NICE INDUCING SCHEMES AND THE THERMODYNAMICS OF RATIONAL MAPS

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Abstract. We consider the thermodynamic formalism of a complex rational map f of degree at least two, viewed as a dynamical system acting on the Riemann sphere. More precisely, for a real parameter t we study the (non-)existence of equilibrium states of f for the potential $-t \ln |f'|$, and the analytic dependence on t of the corresponding pressure function. We give a fairly complete description of the thermodynamic formalism of a rational map that is "expanding away from critical points" and that has arbitrarily small "nice sets" with some additional properties. Our results apply in particular to non-renormalizable polynomials without indifferent periodic points, infinitely renormalizable quadratic polynomials with *a priori* bounds, real quadratic polynomials, topological Collet-Eckmann rational maps, and to backward contracting rational maps. As an application, for these maps we describe the dimension spectrum of Lyapunov exponents, and of pointwise dimensions of the measure of maximal entropy, and obtain some level-1 large deviations results.

Contents

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1. Introduction

The purpose of this paper is to study the thermodynamic formalism of a complex rational map f of degree at least two, viewed as a dynamical system acting on the Riemann sphere $\overline{\mathbb{C}}$. More precisely, for a real parameter t we study the (non-)existence of equilibrium states of f for the potential $-t \ln |f'|$, and the (real) analytic dependence on t of the corresponding pressure function. Our particular choice of potentials is motivated by the close connection between the corresponding pressure function, and the dimension spectrum of pointwise dimensions of the measure of maximal entropy. It is well known that for a polynomial this measure coincides with the harmonic measure of the Julia set. It was recently shown [BJ03, BMS03] that the "universal dimension spectrum", defined over all planar domains, coincides with the supremum of the dimension spectra of polynomials with connected Julia set. See the survey articles [BS05, Mak98, Jon05] for precisions, and [Erë91, BS96] for other applications of the thermodynamic formalism of rational maps to complex analysis.

For $t < 0$ and for an arbitrary rational map f, a complete description of the thermodynamic formalism was given by Makarov and Smirnov in [MS00]. They showed that the corresponding transfer operator is quasi-compact in a suitable Sobolev space, see also [Rue92]. For $t = 0$ and a general rational map f, there is a unique equilibrium state of f for the constant potential equal to 0 [Lju83, FLM83]. To the best of our knowledge it is not known if, for a general rational map f, the pressure function is real analytic on a neighborhood of $t = 0$. For $t > 0$ the only results we are aware of are for generalized polynomial-like maps without recurrent critical points in the Julia set. For such a map the analyticity properties of the pressure function were studied in [MS03, SU03], using a Markov tower extension and an inducing scheme, respectively.

In this paper we give a fairly complete description of the thermodynamic formalism of a rational map that is "expanding away from critical points" and that has arbitrarily small "nice sets" with some additional properties. In particular our results go beyond the non-uniformly hyperbolic setting. The main ingredients are the distinct characterizations of the pressure function given in [PRLS04], and the inducing scheme introduced in [PRL07], which we develop here in a more general setting. We also use a new technique to control distortion along non-univalent backward branches, which is one of the main technical tools introduced in this paper. Finally, we give applications of our results to rigidity, multifractal analysis, and level-1 large deviations.

There have been several recent results on the thermodynamic formalism of multimodal maps with negative Schwarzian derivative, by Bruin and Todd [BT07, BT08], and Pesin and Senti [PS08]. Besides [BT08, Theorem 6], that gives a complete description of the thermodynamic formalism for t close to 0 and for a general transitive multimodal map with negative

Schwarzian derivative, all the results that we are aware of are restricted to non-uniformly hyperbolic maps. We expect that with the approach used here one should be able give a fairly complete description of the thermodynamic formalism of a general transitive multimodal map with negative Schwarzian derivative.

After reviewing some general properties of the pressure function in §1.1, we state our main results in §§1.2, 1.3. The applications to rigidity, multifractal analysis, and level-1 large deviations are given in Appendix B.

Throughout the rest of this introduction we fix a rational map f of degree at least two, we denote by $\mathrm{Crit}(f)$ the set of critical points of f, and by $J(f)$ the Julia set of f.

1.1. **The pressure function and equilibrium states.** We give here the definition of the pressure function and of equilibrium sates, see §2 for references and precisions.

Let $\mathscr{M}(f)$ be the space of all probability measures supported on $J(f)$ that are invariant by f. We endow $\mathscr{M}(f)$ with the weak* topology. For each $\mu \in \mathscr{M}(f)$, denote by $h_\mu(f)$ the *measure theoretic entropy of* μ, and by $\chi_\mu(f) := \int \ln |f'| d\mu$ the *Lyapunov exponent of* μ. Given a real number t we define the *pressure of* $f|_{J(f)}$ *for the potential* $-t \ln |f'|$ by,

$$P(t) := \sup \{h_\mu(f) - t\chi_\mu(f) \mid \mu \in \mathscr{M}(f)\} . \tag{1.1}$$

For each $t \in \mathbb{R}$ we have $P(t) < +\infty$,[1] and the function $P : \mathbb{R} \to \mathbb{R}$ so defined will be called *the pressure function* of f. It is convex, non-increasing and Lipschitz continuous.

An invariant probability measure μ supported on the Julia set of f is called an *equilibrium state of* f *for the potential* $-t \ln |f'|$, if the supremum (1.1) is attained for this measure.

The numbers,

$$\chi_{\inf}(f) := \inf \{\chi_\mu(f) \mid \mu \in \mathscr{M}(f)\} ,$$
$$\chi_{\sup}(f) := \sup \{\chi_\mu(f) \mid \mu \in \mathscr{M}(f)\} ,$$

will be important in what follows. We call

$$t_- := \inf\{t \in \mathbb{R} \mid P(t) + t\chi_{\sup}(f) > 0\} \tag{1.2}$$

$$t_+ := \sup\{t \in \mathbb{R} \mid P(t) + t\chi_{\inf}(f) > 0\} \tag{1.3}$$

the *condensation point* and the *freezing point* of f, respectively. We remark that the condensation (resp. freezing) point can take the value $-\infty$ (resp. $+\infty$). We have the following properties (Proposition 2.1):

- $t_- < 0 < t_+$;

[1]When $t \leq 0$ the number $P(t)$ coincides with the topological pressure of $f|_{J(f)}$ for the potential $-t \ln |f'|$, defined with (n, ε)-separated sets. However, these numbers do not coincide when $t > 0$, and when there are critical points of f in $J(f)$. In fact, since $\ln |f'|$ takes the value $-\infty$ at each critical point of f, in this case the topological pressure of $f|_{J(f)}$ for the potential $-t \ln |f'|$ is equal to $+\infty$.

