of x such that $v(x, \eta)$ is Borel-summable in the direction η such that $\eta^{-1} \in (C_0)^c$ and $x \in U$, where $(C_0)^c$ is the complement of C_0 in the complex plane.

Remark. There exist dense singular directions in C_0.

Σ_0



6. Some geometry.

Let $v_0(x)$ and Σ_0 be given by (9) and (10), respectively. Because Σ_0 is a main analytic set, it has the pure codimension one. Hence, by the well known embedding theorem in several complex variables, for every point b of Σ_0 there exists a complex line L such that $b \in \Sigma_0 \cap L$ is isolated in L. In the following we assume that L is given by $x_j = 0$ $(1 \le j \le n-1)$ and $\Sigma_0 \cap L$ consists of isolated points in L. (See the figure in the above). We denote the variable in L by ζ. We may also assume $\lambda_n = 1$ without loss of generality by dividing the equation with λ_n.

Let $(\nabla R)_j$ be the j-th diagonal component of the matrix ∇R. We consider the system of equations

$$\mathcal{L}w_j - (\lambda_j + (\nabla R)_j(x+v_0))\frac{\partial w_j}{\partial y} = f, \qquad j = 1, 2, \ldots, n, \tag{16}$$

where $f \equiv f(x, y)$ is a holomorphic function of $x \in \mathbb{C}^n$ and $y \in \mathbb{C}$. Consider the characteristic equation corresponding to (16)

$$\frac{d\zeta}{\zeta} = \frac{dx_k}{\lambda_k} = -\frac{dy}{\lambda_j + (\nabla R)_j(x+v_0)}, \qquad k = 1, 2, \ldots, n-1. \tag{17}$$

By integration we have

$$x_k = c_k \zeta^{\lambda_k} \ (k = 1, 2, \ldots, n-1), \ y = y_0 - \Phi(\zeta, b), \tag{18}$$

where c_k's and y_0 are some constants. Fix a branch of v_0 and define

$$\Phi(\zeta, b) \equiv \Phi_j(\zeta, b) = \int_b^\zeta \frac{\lambda_j + (\nabla R)_j(x + v_0(x))}{s} ds, \tag{19}$$

where $x = (x_1, \dots, x_n)$, $x_k = c_k s^{\lambda_k}$ $(k = 1, 2, \dots, n-1)$ and $b \in \mathbb{C}$. Note that the relations (18) give the (multi-valued) change of variable between (x_k, ζ, y) and (c_k, ζ, y_0).

7. Local summability

In Theorem 3 we have proved Borel summability of the formal SP-solution $v(x, \eta)$ in a neighborhood of the origin $x = 0$. We will study Borel summability at other points $\xi \in \mathbb{C}^n \setminus \Sigma_0$, $\xi \neq 0$. Note that $0 \notin \Sigma_0$.

Let $\xi = (\xi', \xi_n)$ and determine $c' = (c_k)_k$ by the relations $\zeta = \xi_n$ and (18) with $x = \xi$. Determine L with $x' = \xi'$. Define the set $T_0 \subset L$ by

$$T_0 := \Sigma_0 \cap \{(\xi', \zeta); \zeta \in L\}.$$

Let $a \in T_0$. With $c' = (c_k)$ define $\Phi(s, a)$ by (19). Define the curve S_a by the set of points s such that $\mathrm{Im}\,\Phi(s, a) = 0$ Clearly, $\xi_n \notin T_0$ because $\xi \in (\mathbb{C}^n \setminus \Sigma_0) \cap \Omega_0$. Hence the following two cases occur:
(a) $\xi_n \notin S_a$ for any $a \in \Sigma_0$.
(b) $\xi_n \in S_a$ for some $a \in \Sigma_0$.
We will show the summability in these two cases.

