GKZ SYSTEMS AND MULTIPLE ZETA VALUES

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Plan

Geometric series and power function

The starting point of the theory of hypergeometric functions is, perhaps, the Eulers' analysis of the function deined by the series

$$(1-t)^{-r} = \sum_{n\geqslant 0} \frac{(r)_n}{n!} t^n, \tag{1}$$

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Although the formula (1) has allready been known before Euler, it was him, who made significant contributions in the study of (1) and noticed, that many known special functions can be put into the similar framework.

Euler-Gauss hypergeometric function

The classical Euler-Gauss hypergeometric function is defined by the series

$$ {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, t\right) = \sum_{n\geqslant 0} \frac{(u)_n (v)_n}{(w)_n} \frac{t^n}{n!} = 1 + \frac{u \cdot v}{w} t + \frac{u(u+1)\cdot v(v+1)}{w(w+1)} \frac{t^2}{2!} + O(t^3), \tag{2} $$

where $|t| < 1$.

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where $|t| < 1$.

It has been introduced by Euler and studied by the leading matematicians of the XIX and the beginning of XX century, including Gauss, Riemann (monodromy, P-function, Riemann surfaces), Kummer (bases of solutions, special values), Shwarz (Shwarz list) and others.

General classical hypergeometric function

One can easily generalize the classical Euler-Gauss hypergeometric function, by the series ($0 < p, q \in \mathbb{Z}$ are parameters, such that $p \leqslant q+1$)

$$ {}_pF_q\left(\begin{matrix} u_1, u_2, \ldots, u_p \\ w_1, w_2, \ldots, w_q \end{matrix} \,\middle|\, t \right) = \sum_{n \geqslant 0} \frac{(u_1)_n (u_2)_n \ldots (u_p)_n}{(w_1)_n (w_2)_n \ldots (w_q)_n} \frac{t^n}{n!}, \qquad (3) $$

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If $p < q+1$, then function (3) is called *confluent* and if $p = q+1$, then it is called *balanced*.

Hypergeometric differential equation

We introduce the following operators: the multiplication operator $f(t) \mapsto tf(t)$, wich we simply denote by t, differential operator $\partial_t := d/dt$ and the Euler operator $\theta_t = t\partial t$.

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$$\begin{aligned} \theta_t \, {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \middle| \, t\right) &= \sum_{n \geqslant 0} \frac{(u)_n (v)_n}{(w)_n} \, n \, \frac{t^n}{n!} \\ &= u \sum_{n \geqslant 0} \left\{ \frac{(u+1)_n (v)_n}{(w)_n} - \frac{(u)_n (v)_n}{(w)_n} \right\} \frac{t^n}{n!}, \\ &= u \left\{ {}_2F_1\left(\begin{matrix} u+1, v \\ w \end{matrix} \middle| \, t\right) - {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \middle| \, t\right) \right\}. \end{aligned} \tag{4}$$

Hypergeometric differential equation

Hence

$$(\theta_t + u)\,{}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \;\middle|\; t\right) = u\,{}_2F_1\left(\begin{matrix} u+1, v \\ w \end{matrix} \;\middle|\; t\right). \tag{5}$$

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And in a similar way

$$(\theta_t + v)\,{}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix}\;\middle|\; t\right) = v\,{}_2F_1\left(\begin{matrix} u, v+1 \\ w \end{matrix}\;\middle|\; t\right) \tag{6}$$

and

$$(\theta_t + w - 1)\,{}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix}\;\middle|\; t\right) = (w-1)\,{}_2F_1\left(\begin{matrix} u, v \\ w-1 \end{matrix}\;\middle|\; t\right). \tag{7}$$

Hypergeometric differential equation

From the above differential-difference relations together with

$$\partial_t \, {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, t\right) = \frac{uv}{w} \, {}_2F_1\left(\begin{matrix} u+1, v+1 \\ w+1 \end{matrix} \,\middle|\, t\right). \tag{8}$$

one obtains

$$(\theta_t + u)(\theta_t + v) \, {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, t\right) = (\theta_t + w)\partial_t \, {}_2F_1\left(\begin{matrix} u+1, v \\ w \end{matrix} \,\middle|\, t\right). \tag{9}$$

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Or in equivalent form:

$$\left\{t(t-1)\partial_t^2 + ((u+v+1)t - w)\partial_t + uv\right\}\, {}_2F_1\left(\left.\begin{matrix} u, v \\ w \end{matrix}\,\right|\, t\right) = 0. \tag{10}$$

General classical hypergeometric equation

The analog of the Euler-Gauss hypergeometric equation for general classical hypergeometric function can be written as

$$t\,P(\theta_t)\,{}_pF_q \;=\; Q(\theta_t)\,{}_pF_q, \tag{11}$$

where

$$\begin{aligned} P(x) &= (x+u_1)(x+u_2)...(x+u_p) \\ Q(x) &= (x+w_1-1)(x+w_2-1)...(x+w_q-1). \end{aligned}$$

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From the above differential equation, one can restore the classical hypergeometric series, as a particular solution. But the hypergeometric equation has order p, so there are $p-1$ independent solutions to (11). Usually they are also called hypergeometric functions. However they may not be representable by hypergeometric series.