- for all $t \in \mathbb{R} \setminus (t_-, t_+)$ we have $P(t) = \max\{-t\chi_{\sup}(f), -t\chi_{\inf}(f)\}$;
- for all $t \in (t_-, t_+)$ we have $P(t) > \max\{-t\chi_{\inf}(f), -t\chi_{\sup}(f)\}$.

1.2. **Nice sets and the thermodynamics of rational maps.** A neighborhood V of $\mathrm{Crit}(f) \cap J(f)$ is a *nice set for* f, if for every $n \geq 1$ we have $f^n(\partial V) \cap V = \emptyset$, and if each connected component of V is simply connected and contains precisely one critical point of f in $J(f)$. A *nice couple for* f is a pair of nice sets $(\widehat{V}, V)$ for f such that $\overline{V} \subset \widehat{V}$ and such that for every $n \geq 1$ we have $f^n(\partial V) \cap \widehat{V} = \emptyset$. We will say that a nice couple $(\widehat{V}, V)$ is *small*, if there is a small $r > 0$ such that $\widehat{V} \subset B(\mathrm{Crit}(f) \cap J(f), r)$.

The following is our main result. We say that a rational map f is *expanding away from critical points*, if for every neighborhood V' of $\mathrm{Crit}(f) \cap J(f)$ the map f is uniformly expanding on the set

$$\{z \in J(f) \mid \text{ for every } n \geq 0,\ f^n(z) \notin V'\}.$$

Theorem A. *Let f be a rational map of degree at least two that is expanding away from critical points, and that has arbitrarily small nice couples. Then following properties hold.*

Analyticity of the pressure function: *The pressure function of f is real analytic on (t_-, t_+), and linear with slope $-\chi_{\sup}(f)$ (resp. $-\chi_{\inf}(f)$) on $(-\infty, t_-]$ (resp. $[t_+, +\infty)$).*

Equilibrium states: *For each $t_0 \in (t_-, t_+)$ there is a unique equilibrium state of f for the potential $-t_0 \ln |f'|$. Furthermore this measure is ergodic and mixing.*

Remark 1.1. In the proof of Theorem A we construct the equilibrium states through an inducing scheme with an exponential tail estimate, that satisfies some additional technical properties; see §4.3 for precise statements. The results of [You99] imply that the equilibrium states in Theorem A are exponentially mixing and that the Central Limit Theorem holds for these measures. It also follows that these equilibrium states have other statistical properties, such as the "almost sure invariant principle", see e.g. [MN05, MN06, TK05].

We now list some classes of rational maps for which the hypotheses of Theorem A hold.

- Using [KvS06] we show that each at most finitely renormalizable polynomial without indifferent periodic orbits satisfies the hypotheses of Theorem A, see Theorem D in §A.1.
- *Topological Collet-Eckmann rational maps* have arbitrarily small nice couples [PRL07, Theorem E] and are expanding away of critical points. *Collet-Eckmann rational maps*, as well as maps without recurrent critical points and without parabolic periodic points, satisfy the Topological Collet-Eckmann condition; see [PR98] and also [PRLS03, Main Theorem].

- Each *backward contracting rational map* has arbitrarily small nice couples [RL07, Proposition 6.6]. If in addition the Julia set is different from $\overline{\mathbb{C}}$, such a map is also expanding away of critical points [RL07, Corollary 8.3]. In [RL07, Theorem A] it is shown that a rational map f of degree at least two satisfying the *summability condition with exponent 1*:

  f does not have indifferent periodic points and for each critical value v in the Julia set of f we have

  $$\sum_{n \geq 0} |(f^n)'(v)|^{-1} < +\infty.$$

  is backward contracting, and it thus has arbitrarily small nice couples. In [Prz98] it is shown that each rational map satisfying the summability condition with exponent 1 is expanding away of critical points.

Using a stronger version of Theorem A (Theorem A' in §7), we show in Appendix A that each infinitely renormalizable quadratic polynomial for which the diameters of the small Julia sets converge to 0 satisfies the conclusions of Theorem A, see §A.2. In particular the conclusions of Theorem A hold for each infinitely renormalizable polynomial with *a priori* bounds; see [KL08, McM94] and references therein for results on *a priori* bounds.

In §A.3 we show the following corollary of Theorem A.

Corollary 1.2. *The conclusions of Theorem A hold for every real quadratic polynomial.*

We will now consider several known related results.

As mentioned above, Makarov and Smirnov showed in [MS00] that the conclusions of Theorem A hold for every rational map on $(-\infty, 0)$. Furthermore, they characterized all those rational maps whose condensation point t_- is finite; see §B.1.

For a uniformly hyperbolic rational map we have $t_- = -\infty$ and $t_+ = +\infty$, and for a sub-hyperbolic polynomial with connected Julia set we have $t_+ = +\infty$ [MS96]. The freezing point t_+ is finite whenever f does not satisfy the Topological Collet-Eckmann Condition[2] (Proposition 2.1). In fact, in this case the freezing point t_+ is the first zero of the pressure function. On the other hand, there is an example in [MS03, §3.4] of a generalized polynomial-like map satisfying the Topological Collet-Eckmann Condition[3] and whose freezing point t_+ is finite.

When f is a generalized polynomial-like map without recurrent critical points, the part of Theorem A concerning the analyticity of the pressure

[2]By [PRLS03, Main Theorem] f satisfies the Topological Collet-Eckmann Condition if, and only if, $\chi_{\mathrm{inf}}(f) > 0$.

[3]In fact this map has the stronger property that no critical point in its Julia set is recurrent.

function was shown in [MS00, MS03, SU03]. Note that the results of [SU03] apply to maps with parabolic periodic points.