Theorem 4 Assume that $R(x)$ is an entire function on $\mathbb{C}^n$ and that $\nabla R(x + v_0)$ is a diagonal matrix. Suppose $0 \notin \Sigma_0$. Moreover, assume that $\mathrm{Re}\,\lambda_j > 0$ for $j = 1, 2, \dots, n$. Let $\xi \in \mathbb{C}^n \setminus \Sigma_0$. Then we have
The case (a). There exist an $\varepsilon > 0$ and a neighborhood D of ξ such

that if $\|v_0\| < \varepsilon$, then $v(x, \eta)$ is Borel-summable in the direction η with $\eta^{-1} \in (C_0)^c$ for any $x \in D$.
The case (b). Assume that $\xi_n \in S_a$ for some S_a which is not a branch cut of $\Phi'(s, \cdot) = (d/ds)\Phi(s, \cdot)$. Then there exist an $\varepsilon > 0$ and a neighborhood D of ξ such that if $\|v_0\| < \varepsilon$, then $v(x, \eta)$ is Borel-summable in the direction η with $\eta^{-1} \in (C_0)^c$ for any $x \in D$.

8. Global summability

By using the results in the preceeding sections we will show the following fact. Given a domain K whose closure is compact. Then there exists an $\varepsilon > 0$ such that if $\|v_0\| < \varepsilon$ and v_0 is holomorphic in K, then the SP-solution is Borel summable in the direction η with $\eta^{-1} \in (C_0)^c$ for any $x \in K$. Indeed, one can make analytic continuation by covering K with a finite number of open sets. We note that the Borel sum gives the desired solution of our equation.

9. Connection problem across singular directions and Poincaré's theorem

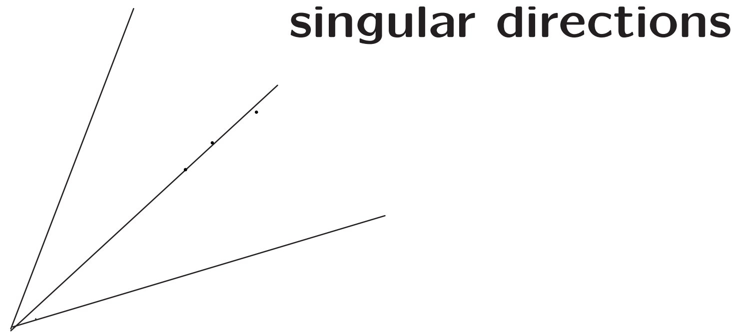
This is the main topic in this talk. Consider the connection problem with respect to η of the summed SP-solution $V(x, \eta)$ of (6) across every singular direction in C_0. Let E_0 be given by

$$E_0 := \left\{ \frac{\langle \lambda, \alpha \rangle}{\lambda_k}; k = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^n_{\geq 0}, |\alpha| \geq 2 \right\}. \tag{20}$$

We can show:
E_0 is contained in the right half-plane $\mathrm{Re}\,\eta > 0$ and $\eta/|\eta|$ $(\eta \in E_0)$ are dense in some sector of the complex plane.

Note that E_0 gives the singular directions for the Borel sum $V(x, \eta)$ which is dense in C_0. A connection problem occurs at a singular direction in C_0. We shall study the analytic continuation of $V(x, \eta)$ from the negative real axis to the point $\eta = 1$.

Theorem 5 Assume there exists a real ξ such that $|\arg \lambda_j - \xi| < \pi/4$ for $j = 1, 2, \ldots, n$ and that λ_j $(j = 1, 2, \ldots, n)$ be linearly independent over $\mathbb{Z}$. Then there exists a neighborhood W of the origin of $x \in \mathbb{C}^n$ such that the connection coefficient across every singular direction in C_0 vanishes. Especially, $V(x, \eta)$ is a single-valued meromorphic function with respect to η and analytic in x when $(x, \eta) \in W \times \mathbb{C} \setminus E_0$.



We say that a solution of (6) with $\eta = 1$ is said to be a classical Poincaré solution if it is constructed as a power series of x at the origin $x = 0$, which is convergent provided the Poincaré condition is satisfied. By Theorem 5 we have the corollary.

Theorem 6 Assume there exists a real ξ such that $|\arg \lambda_j - \xi| < \pi/4$ for $j = 1, 2, \ldots, n$ and that λ_j $(j = 1, 2, \ldots, n)$ be linearly independent over $\mathbb{Z}$. Then the analytic continuation of the Borel summed SP-solution $V(x, \eta)$ to $\eta = 1$ in $\mathbb{C} \setminus E_0$ coincides with the classical Poincaré solution of (6) with $\eta = 1$.

Open problem

We hope that the Birkoff normalizing transformation of the Hamiltonian system is also obtained in the same manner as above.

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Thank you very much

for attending to the conference and to my lecture