Balanced and confluent differential equations

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Balanced equation has only regular singular points, i.e. solutions in zeros of the indical equations are at most of polynomial growth. This implies significant differences between the two above cases.

Integral representations

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It is also closely related to harmonic analysis on S^1.

Integral representations

The other useful formula is the Mellin-Barnes integral:

$$\begin{aligned}&\frac{\Gamma(u)\Gamma(v)}{\Gamma(w)}\,{}_2F_1\left(\left.\begin{matrix}u,v\\w\end{matrix}\right|\,t\right)\\=\quad&\frac{1}{2\pi i}\int_C \frac{\Gamma(u+s)\Gamma(v+s)}{\Gamma(w+s)}\,\Gamma(-s)(-t)^s\,ds,\end{aligned}\tag{12}$$

where the contour C is a line from $-i\infty + s_0$ to $-i\infty + s_0$, for some $s_0 \in \mathbb{R}$, separating poles of $\Gamma(-s)$ from the poles of the other Γ-factors.

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There are also other very interesting integral formulas which follow from the integral geometry.

Integral representations and Meijer G-function

General hypergeometric series also admits the Mellin-Barnes integral representation:

$$\begin{aligned} & \frac{\Gamma(u_1)\ldots\Gamma(u_p)}{\Gamma(w_1)\ldots\Gamma(w_q)}\, {}_pF_q\left(\begin{matrix} u_1, \ldots, u_p \\ w_1, \ldots, w_q \end{matrix} \,\middle|\, t\right) \\ = \quad & \frac{1}{2\pi i}\int_C \frac{\Gamma(u_1+s)\ldots\Gamma(u_p+s)}{\Gamma(w_1+s)\ldots\Gamma(w_q+s)}\,\Gamma(-s)(-t)^s\,ds. \end{aligned} \tag{13}$$

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&= \frac{1}{2\pi i}\int_C \frac{\Gamma(u_1+s)\ldots\Gamma(u_p+s)}{\Gamma(w_1+s)\ldots\Gamma(w_q+s)}\,\Gamma(-s)(-t)^s\,ds. \qquad (13)
\end{aligned}$$

with appropriately chosen contour C. Formula (13) led Cornelis Simon Meijer to the definition of the Meijer G-function:

$$\begin{aligned}
&G^{m,n}_{p,q}\left(\begin{matrix}u_1,\ldots,u_p\\ w_1,\ldots,w_q\end{matrix}\,\middle|\,t\right)\\
&= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma(w_j-s)\prod_{j=1}^{n}\Gamma(1-u_j+s)}{\prod_{j=1}^{p}\Gamma(u_j-s)\prod_{j=1}^{q}\Gamma(1-w_j+s)}\,\Gamma(s)t^s\,ds. \qquad (14)
\end{aligned}$$

Appell functions

In 1880 P. Appell defined the following list of hypergeometric functions of two variables:

$$F_1\left(\begin{matrix} u, v_1, v_2 \\ w \end{matrix} \middle| \, x, y\right) = \sum_{n,m\geqslant 0} \frac{(u)_{m+n}(v_1)_m(v_1)_n}{(w)_{m+n}} \frac{x^m y^n}{m!n!}, \qquad (15)$$

$$F_2\left(\begin{matrix} u, v_1, v_2 \\ w_1, w_2 \end{matrix} \middle| \, x, y\right) = \sum_{n,m\geqslant 0} \frac{(u)_{m+n}(v_1)_m(v_1)_n}{(w_1)_m(w_2)_n} \frac{x^m y^n}{m!n!}, \qquad (16)$$

$$F_3\left(\begin{matrix} u_1, u_2, v_1, v_2 \\ w \end{matrix} \middle| \, x, y\right) = \sum_{n,m\geqslant 0} \frac{(u_1)_m(u_2)_n(v_1)_m(v_1)_n}{(w)_{m+n}} \frac{x^m y^n}{m!n!}, \qquad (17)$$

$$F_4\left(\begin{matrix} u, v \\ w_1, w_2 \end{matrix} \middle| \, x, y\right) = \sum_{n,m\geqslant 0} \frac{(u)_{m+n}(v)_{m+n}}{(w_1)_m(w_2)_n} \frac{x^m y^n}{m!n!}. \qquad (18)$$