In the case of a general transitive multimodal map, a result analogous to Theorem A was shown by Bruin and Todd in [BT08, Theorem 6] for t in a neighborhood of 0. Similar results for t in a neighborhood of $[0, 1]$ were shown by Pesin and Senti in [PS08] for multimodal maps satisfying the Collet-Eckmann condition and some additional properties (see also [BT07, Theorem 2]) and by Bruin and Todd in [BT07, Theorem 1], for t in a one-sided neighborhood of 1, and for multimodal maps with polynomial growth of the derivatives along the critical orbits; see also [BK98].

In [Dob07, Proposition 7], Dobbs shows that there is a real quadratic polynomial f_0 such that the pressure function, *defined for the restriction of f_0 to a certain compact interval*, has infinitely many phase transitions before it vanishes. This behavior of f_0 as an interval map is in sharp contrast with its behavior as a map of $\overline{\mathbb{C}}$: our results imply that the pressure function of f_0, viewed as a map acting on $\overline{\mathbb{C}}$, is real analytic before it vanishes.

1.3. **On equilibria after the freezing point.** For a rational map f whose freezing point t_+ is finite, and for $t \in [t_+, +\infty)$, we now consider the problem of the existence and uniqueness of equilibrium states of f for the potential $-t \ln |f'|$. We first consider the following result in the case when f satisfies the Topological Collet-Eckmann Condition.

Theorem B. *Let f be a rational map satisfying the Topological Collet-Eckmann Condition and whose freezing point t_+ is finite. Then the following properties hold.*

1. *For every $\mu \in \mathscr{M}(f)$ we have $\chi_\mu(f) > \chi_{\inf}$.*
2. *For each $t \in (t_+, +\infty)$ there is no equilibrium state of f for the potential $-t \ln |f'|$.*
3. *There is at most one equilibrium state of f for the potential $-t_+ \ln |f'|$, and if such a measure exists then it has positive measure theoretic entropy, and the pressure function of f is not differentiable at $t = t_+$.*

Remark 1.3. We wrote Theorem B for rational maps, but the proof applies without change to (generalized) polynomial-like maps. In particular this theorem applies to the example given by Makarov and Smirnov in [MS03, §3.4]. This is the only example we know of a map f satisfying the Topological Collet-Eckmann Condition, and whose freezing point t_+ is finite. It is not clear to us if this map has an equilibrium state for the potential $-t_+ \ln |f'|$.

Recall that for a rational map that does not satisfy the Topological Collet-Eckmann Condition, the freezing point t_+ is always finite (Proposition 2.1). There is an example given by Bruin and Todd in [BT06, Corollary 2], of a complex quadratic polynomial f_0 that does not satisfy the Topological Collet-Eckmann Condition, and such that for each $t \in [t_+, +\infty)$ there is no equilibrium state of f_0 for the potential $-t \ln |f_0'|$. In a sharp contrast, in [CRL08, Corollary 2] it is shown that there is a complex quadratic polynomial f_1 having an uncountable number of distinct ergodic probability measures μ in $\mathscr{M}(f_1)$ such that $\chi_\mu(f_1) = 0$, see also [Bru03]. In particular, for this f_1 we have $\chi_{\inf}(f_1) = 0$ and $t_+ < +\infty$ (Proposition 2.1), and for each $t \in [t_+, +\infty)$ there is an uncountable number of distinct ergodic equilibrium states of f_1 for the potential $-t \ln |f_1'|$ (Lemma 8.1).[4]

1.4. **Notes and references.** See the book [Rue04] for an introduction to the thermodynamic formalism, and [PU02, Zin96] for an introduction in the case of rational maps.

For results concerning other potentials, see [DU91, GW07, Prz90, Urb03] for the case of rational maps, and [BT08, PS08] and references therein for the case of multimodal maps.

For a rational map f satisfying the Topological Collet-Eckmann Condition, and for $t = \mathrm{HD}_{\mathrm{hyp}}(f)$, the construction of the corresponding equilibrium state given here gives a new proof of the existence of an absolutely continuous invariant measure. More precisely, it gives a new proof of [PRL07, Key Lemma].

Part 1 of Theorem B implies that for f satisfying the Topological Collet-Eckmann Condition and whose freezing point t_+ is finite, the function $\mu \mapsto \chi_\mu(f)$ is discontinuous. This is not so surprising, since Bruin and Keller showed in [BK98, Proposition 2.8] that this holds for *every* S-unimodal map satisfying the Collet-Eckmann condition.

1.5. **Organization.** We now describe the organization of the paper. Our results are either well-known or vacuous for rational maps without critical points in the Julia set, so we will (implicitly) assume that all the rational maps we consider have at least one critical point in the Julia set.

In §2 we review some general results concerning the pressure function, including some of the different characterizations of the pressure function given in [PRLS04]. We also review some results concerning the asymptotic behavior of the derivative of the iterates of a rational map. These results are mainly taken or deduced from results in [Prz99, PRLS03, PRLS04].

To prove Theorem A and its stronger version (Theorem A' in §7) we make use of the inducing scheme introduced in [PRL07], which is developed in the more general setting considered here in §§3, 4. In §3.1 we recall the definitions of nice sets and couples, and introduce a weaker notion of nice couples that we call "pleasant couples". Then we recall in §3.2 the definition of the canonical induced map associated to a nice (or pleasant) couple. We also review the decomposition of its domain of definition into "first return", and "bad pull-backs" (§3.3). In §3.4 we consider a two variable pressure function associated to such an induced map, that will be very important

[4] Let us also mention that, if f_2 is the interval map which is fixed by the period 3 renormalization operator and if t_0 is the first zero of the corresponding pressure function, then f_2 has a countably infinite number of pairwise distinct equilibrium states for the potential $-t_0 \ln |f_2'|$, see [Dob07] for details.

for the rest of the paper. This pressure function is analogous to the one introduced by Stratmann and Urbanski in [SU03].

In §4 we give sufficient conditions on a nice couple so that the conclusions of Theorem A hold for values of t in a neighborhood of an arbitrary $t_0 \in (t_-, t_+)$ (Theorem C). These conditions are formulated in terms of the two variable pressure function defined in §3.4. We follow the method of [PRL07] for the construction of the conformal measures and the equilibrium states, which is based on the results of Mauldin and Urbanski in [MU03]. As in [PS08], we use a result of Zweimüller in [Zwe05] to show that the invariant measure we construct is in fact an equilibrium state. The uniqueness is a direct consequence of the results of Dobbs in [Dob08], generalizing [Led84]. Finally, we use the method introduced by Stratmann and Urbanski in [SU03] to show that the pressure function is real analytic. Here we make use of the fact that the two variable pressure function is real analytic on the interior of the set where it is finite, a result shown by Mauldin and Urbanski in [MU03].