Appell and Horn functions

Series defining functions F_1, F_2, F_3, F_4 converge in regions

$$\begin{aligned}
D_1 &= \{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}, \\
D_2 &= \{(x, y) \in \mathbb{C}^2 : |x| + |y| < 1\}, \\
D_3 &= \{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}, \\
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In addition to the list of four Appell functions, there are 10 other balanced hypergeometric series and futher 20 confluent series, that have been enumerated by Horn (1931) and corrected by Borngässer (1933).

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Lauricella (1893) generalized the notion of Appell's functions to n variables.

Differential equations and integral representations

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$$\frac{\Gamma(v_1)\Gamma(v_2)\Gamma(w_1-v_1)\Gamma(w_2-v_2)}{\Gamma(w_1)\Gamma(w_2)}\, F_2\left(\begin{matrix} u, v_1, v_2 \\ w_1, w_2 \end{matrix}\,\middle|\, x, y\right)$$
$$= \int_0^1\int_0^1 t_1^{v_1-1} t_2^{v_2-1}(1-t_1)^{w_1-v_1-1}(1-t_2)^{w_2-v_2-1} \times$$
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Surprisingly, the function F_1 can also be expressed by the simple integral

$$\frac{\Gamma(w)\Gamma(w-u)}{\Gamma(v_1)\Gamma(v_2)}\, F_1\left(\begin{matrix} u, v_1, v_2 \\ w \end{matrix}\,\middle|\, x, y\right)$$
$$= \int_0^1 t^{u-1}(1-t)^{w-u-1}(1-tx)^{-v_1}(1-ty)^{-v_1}\,dt.$$

Mellin-Barnes integral representations

For general complex parameters, the F_1 function can be written as the following contour integral

$$\begin{aligned} & \frac{\Gamma(u)\Gamma(v_1)\Gamma(v_2)}{\Gamma(w)} F_1 \left(\begin{matrix} u, v_1, v_2 \\ w \end{matrix} \middle| \, x, y \right) \\ = \; & \frac{1}{(2\pi i)^2} \int_C \frac{\Gamma(u - s_1 - s_2)\Gamma(v_1 - s_1)\Gamma(v_2 - s_2)}{\Gamma(w - s_1 - s_2)} \\ & \quad \times \quad \Gamma(s_1)\Gamma(s_2)(-x)^{-s_1}(-y)^{-s_2} \, ds_1 \, ds_2, \end{aligned} \tag{19}$$

where C is an appropriately chosen 2-cycle in $\mathbb{C}^2$.

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where C is an appropriately chosen 2-cycle in $\mathbb{C}^2$.

Analogous formulas exist for all other Appel and Horn functions.

Horn's condition

Common properties of the classical, Appell's and Lauricella's hypergeometric functions led Horn to the following definition.

Definition (Horn function)

Consider a Taylor series of the form (we use standard multiindex notation)

$$f(t) = \sum_{n \in \mathbb{N}^p} a_n t^n, \tag{20}$$

where a_n are such that $a_{n+e_j}/a_n \in \mathbb{C}(n_1, \ldots, n_p)$.

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All of the above functions (Euler-Gauss, Appell, Lauricella) are special cases of Horn hypergeometric functions. Horn functions can also be divided into confluent and balanced ones. All of them possess integral representations and satisfy meromorphic differential equations generalizing the one-dimensional case.

Euler-Gauss function in the multivariable context

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which can be rewritten as

$$\partial_1 \partial_2 - \partial_3 \partial_4 \tag{22}$$

in coordinates $x_1 = t_1 + t_2$, $x_2 = i(t_1 - t_2)$, $x_3 = t_3 - t_4$ and $x_4 = i(t_3 + t_4)$.

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Here (and further on) $\partial_i := \partial_{t_i}$; the same convention applies to $\theta_i := \theta_{t_i}$.

PDE satisfied by multivariable Euler-Gauss function

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Proposition

The function

$$\Phi\left(\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, t_1, t_2, t_3, t_4\right) := t_1^{-u} t_2^{-v} t_3^{w-1} \, {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, \frac{t_3 t_4}{t_1 t_2}\right) \tag{23}$$

satisfies the equation

$$(\partial_1 \partial_2 - \partial_3 \partial_4)\Phi = 0. \tag{24}$$

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Proposition

The function

$$\Phi\left(\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, t_1, t_2, t_3, t_4\right) := t_1^{-u} t_2^{-v} t_3^{w-1} \, {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, \frac{t_3 t_4}{t_1 t_2}\right) \tag{23}$$

satisfies the equation

$$(\partial_1 \partial_2 - \partial_3 \partial_4)\Phi = 0. \tag{24}$$

This approach, which was motivated by the desire to find systems of PDEs whose solutions could be expressed in terms of generalized hypergeometric series, leads to the notion of GKZ systems.

Euler-Gauss PDE; towards GKZ system

For Euler-Gauss hypergeometric series, the operator $\theta_t + u$ may be viewed as an index-raising operator. Similar is true in the case of $\theta_t + v$, while and $\theta_t + w$ lowers the third index.

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One may introduce other raisingand lowering operators using the recursion properties of the Pochhammer symbols. However, the dependence of these operators on the parameters makes it difficult to study, for example, their composition properties and thus the algebra they generate.

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Miller's idea is to replace the multiplication operators u, v, w by by Euler operators $\theta_x, \theta_y, \theta_z$, corresponding to new variables x, y, z, respectively. Now, since the Euler operator, θ_x acts as multiplication by u on x^u, it is natural, to define

$$\Phi_{u,v,w}(t) := x^u y^v z^{w-1} \, {}_2F_1 \left(\begin{matrix} u, v \\ w \end{matrix} \middle| \, t \right). \tag{25}$$

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The following relations are immediate consequence of the above:

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\theta_x \, \Phi_{u,v,w} &= u \, \Phi_{u,v,w} \\
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So

$$(\theta_t + \theta_x) \, \Phi_{u,v,w} = x^u y^v z^{w-1} \, (\theta_t + u) \, {}_2F_1 \left(\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, t \right) \tag{26}$$

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There are similar formulas for v and w.

Euler-Gauss PDE; towards GKZ system

Together with relation

$$xyz\partial_t \Phi_{u,v,w} = \frac{uv}{w}\Phi_{u,v,w} \tag{28}$$

we get all relations leading to the Euler-Gauss equation (i.e. one can reconstruct ${}_2F_1$).

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However, there are several advantages. Identities (27), (28) and two remaining ones are in parameter free form and they make sense for any (appropriately regular) function on the variables t, x, y, z, while the previous (the classica) form depended on the non-intrinsic parameters u, v and w.

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Another advantage is provided by the following

Proposition

The operators $L_1 := x(\theta_t + \theta_x)$, $L_2 := y(\theta_t + \theta_y)$, $L_3 := z^{-1}(\theta_t + \theta_z)$ *and* $L_4 := xyz\partial_t$ *commute. Consequently, there exist coordinates* $\xi_1, ..., \xi_4$ *on* $\mathbb{C}^4$ *such that* $L_j = \partial_{\xi_j}$.

Euler-Gauss PDE; towards GKZ system

Existence of such $\partial_{\xi_1}, \partial_{\xi_2}, \partial_{\xi_3}$ and ∂_{ξ_4} follows from Frobenius' Theorem. However, in this case we can write them explicitly as:

$$\begin{aligned}
\xi_1 &= -x^{-1}, \\
\xi_2 &= -v^{-1}, \\
\xi_3 &= z, \\
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Therefore

$$\begin{aligned} \theta_x &= -\theta_{\xi_1} - \theta_{\xi_4} \\ \theta_y &= -\theta_{\xi_2} - \theta_{\xi_4} \\ \theta_z &= \theta_{\xi_3} - \theta_{\xi_4} \\ \theta_t &= \theta_{\xi_4}. \end{aligned}$$

Euler-Gauss PDE; towards GKZ system

The above analysis leads to the following

Theorem

Given complex numbers u, v and $w \notin -\mathbb{N}$, the function $\Phi_{u,v,w}$ defined in (62) satisfies the system of partial differential equations:

$$\begin{aligned}
(\theta_1 + \theta_4 + u)\Phi_{u,v,w} &= 0 \\
(\theta_2 + \theta_4 + v)\Phi_{u,v,w} &= 0 \\
(-\theta_3 + \theta_4 + w - 1)\Phi_{u,v,w} &= 0 \\
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First three equations can be written more simply, as

$$A\theta\Phi_{u,v,w} = 0, \tag{29}$$

with matrix A and $\theta := (\theta_1, ..., \theta_4)$.

GKZ hypergeometric system

Let A denote $d \times n$ matrix of rank d with coefficients in $\mathbb{Z}$.