The proof Theorem A' (a stronger version of Theorem A) is contained in §§5, 6, 7. The proof is divided into two parts. The first, and by far the most difficult part, is to show that for $t_0 \in (t_-, t_+)$ the two variable pressure associated to a sufficiently small nice couple is finite on a neighborhood of $(t, p) = (t_0, P(t_0))$. To do this we use the strategy of [PRL07]: we use the decomposition of the domain of definition of the induced map associated to a nice couple, into "first return" and "bad-pull-backs" evoked in §3.3. Unfortunately, for values of t such that $P(t) < 0$, there does not seem to be a natural way to adapt the "density" introduced in [PRL07] to estimate the contribution of a bad pull-back. Instead we use a different argument involving a Whitney type decomposition of a pull-back, which is one of the main technical tools introduced in this paper. Roughly speaking we have replaced the "annuli argument" of [PRL07, Lemma 5.4] by an argument involving "Whitney squares", that allows us to make a direct estimate avoiding an induction on the number visits to the critical point. The Whitney type decomposition is introduced in §5, and the estimate on the contribution of a (bad) pull-back is given in §6. The finiteness of the two variable pressure function is shown in §7.1. The second part of the proof, that for each t close to t_0 the two variable pressure function vanishes at $(t, p) = (t, P(t))$, is given in §7.2. Here we have replaced the analogous (co-)dimension argument of [PRL07], with an argument involving the pressure function of the rational map.

The proof of Theorem B is given in §8. It is based on the generating series technique used by Makarov and Smirnov in [MS03]. The main idea is to estimate the pressure function from below by finding a suitable "Iterated Function System" of iterated inverse branches of the rational map.

Appendix A is devoted to show that the conclusions of Theorem A hold for several classes of polynomials. In §A.1 we show that each at most finitely renormalizable polynomial without indifferent periodic points satisfies the hypotheses of Theorem A (Theorem D). Then in §A.2 we show that each infinitely renormalizable quadratic polynomial for which the diameters of small Julia sets converge to 0 satisfies the hypotheses of Theorem A'. Finally §A.3 is devoted to the proof of Corollary 1.2, treating the case of real quadratic polynomials.

In Appendix B we give applications of our main results to rigidity, multifractal analysis, and level-1 large deviations.

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2. Preliminaries

The purpose of this section is to give some some general properties of the pressure function (§§2.2, 2.3), and some characterizations of $\chi_{\inf}$ and $\chi_{\sup}$ (§2.4). These results are mainly taken or deduced from the results in [Prz99, PRLS03, PRLS04]. We also fix some notation and terminology in §2.1, that will be used in the rest of the paper.

Throughout the rest of this section we fix a rational map f of degree at least two. We will denote $h_\mu(f), \chi_\mu(f), \ldots$ just by $h_\mu, \chi_\mu, \ldots$. For simplicity we will assume that no critical point of f in the Julia set is mapped to another critical point under forward iteration. The general case can be handled by treating whole blocks of critical points as a single critical point; that is, if the critical points $c_0, \ldots, c_k \in J(f)$ are such that c_i is mapped to c_{i+1} by forward iteration, and maximal with this property, then we treat this block of critical points as a single critical point.

2.1. **Notation and terminology.** We will denote the extended real line by $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

Distances, balls, diameters and derivatives are all taken with respect to the spherical metric. For $z \in \overline{\mathbb{C}}$ and $r > 0$, we denote by $B(z, r) \subset \overline{\mathbb{C}}$ the ball centered at z and with radius r.

For a given $z \in \overline{\mathbb{C}}$ we denote by $\deg_f(z)$ the local degree of f at z, and for $V \subset \overline{\mathbb{C}}$ and $n \geq 0$, each connected component of $f^{-n}(V)$ will be called *a pull-back of V by f^n*. For such a set W we put $m_W = n$. When $n = 0$ we obtain that each connected component W of V is a pull-back of V with $m_W = 0$. Note that the set V is not assumed to be connected.

We will abbreviate "Topological Collet-Eckmann" by TCE.

2.2. **General properties of the pressure function.** Given a positive integer n let $\Lambda_n : \overline{\mathbb{C}} \times \mathbb{R} \to \overline{\mathbb{R}}$ be the function defined by

$$\Lambda_n(z_0, t) := \sum_{w \in f^{-n}(z_0)} |(f^n)'(z_0)|^{-t}.$$

Then for every $t \in \mathbb{R}$ and every z_0 in $\overline{\mathbb{C}}$ outside a set of Hausdorff dimension 0, we have

$$\limsup_{n \to +\infty} \frac{1}{n} \ln \Lambda_n(z_0, t) = P(t), \tag{2.1}$$

see [Prz99, PRLS04].

In the following proposition,

$$\mathrm{HD}_{\mathrm{hyp}}(f) := \sup\{\mathrm{HD}(X) \mid X \text{ compact and invariant subset of } \overline{\mathbb{C}} \text{ where } f \text{ is uniformly expanding}\}.$$

Proposition 2.1. *Given a rational map f of degree at least two, the function*

$$t \mapsto P(t) + t\chi_{\mathrm{inf}} \ \textit{(resp. } t \mapsto P(t) + t\chi_{\mathrm{sup}}\textit{)},$$

is convex, non-increasing, and non-negative on $[0, +\infty)$ (resp. $(-\infty, 0]$). Moreover $t_- < 0$, and we have $t_+ \geq \mathrm{HD}_{\mathrm{hyp}}(f)$ with strict inequality if, and only if, f satisfies the TCE condition.

In particular for all t in (t_-, t_+) we have $P(t) > \max\{-t\chi_{\mathrm{inf}}, -t\chi_{\mathrm{sup}}\}$, and for all t in $\mathbb{R} \setminus (t_-, t_+)$ we have $P(t) = \max\{-t\chi_{\mathrm{inf}}, -t\chi_{\mathrm{sup}}\}$.