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Definition

Let $u \in \mathbb{C}^d$. Define

$$I_A = \{\partial^\alpha - \partial^\beta : A\alpha = A\beta; \alpha, \beta \in \mathbb{N}^d\}. \tag{30}$$

The GKZ hypergeometric system is the left ideal $H(A, u)$ in the Weyl algebra generated by the union of I_A and $A\theta - u$. Solutions of GKZ systams are called A-hypergeometric functions.

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GKZ stands for Gelfand, Kapranov and Zelevinsky, who first studied the general multivariable hypergeometric systems associated to A, u.

Euler-Gauss function as GKZ hypergeometric system

As it has been allready seen before, the multivariable Euler-Gauss function satisfies GKZ system associated to the data

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad (31)$$

and $\bar{u} = (-u, -v, 1 - w)$.

Appell function as GKZ hypergeometric system

Consider a GKZ system associated to the following data:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix} \tag{32}$$

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Theese data correspond to the function Φ associated with Appell F_1.

Note, that here I_A is not principal, i.e. we have

$$I_A = \langle \partial_1\partial_2 - \partial_3\partial_2, \partial_1\partial_5 - \partial_3\partial_6, \partial_2\partial_6 - \partial_4\partial_5 \rangle \,. \tag{33}$$

'Hypergeometric properties' of GKZ system

Solutions of GKZ system have properties analogous to the classical (including Euler-Gauss) hypergeometric functions. In "*Generalized Euler integrals and A-hypergeometric functions* (Adv. Math. 84, 255–271), Gelfand, Kapranov and Zelevinsky proved the following

Theorem (GKZ)

Let $f_1, f_2, ..., f_n \in \mathbb{C}[x_1, x_2, ..., x_m]$, $x, \beta \in \mathbb{C}^m$ *and* $\alpha \in \mathbb{C}^n$. *Then*

$$\int_C f_1^{\alpha_1} f_2^{\alpha_2} ... f_n^{\alpha_n} x^\beta dx. \tag{34}$$

where C *is an* m*-dimensional real cycle, are A-hypergeometric functions of the coefficients of the polynomials* $f_1, f_2, ..., f_n$.

The Γ-series and Mellin-Barnes integral

Solutions of GKZ system can be represented as Γ-series

$$\sum_{m} \prod_{j \in J} \frac{t^{m_j}}{m_j!} \prod_{i \in I} \frac{t^{(Am)_i + u_i}}{\Gamma((Am)_i + u_i + 1)}. \tag{35}$$

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There is also a Mellin-Barnes representation, which can be regarded as continuous analog of the Γ-series.

Grassmannian manifolds

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From now on the base field is going to be $\mathbb{C}$. We will also write $Gr(k, n)$ instead of $Gr(k, \mathbb{C}^n)$.

Description of $Gr(1, n)$

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Thus we have $Gr(1, n) = \mathbb{C}^* \backslash (\mathbb{C}^n - \{0\})$, or in more fancy way: $Gr(1, n) = GL(1, \mathbb{C}) \backslash hom_1(\mathbb{C}^n, \mathbb{C})$, where the action is described by multiplication $GL(1, \mathbb{C}) \ni t.\phi \in hom_1(\mathbb{C}^n, \mathbb{C})$.

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Using hermitian form on $\mathbb{C}^n$ one can proove that taking unitary projections in place of space $hom_1(\mathbb{C}^n, \mathbb{C})$ and the unitary group $U(1)$ in place of $GL(1, \mathbb{C})$, one gets the same space, i.e. we have $Gr(1, n) = U(1) \times U(n-1) \backslash U(n)$.

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Atlas on $Gr(1, n)$

For homogeneous coordinates $[x_0, x_1, ..., x_{n-1}]$ on $Gr(1, n)$ there are n natural maps $\varphi_i : Gr(1, n) \supset U_i \to \mathbb{C}^{n-1}$, where $i \in \{0, 1, ..., n-1\}$,
$U_i = \{[x_0, x_1, ..., x_{n-1}] : x_i \neq 0\}$ and
$\varphi_i([x_0, x_1 ..., x_{n-1}]) = (x_1, ..., x_{1-i}, x_{1+i}, ..., x_{n-1})/x_i$.

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On the intersections $U_i \cap U_j$ map $\varphi_i \circ \varphi_j^{-1} : U_i \cap U_j \rightarrow U_i \cap U_j$ is a diffeomorphism, thus we have an atlas on $Gr(1,n)$.

$Gr(2, 4)$

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Analogously to homogeneous coordinates on the projective spaces, we can consider equiv. classes of matrices $x \in \mathrm{hom}_2(\mathbb{C}^4,\mathbb{C}^2)$, given by

$$\begin{bmatrix} x_{00} & x_{01} & x_{02} & x_{03} \\ x_{00} & x_{01} & x_{02} & x_{03} \end{bmatrix}, \tag{36}$$

where we identify $x, y \in \mathrm{hom}_2(\mathbb{C}^4,\mathbb{C}^2)$, if $x = ty$ for some $t \in GL(2,\mathbb{C})$.

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Thus we have $Gr(2,4) = GL(2,\mathbb{C})\backslash \mathrm{hom}_2(\mathbb{C}^4,\mathbb{C}^2)$.

$Gr(2, 4, \mathbb{R})$ as a homogeneous space

As in the case of projective space, $Gr(2, 4, \mathbb{R})$ can be described as a homogeneous space. We can repace $GL(2, \mathbb{C})\backslash \hom_2(\mathbb{C}^4, \mathbb{C}^2)$ by $Gr(2, 4) = U(2) \times U(2)\backslash U(4)$.

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One can also construct open covering $U_{ij} \subset Gr(2, 4)$ and family of diffeomorphisms $\varphi_{ij} : U_{ij} \to \mathbb{C}^4$. Sets U_{ij} are defined so that the square matrix consisting of i-th and j-th collumn must be invertible.

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For example, we have

$$\varphi_{01}\left(\begin{bmatrix} 1 & 0 & x_{02} & x_{03} \\ 0 & 1 & x_{02} & x_{03} \end{bmatrix}\right) = \begin{pmatrix} x_{02} & x_{03} \\ x_{02} & x_{03} \end{pmatrix}. \tag{37}$$

General $Gr(k, n, \mathbb{F})$

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We have

$$\begin{aligned} Gr(k,n,\mathbb{R}) &= O(k)\times O(n-k)\backslash O(n), & (38)\\ Gr(k,n,\mathbb{C}) &= U(k)\times U(n-k)\backslash U(n), & (39)\\ Gr(k,n,\mathbb{H}) &= Sp(k)\times Sp(n-k)\backslash Sp(n). & (40)\end{aligned}$$

Vector bundles over Grassmannians

Consider submanifold τ in $Gr(k, n, \mathbb{C}) \times \mathbb{C}^n$, given by set of pairs $x, V(x)$, where $x \in Gr(k, n, \mathbb{C})$ and $V(x) \subset \mathbb{C}^n$ is the vector space cooresponding to x.

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Plücker embedding

Assume $x \in Gr(k, n, \mathbb{C})$ corresponds to $V = V(x) \subset \mathbb{C}^n$, spanned by $v_1, v_2, ..., v_k$. Then the mapping $(v_1, v_2, ..., v_k) \mapsto v_1 \wedge v_2 \wedge ... \wedge v_k$ induces an embedding $p : Gr(k, n, \mathbb{C}) \to P(\wedge^k \mathbb{C}^n)$.

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We have $(\tau \to Gr(k, n, \mathbb{C})) = p^*(\tau \to P(\wedge^k \mathbb{C}^n))$, where $(\tau \to P(\wedge^k \mathbb{C}^n)) \simeq \mathcal{O}_{P(\wedge^k \mathbb{C}^n)}(1)$.

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The GKZ connection

From the integral representation of Φ it follows that under the action of $g \in GL(k, \mathbb{C})$ on $Gr(k, n, \mathbb{C})$ the formula transforms as follows

$$\Phi(\alpha, gx) = (\det g)^{-1} \Phi(\alpha, x), \tag{41}$$

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Hypergeometric function on $Gr(k, n, \mathbb{C})$

There are natural actions of $GL(k, \mathbb{C})$ and $GL(n, \mathbb{C})$ on $hom(\mathbb{C}^n, \mathbb{C}^k)$. So is the action of maximal torus $T^n \subset GL(n, \mathbb{C})$.

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$$\sum_{i=1}^{k} x_{ir} \frac{\partial \Phi}{\partial x_{ir}} = (\alpha_r - 1)\Phi, \tag{42}$$

$$\sum_{i=1}^{n} x_{ir} \frac{\partial \Phi}{\partial x_{jr}} = -\delta_{ij}\Phi, \tag{43}$$

$$\frac{\partial^2 \Phi}{\partial x_{ir} \partial x_{js}} = \frac{\partial^2 \Phi}{\partial x_{is} \partial x_{jr}}. \tag{44}$$

Coordinates x_{ir} are entries of a matrix $x \in hom(\mathbb{C}^n, \mathbb{C}^k)$.