Proof. For each $\mu \in \mathscr{M}(f)$ the function $t \mapsto h_\mu(f) - t(\chi_\mu - \chi_{\mathrm{inf}})$ (resp. $t \mapsto h_\mu - t(\chi_\mu - \chi_{\mathrm{sup}})$) is affine and non-increasing $[0, +\infty)$ (resp. $(-\infty, 0]$). As by definition

$$P(t) = \sup\{h_\mu - t\chi_\mu \mid \mu \in \mathscr{M}(f)\},$$

we conclude that the function $t \mapsto P(t) + t\chi_{\mathrm{inf}}$ (resp. $t \mapsto P(t) + t\chi_{\mathrm{inf}}$) is convex and non-increasing $[0, +\infty)$ (resp. $(-\infty, 0]$). It also follows from the definition that $t \mapsto P(t) + t\chi_{\mathrm{inf}}$ (resp. $t \mapsto P(t) + t\chi_{\mathrm{inf}}$) is non-negative on this set.

The inequalities $t_- < 0$ and $t_+ \geq \mathrm{HD}_{\mathrm{hyp}}(f)$ follow from the fact that χ_{inf} is non-negative and from the fact that the pressure function P is strictly positive on $(0, \mathrm{HD}_{\mathrm{hyp}}(f))$ [Prz99]. When f satisfies the TCE condition, then $\chi_{\mathrm{inf}} > 0$ [PRLS03, Main Theorem] and thus $t_+ > \mathrm{HD}_{\mathrm{hyp}}(f)$. When f does not satisfy the TCE condition, then $\chi_{\mathrm{inf}} = 0$ [PRLS03, Main Theorem] and therefore the equality $t_+ = \mathrm{HD}_{\mathrm{hyp}}(f)$ follows from the fact that $\mathrm{HD}_{\mathrm{hyp}}(f)$ is the first zero of the function P [Prz99]. □

2.3. **The pressure function and conformal measures.** For real numbers t and p we will say that a finite Borel measure μ is (t, p)-*conformal* for f, if for each Borel subset U of $\overline{\mathbb{C}}$ on which f is injective we have

$$\mu(f(U)) = \exp(p) \int_U |f'|^t d\mu.$$

By the locally eventually onto property of f on $J(f)$ it follows that if the topological support of a (t, p)-conformal measure is contained in $J(f)$, then it is in fact equal to $J(f)$.

Proposition 2.2. *Let f be a rational map of degree at least two. Then for each $t \in (t_-, +\infty)$ there exists a $(t, P(t))$-conformal measure for f supported on $J(f)$, and for each real number p for which there is a (t, p)-conformal measure for f supported on $J(f)$ we have $p \geq P(t)$.*

Proof. When $t = 0$, the assertions are well known, see for example [DU91, p. 104]. The case $t > 0$ is given by [PRLS04, Theorem A]. In the case $t \in (t_-, 0)$ the existence is given by [MS00, §3.5] (see also [PRLS04, Theorem A.7]), and in [PRLS04, Proposition A.11] it is shown that if for some real number p there is a (t, p)-conformal measure, then in fact $p = P(t)$. □

2.4. **Characterizations of $\chi_{\inf}$ and $\chi_{\sup}$.** The following proposition gives some characterizations of $\chi_{\inf}$ and $\chi_{\sup}$, which are obtained as direct consequences of the results in [PRLS03].

For each $\alpha > 0$ put

$$E_\alpha = \bigcap_{n_0 \geq 1} \bigcup_{n \geq 1} B\left(f^n(\mathrm{Crit}(f)), \max\{n_0, n\}^{-\alpha}\right).$$

Observe that the Hausdorff dimension of E_α is less than or equal to α^{-1}. It thus follows that the Hausdorff dimension of the set $E_\infty := \bigcap_{\alpha > 0} E_\alpha$ is equal to 0.

Proposition 2.3. *For a rational map f of degree at least two, the following properties hold.*

1. *Given a repelling periodic point p of f, let m be its period and put $\chi(p) := \frac{1}{m} \ln |((f^m)'(p)|$. Then we have*

$$\inf\{\chi(p) \mid p \text{ is a repelling periodic point of } f\} = \chi_{\inf},$$

$$\sup\{\chi(p) \mid p \text{ is a repelling periodic point of } f\} = \chi_{\sup}.$$

2.
$$\lim_{n \to +\infty} \frac{1}{n} \ln \sup\{|(f^n)'(z)| \mid z \in \overline{\mathbb{C}}\} = \chi_{\sup}.$$

3. *For each $z_0 \in \overline{\mathbb{C}} \setminus E_\infty$ we have*

$$\lim_{n \to +\infty} \frac{1}{n} \ln \min\{|(f^n)'(w)| \mid w \in f^{-n}(z_0)\} = \chi_{\inf}, \tag{2.2}$$

$$\lim_{n \to +\infty} \frac{1}{n} \ln \max\{|(f^n)'(w)| \mid w \in f^{-n}(z_0)\} = \chi_{\sup}. \tag{2.3}$$

Proof.

1. The equality involving $\chi_{\inf}$ was shown in [PRLS03, Main Theorem]. To prove the equality involving $\chi_{\sup}$, first note that if p is a repelling periodic of f, and if we denote by m its period, then the measure $\mu := \sum_{j=0}^{m-1} \delta_{f^j(p)}$ is invariant by f and its Lyapunov exponent is equal to $\chi(p)$. It thus follows that

$$\sup\{\chi(p) \mid p \text{ repelling periodic point of } f\} \leq \chi_{\sup}.$$

The reverse inequality follows from the fact, shown using Pesin theory, that for every ergodic and invariant probability measure μ whose Lyapunov exponent is positive and every $\varepsilon > 0$ one can find a repelling periodic point p such that $|\chi_\mu - \chi(p)| < \varepsilon$; see for example [PU02, Theorem 10.6.1].