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Solutions $\Phi = \Phi(\alpha, x)$ depend on variables $x \in hom(\mathbb{C}^n, \mathbb{C}^k)$ and parameters $\alpha \in \mathbb{C}^n$.

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Gerlfand, Graev, Kapranov and Zalevinsky studied properties of systems (42)-(44) in papers:

"Holonomic systems of equations and series of hypergeometric type",

"Hypergeometric functions and toric varietes",

"Generalized Euler integrals and A-hypergeometric functions".

Multiple ζ function

Definition

The multiple zeta function is defined by the series

$$\sum_{n_p > \ldots > n_2 > n_1 > 0} n_1^{-s_1} n_2^{-s_2} \ldots n_p^{-s_p} := \zeta(s_1, s_2, \ldots, s_p), \tag{45}$$

whenever (45) converges. Number p is called depth, and $|s| := s_1 + s_2 + \ldots + s_p$ *- weight of* $\zeta(s_1, s_2, \ldots, s_p)$. *Multiple zeta values (in short MZV), are values of multiple zeta function at integral points.*

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To simplify nontation, one writes $(\{s_1, \ldots, s_q\}^n)$, meaning $(s_1, \ldots, s_q, s_1, \ldots, s_q, \ldots, s_1, \ldots, s_q)$, where $(s_1, \ldots, s_q)$ is repeated n times.

Multiple ζ values

Multiple zeta values apeared for the first time in Euler's *Meditationes circa singulare serierum genus* (1775), where he found the following formula relating Multiple Zeta Values to 'single' ones:

$$\sum_{n>0} \frac{H_n}{(n+1)^2} = \zeta(2,1) = \zeta(3) = \sum_{n>0} \frac{1}{n^3}, \tag{46}$$

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where H_m is the m-th harmonic number.

If $p = 1$, then multiple zeta function is simply the Riemann zeta function

$$\sum_{n>0} n^{-s} = \zeta(s), \tag{47}$$

function which is a fundamental object of study in number theory.

Relations between multiple zeta values

MZV satisfy a lot of relations. For example

$$\begin{aligned}\zeta(r)\zeta(s) &= \sum_{m,n>0} m^{-r} n^{-s} \\ &= \left(\sum_{m>n>0} + \sum_{n>m>0} + \sum_{m=n>0}\right) m^{-r} n^{-s} \\ &= \zeta(r,s) + \zeta(s,r) + \zeta(r+s). \end{aligned} \tag{48}$$

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Other nontrivial relations can be obtained from Drinfeld-Kontsevich integral:

$$I_{\epsilon_1,\dots,\epsilon_{|p|}}(t) := \int_T \frac{dt_1}{A_{\epsilon_1}(t_1)} \dots \frac{dt_k}{A_{\epsilon_{|s|}}(t_{|s|})}, \tag{49}$$

where $T = \{(t_1, t_2, \dots, t_{|s|}) \in \mathbb{R}^k : 0 < t_1 < \dots < t_{|s|} < t < 1\}$, $\epsilon_i \in \{0, 1\}$, $A_0(x) = x$ and $A_1(x) = 1 - x$. Morover we assume that $\epsilon_1 = 1$ i $\epsilon_{|s|} = 0$.

Generating function and associated differential equation

The Drinfeld-Kontsevich integral can be used to construct Fuchsian differential equation associaded to generating function of the sequence $\zeta(\{(s_1, s_2, ..., s_p)\}^n)$.

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Holomorphic solution $F(t, \lambda)$ of the eigenequation

$$(T + \lambda^{|s|})f = 0, \tag{51}$$

such that $F(1, 0) = 1$, has the following expansion around $t = 1$:

$$F(1, \lambda) = \sum_{n \geqslant 0} (-1)^n \zeta(\{s_1, ..., s_p\}^n)\lambda^{|s|n}. \tag{52}$$

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In other words, function $F(1, \lambda)$ is a generating function of the sequence $\zeta(\{s_1, ..., s_p\}^n)$.

Particular solutions associated to $\zeta(\{s\}^n)$

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$$F(t,\lambda) = {}_sF_{s-1}\left(\begin{matrix}\mu\lambda, \mu^2\lambda, ..., \mu^s\lambda \\ 1, .., 1\end{matrix}\;\middle|\; t\right). \tag{55}$$

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For exmaple, in three papers written together with proffesor Żołądek:

Z. Ż., *Linear meromorphic differential equations and multiple zeta-values I. Zeta (2)*, Fund. Math. 210 (2010), 207-242.

Z. Ż., *Linear meromorphic differential equations and multiple zeta-values II. Generalization of the WKB method*, J. Math. Anal. Appl. 383 (2011), 55-70.

Z. Ż., *Linear meromorphic differential equations and multiple zeta-values I. Zeta (3)*, J. Math. Phys. 53 (2012), 1-40.

we give new proofs of certain MZV-identities, examining equation (51) asymptotic methods (WKB series, stationary phase approximation).

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converges, then we will call it multivariable polylogarithm (associated to $q_1, q_2, ..., q_r$).

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converges, then we will call it multivariable polylogarithm (associated to $q_1, q_2, ..., q_r$).

If $r = 1$, $t_1 = t$ and $s_1 = s$, then it reduces to familiar polylogarithm

$$Li_{q_1}\begin{pmatrix} t \\ s \end{pmatrix} = \sum_{n_i>0} \frac{t^n}{n^s} = Li_s(t). \tag{57}$$

Multivariable polylogarithms associated to MZV's

Of particular importance is MPL associated to forms $q_1 = n_1$, $q_2 = n_1 + n_2$, ..., $q_r = n_1 + ... + n_r$.

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$$Li_{q_1,q_2,\dots,q_r}\begin{pmatrix}1,1,...,1\\ s_1,s_2,...,s_r\end{pmatrix} = \sum_{n_i>0}\frac{1}{n_1^{s_1}(n_1+n_2)^{s_2}...(n_1+...+n_r)^{s_r}} = \zeta(s_1,s_2,...,s_r). \quad (58)$$

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We will denote multivariable polylogarithm associated to the above choice by

$$Li\begin{pmatrix}t_1,t_2,...,t_r\\ s_1,s_2,...,s_r\end{pmatrix} = \sum_{n_i>0}\frac{t_1^{n_1}t_2^{n_2}...t_r^{n_r}}{n_1^{s_1}(n_1+n_2)^{s_2}...(n_1+...+n_r)^{s_r}}. \tag{59}$$

Multivariable polylogarithms associated to MZV's

It is well known, that all classical (one variable) hypergeometric functions associated to multiple zeta values admit polylogarythmic series representations. For example if $r = 1$ and $s = 2$ (i.e. we deal with generating function of $\zeta(\{2\}^n)$), then

$$ {}_2F_1\left(\begin{matrix}\lambda, -\lambda\\ 1\end{matrix}\,\middle|\, t\right) = \sum_{n\geqslant 0}(-1)^n\lambda^{2n}Li_{\{2\}^n}(t) \tag{60} $$

and (from Gauss formula) we get

$$ {}_2F_1\left(\begin{matrix}\lambda, -\lambda\\ 1\end{matrix}\,\middle|\, 1\right) = \frac{1}{\Gamma(1+\lambda)\Gamma(1-\lambda)} = \sum_{n\geqslant 0}(-1)^n\lambda^{2n}\zeta(\{2\}^n). \tag{61} $$

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Multivariable polylogarithms associated to MZV's

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$$\int_0^u \frac{du}{u} \cdot Li\begin{pmatrix} u, v \\ r, s \end{pmatrix} = \cdot Li\begin{pmatrix} u, v \\ r+1, s \end{pmatrix} \quad (62)$$

$$\int_0^u \frac{du}{u} \cdot Li\begin{pmatrix} u, v \\ r, s \end{pmatrix} + \int_0^v \frac{dv}{v} \cdot Li\begin{pmatrix} u, v \\ r, s \end{pmatrix} = \cdot Li\begin{pmatrix} u, v \\ r, s+1 \end{pmatrix}. \quad (63)$$

Relations between MZV's of geometric origin

With use of symmetry properties of GKZ hypergeometric functions associated to Grassmannians one may find a lot of relations between multiple zeta values. For example it is possible to obtain Euler's identity

$$\zeta(3) = \zeta(2,1), \tag{64}$$

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Further study may reveal new identities. Furthermore, from Euler-type integral representations of GKZ functions it may be possible to deliver analogs of the Gauss identity:

$$ {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, 1\right) = \frac{\Gamma(w)\Gamma(w-u-v)}{\Gamma(w-u)\Gamma(w-v)} \tag{65}$$

for certain types of GKZ functions.

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which follows from the geometry of Grassmannians.

Further study may reveal new identities. Furthermore, from Euler-type integral representations of GKZ functions it may be possible to deliver analogs of the Gauss identity:

$$ {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \,\middle|\, 1\right) = \frac{\Gamma(w)\Gamma(w-u-v)}{\Gamma(w-u)\Gamma(w-v)} \tag{65}$$

for certain types of GKZ functions. And those could be used to obtain the generating functions not only for multiple zeta values, but their generalizations.

THANK YOU!

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