2. For each positive integer n put

$$M_n := \sup\{|(f^n)'(z)| \mid z \in \overline{\mathbb{C}}\}.$$

Note that for positive integers m, n we have $M_{m+n} \leq M_m \cdot M_n$, so the limit

$$\chi := \lim_{n \to +\infty} \tfrac{1}{n} \ln M_n$$

exists. The inequality $\chi \geq \chi_{\text{sup}}$ follows from part 1. To prove the reverse inequality, for each positive integer n let $z_n \in \overline{\mathbb{C}}$ be such that $|(f^n)'(z_n)| = M_n$ and put

$$\mu_n := \tfrac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(z_n)}.$$

Let $(n_j)_{j \geq 0}$ be a diverging sequence of integers so that μ_{n_j} converges to a measure μ, which is invariant by μ. Since the function $\ln |f'|$ is bounded from above, the monotone convergence theorem implies that

$$\lim_{A \to -\infty} \int \max\{A, \ln |f'|\} d\mu = \int \ln |f'| d\mu.$$

On the other hand, for each real number A we have

$$\int \max\{A, \ln |f'|\} d\mu = \lim_{j \to +\infty} \int \max\{A, \ln |f'|\} d\mu_{n_j} \geq \limsup_{j \to +\infty} \int \ln |f'| d\mu_{n_j}.$$

We thus conclude that

$$\chi_{\text{sup}} \geq \int \ln |f'| d\mu \geq \limsup_{j \to +\infty} \int \ln |f'| d\mu_{n_j} = \chi.$$

3. For a point $z_0 \in \overline{\mathbb{C}}$ that is not in the forward orbit of a critical point of f, the inequalities

$$\limsup_{n \to +\infty} \tfrac{1}{n} \ln \min\{|(f^n)'(w)| \mid w \in f^{-n}(z_0)\} \leq \chi_{\text{inf}},$$

$$\liminf_{n \to +\infty} \tfrac{1}{n} \ln \max\{|(f^n)'(w)| \mid w \in f^{-n}(z_0)\} \geq \chi_{\text{sup}}.$$

are a direct consequence of part 1 and the following property: For each repelling periodic point p there is a constant $C > 0$ such that for every positive integer n there is $w \in f^{-n}(z_0)$ satisfying

$$C^{-1} \exp(n\chi(p)) \leq |(f^n)'(w)| \leq C \exp(n\chi(p)).$$

Part 2 shows that for each $z_0 \in \overline{\mathbb{C}}$ we have

$$\limsup_{n \to +\infty} \tfrac{1}{n} \ln \max\{|(f^n)'(w)| \mid w \in f^{-n}(z_0)\} \leq \chi_{\text{sup}}.$$

It remains to show that for every $z_0 \in \overline{\mathbb{C}} \setminus E_\infty$ we have

$$\limsup_{n\to+\infty} \tfrac{1}{n} \ln \min\{|(f^n)'(w)| \mid w \in f^{-n}(z_0)\} \geq \chi_{\inf}.$$

To do this it is enough to show that there exists one point in $\overline{\mathbb{C}} \setminus E_\infty$ for which this holds, see [Prz99, §3] or [PRLS03, §1]. This follows from [PRLS03, Lemma 3.1], taking for each integer n,

$$\lambda_n := \min\{|(f^n)'(p)| \mid \text{repelling periodic point of period } n\}.$$

□

3. Nice sets, pleasant couples and induced maps

In §3.1 we recall the definition and review some properties of nice sets and couples, and we introduce a notion weaker than nice couple, that we call "pleasant couple". Then we consider the canonical induced map associated to a pleasant couple in §3.2, as it was introduced in [PRL07, §4], and review some of its properties (§3.3). Finally, we introduce in §3.4 a two variable pressure function associated to the a canonical induced map, that will be important in what follows.

Throughout all this section we fix a rational map f of degree at least two.

3.1. **Nice sets, nice couples, and pleasant couples.**

Recall that a neighborhood V of $\mathrm{Crit}(f) \cap J(f)$ is a *nice set for* f, if for every $n \geq 1$ we have $f^n(\partial V) \cap V = \emptyset$, and if each connected component of V is simply connected and contains precisely one critical point of f in $J(f)$.

Let $V = \bigcup_{c \in \mathrm{Crit}(f) \cap J(f)} V^c$ be a nice set for f. Then for every pull-back W of V we have either

$$W \cap V = \emptyset \text{ or } W \subset V.$$

Furthermore, if W and W' are distinct pull-backs of V, then we have either,

$$W \cap W' = \emptyset, \, W \subset W' \text{ or } W' \subset W.$$

For a pull-back W of V we denote by $c(W)$ the critical point in $\mathrm{Crit}(f) \cap J(f)$ and by $m_W \geq 0$ the integer such that $f^{m_W}(W) = V^{c(W)}$. Moreover we put,

$$K(V) = \{z \in \overline{\mathbb{C}} \mid \text{ for every } n \geq 0 \text{ we have } f^n(z) \notin V\}.$$

Note that $K(V)$ is a compact and forward invariant set and for each $c \in \mathrm{Crit}(f) \cap J(f)$ the set V^c is a connected component of $\overline{\mathbb{C}} \setminus K(V)$. Moreover, if W is a connected component of $\overline{\mathbb{C}} \setminus K(V)$ different from the V^c, then $f(W)$ is again a connected component of $\overline{\mathbb{C}} \setminus K(V)$. It follows that W is a pull-back of V and that f^{m_W} is univalent on W.

Given a nice set V for f and a neighborhood $\widehat{V}$ of $\overline{V}$ in $\overline{\mathbb{C}}$ we will say that $(\widehat{V}, V)$ is a *pleasant couple for* f if the following property holds: For every pull-back W of V, the pull-back of $\widehat{V}$ by f^{m_W} that contains W is either contained in $\widehat{V}$ when W is contained in V, and it is disjoint from $\mathrm{Crit}(f)$ when W is disjoint from V.

If $(\widehat{V}, V)$ is a pleasant couple for f, then for each $c \in \mathrm{Crit}(f) \cap J(f)$ we denote by $\widehat{V}^c$ the connected component of $\widehat{V}$ containing c. Furthermore, for each pull-back W of V we will denote by $\widehat{W}$ the pull-back of $\widehat{V}$ by f^{m_W} that contains W, and put $m_{\widehat{W}} := m_W$ and $c(\widehat{W}) := c(W)$. If W is a connected component of $\overline{\mathbb{C}} \setminus K(V)$, then for every $j = 0, \ldots, m_W - 1$, the set $f^j(W)$ is a connected component of $\overline{\mathbb{C}} \setminus K(V)$ different from the V^c, and $\widehat{f^j(W)}$ is disjoint from $\mathrm{Crit}(f)$. It follows that f^{m_W} is univalent on $\widehat{W}$.

A *nice couple for* f is a pair $(\widehat{V}, V)$ of nice sets for f such that $\overline{V} \subset \widehat{V}$, and such that for every $n \geq 1$ we have $f^n(\partial V) \cap \widehat{V} = \emptyset$.

If $(\widehat{V}, V)$ is a nice couple for f, then for every pull-back $\widehat{W}$ of $\widehat{V}$ we have either

$$\widehat{W} \cap V = \emptyset \text{ or } \widehat{W} \subset V.$$

It thus follows that $(\widehat{V}, V)$ is a pleasant couple.

Remark 3.1. The definitions of nice sets and couples given here is slightly weaker than that of [PRL07, RL07]. For a set $V = \bigcup_{c \in \mathrm{Crit}(f) \cap J(f)} V^c$ to be nice, in those papers we required the stronger condition that for each integer $n \geq 1$ we have $f^n(\partial V) \cap \overline{V} = \emptyset$, and that the closures of the sets V^c are pairwise disjoint. Similarly, for a pair of nice sets $(\widehat{V}, V)$ to be a nice couple we required the stronger condition that for each $n \geq 1$ we have $f^n(\partial V) \cap \overline{\widehat{V}} = \emptyset$. The results we need from [PRL07] still hold with the weaker property considered here.

Observe that if $(\widehat{V}, V)$ is a nice couple as defined here, then V is a nice set in the sense of [PRL07, RL07].

The following proposition (which we owe to Shen), though not used later on, sheds some light on the definitions above; compare with the construction of nice couples in §A.1 (Theorem D), and in [RL07, §6].

Proposition 3.2. *Suppose that for a rational map f there exists a nice set $U = \bigcup_{c \in \mathrm{Crit}(f) \cap J(f)} U^c$ such that for every integer $n \geq 1$*

$$f^n(\partial U) \cap \overline{U} = \emptyset. \tag{3.1}$$

Suppose furthermore that for an integer $k \geq 1$ the maximal diameter of a connected component of $f^{-k}(U)$ converges to 0 as $k \to +\infty$. Then there exists a nice set V for f that is compactly contained in U such that (U, V) is a nice couple for f.

Proof. Since U is a nice set each connected component of the set $A := \overline{\mathbb{C}} \setminus f^{-1}(K(U))$ is a pull-back of U. Furthermore, by (3.1) each connected component W of A intersecting U is compactly contained in U, and m_W is the first return time to U of points in W.

If the forward trajectory of c visits U, take as V^c the connected component containing c of A. Since U is a nice set, V^c is a first return pull-back of U, and by (3.1) the set V^c is compactly contained in U. In particular for each integer $n \geq 1$ we have $f^n(\partial V^c) \cap U = \emptyset$. For each critical point $c \in \text{Crit}(f) \cap J(f)$ whose forward trajectory never returns to U, take a preliminary disc D compactly contained in U^c. By (3.1) each connected component of A intersecting D is compactly contained in U^c. Let now $\widetilde{V}^c$ be the union of D and all those connected components of A intersecting D. The hypothesis on diameters of pull-backs implies that V^c is compactly contained in U, and that each point in $\partial\widetilde{V}^c$ is either contained in $\partial D \cap (\overline{\mathbb{C}} \setminus A)$, or in the boundary of a connected component of A intersecting D (which is a first return pull-back of U). Therefore for each integer $n \geq 1$ we have $f^n(\partial\widetilde{V}^c) \cap U = \emptyset$. Finally let V^c be the union of $\widetilde{V}^c$ and all connected components of $\overline{\mathbb{C}} \setminus \widetilde{V}^c$ contained in U^c (We do this "filling holes" trick since *a priori* it could happen that the union of D and one of the connected components of A, and consequently $\widetilde{V}^c$, might not be simply-connected). We have $\partial V^c \subset \partial\widetilde{V}^c$, so for each integer $n \geq 1$ we have $f^n(\partial V^c) \cap U = \emptyset$.

Set $V = \bigcup_{c \in \text{Crit}(f) \cap J(f)} V^c$. We have shown that for each integer $n \geq 1$ we have $f^n(\partial V) \cap U = \emptyset$, so (U, V) is a nice couple. □

3.2. **Canonical induced map.** Let $(\widehat{V}, V)$ be a pleasant couple for f. We say that an integer $m \geq 1$ is a *good time for a point* z in $\overline{\mathbb{C}}$, if $f^m(z) \in V$ and if the pull-back of $\widehat{V}$ by f^m to z is univalent. Let D be the set of all those points in V having a good time and for $z \in D$ denote by $m(z) \geq 1$ the least good time of z. Then the map $F : D \to V$ defined by $F(z) := f^{m(z)}(z)$ is called *the canonical induced map associated to* $(\widehat{V}, V)$. We denote by $J(F)$ the maximal invariant set of F.

As V is a nice set, it follows that each connected component W of D is a pull-back of V. Moreover, f^{m_W} is univalent on $\widehat{W}$ and for each $z \in W$ we have $m(z) = m_W$. Similarly, for each positive integer n, each connected component W of the domain of definition of F^n is a pull-back of V and f^{m_W} is univalent on $\widehat{W}$. Conversely, if W is a pull-back of V contained in V such that f^{m_W} is univalent on $\widehat{W}$, then there is $c \in \text{Crit}(f) \cap J(f)$ and a positive integer n such that F^n is defined on W and $F^n(W) = V^c$. In fact, in this case m_W is a good time for each element of W and therefore $W \subset D$. Thus, either we have $F(W) = V^{c(W)}$, and then W is a connected component of D, or $F(W)$ is a pull-back of V contained in V such that $f^{m_{F(W)}}$ is univalent on $\widehat{F(W)}$. Thus, repeating this argument we can show by induction that there is a positive integer n such that F^n is defined on W and that $F^n(W) = V^{c(W)}$.

Lemma 3.3 ([PRL07], Lemma 4.1)**.** *For every rational map f there is $r > 0$ such that if $(\widehat{V}, V)$ is a pleasant couple satisfying*

$$\max_{c \in \text{Crit}(f) \cap J(f)} \text{diam}\left(\widehat{V}^c\right) \leq r, \tag{3.2}$$

then the canonical induced map $F : D \to V$ associated to $(\widehat{V}, V)$ is topologically mixing on $J(F)$. Moreover there is $\widetilde{c} \in \mathrm{Crit}(f) \cap J(f)$ such that the set

$$\{m_W \mid W \text{ c.c. of } D \text{ contained in } V^{\widetilde{c}} \text{ such that } F(W) = V^{\widetilde{c}}\} \tag{3.3}$$

is non-empty and its greatest common divisor is equal to 1.

3.3. **Bad pull-backs.** Let $(\widehat{V}, V)$ be a pleasant couple for f. For an integer $n \geq 1$ we will say that a connected component $\widetilde{W}$ of $f^{-n}(\widehat{V})$ is a *bad pull-back of* $\widehat{V}$ *of order* n, if f^n is not univalent on $\widetilde{W}$ and if for every $m = 1, \ldots, n-1$ such that $f^m(\widetilde{W}) \subset V$, the map f^m is not univalent on the connected component of $f^{-m}(\widehat{V})$ containing $\widetilde{W}$.

The following lemma will be used in the proof of Theorem A in §7. Although it is essentially the same as part 1 of Lemma 7.4 of [PRL07], we have included a proof because it is stated in a slightly different way. We denote by $\mathfrak{D}_V$ the collection of all the connected components of $\overline{\mathbb{C}} \setminus K(V)$ Furthermore, for a pull-back $\widetilde{W}$ of $\widehat{V}$ we denote by $\mathfrak{D}_{\widetilde{W}}$ the collection of all the pull-backs W of V that are contained in $\widetilde{W}$, such that $f^{m_{\widetilde{W}}+1}$ is univalent on $\widehat{W}$, and such that $f^{m_{\widetilde{W}}+1}(W) \in \mathfrak{D}_V$.

Lemma 3.4 ([PRL07], part 1 of Lemma 7.4)**.** *If we denote by* $\mathfrak{D}$ *the collection of the connected components of* D, *then we have*

$$\mathfrak{D} = \bigcup_{\substack{\widetilde{W} \text{ bad pull-back of } \widehat{V} \\ \text{or } \widetilde{W} = \widehat{V}^c,\ c \in \mathrm{Crit}(f) \cap J(f)}} \mathfrak{D}_{\widetilde{W}}.$$

Proof. Clearly, for each $c \in \mathrm{Crit}(f) \cap J(f)$ we have $\mathfrak{D}_{\widehat{V}^c} \subset \mathfrak{D}$. Let $\widetilde{W}$ be a bad pull-back of $\widehat{V}$ and let $W \in \mathfrak{D}_{\widetilde{W}}$. To show that W belongs to $\mathfrak{D}$ we need to show that f^{m_W} is univalent on $\widehat{W}$, and that for each $j \in \{1, \ldots, m_W - 1\}$ such that $f^j(W) \subset V$, the map f^j is not univalent on the pull-back of $\widehat{V}$ by f^j that contains W. That f^{m_W} is univalent on $\widehat{W}$ follows from the fact that $f^{m_{\widetilde{W}}+1}$ is univalent on $\widehat{W}$, and that $f^{m_{\widetilde{W}}+1}(W) \in \mathfrak{D}_V$. Let $j \in \{1, \ldots, m_W - 1\}$ be such that $f^j(W) \subset V$. As $f^{m_{\widetilde{W}}+1}(W) \in \mathfrak{D}_V$, we have $j \leq m_{\widetilde{W}}$. Since $\widetilde{W}$ is a bad pull-back, the map f^j is not univalent on the pull-back of $\widehat{V}$ by f^j that contains W. This completes the proof that $W \in \mathfrak{D}$.

Let $W \in \mathfrak{D}$. If $f(W) \in \mathfrak{D}_V$, then there is $c \in \mathrm{Crit}(f) \cap J(f)$ such that $W \in \mathfrak{D}_{\widehat{V}^c}$. If $f(W) \notin \mathfrak{D}_V$, then there is a positive integer j such that $j < m_W$ and $f^j(W) \subset V$, and for every such integer j the map f^j is not univalent on the pull-back of $\widehat{V}$ by f^j that contains W. Thus, if we denote by n the largest of such j, then the pull-back $\widetilde{W}$ of $\widehat{V}$ by f^n containing W is a bad pull-back and $W \in \mathfrak{D}_{\widetilde{W}}$. □

3.4. **Pressure function of the canonical induced map.** Let $(\widehat{V}, V)$ be a pleasant couple for f and let $F : D \to V$ be the canonical induced map associated to $(\widehat{V}, V)$. Furthermore, denote by $\mathfrak{D}$ the collection of connected components of D and for each $c \in \text{Crit}(f) \cap J(f)$ denote by $\mathfrak{D}^c$ the collection of all elements of $\mathfrak{D}$ contained in V^c, so that $\mathfrak{D} = \bigsqcup_{c \in \text{Crit}(f) \cap J(f)} \mathfrak{D}^c$. A word on the alphabet $\mathfrak{D}$ will be called *admissible* if for every pair of consecutive letters $W, W' \in \mathfrak{D}$ we have $W \in \mathfrak{D}^{c(W')}$. For a given integer $n \geq 1$ we denote by E^n the collection of all admissible words of length n. Given $W \in \mathfrak{D}$, denote by ϕ_W the holomorphic extension to $\widehat{V}^{c(W)}$ of the inverse of $F|_W$. For a finite word $\underline{W} = W_1 \ldots W_n \in E^*$ put $c(\underline{W}) := c(W_n)$ and $m_{\underline{W}} = m_{W_1} + \cdots + m_{W_n}$. Note that the composition

$$\phi_{\underline{W}} := \phi_{W_1} \circ \cdots \circ \phi_{W_n}$$

is well defined and univalent on $\widehat{V}^{c(\underline{W})}$ and takes images in V.

For each $t, p \in \mathbb{R}$ and $n \geq 1$ put

$$Z_n(t,p) := \sum_{\underline{W} \in E^n} \exp(-m_{\underline{W}} p) \left(\sup \left\{ |\phi'_{\underline{W}}(z)| \mid z \in V^{c(\underline{W})} \right\} \right)^t .$$

It is easy to see that for a fixed $t, p \in \mathbb{R}$ the sequence $(\ln Z_n(t,p))_{n \geq 1}$ is sub-additive, and hence that we have

(3.4)
$$P(F, -t \ln |F'| - pm) := \lim_{n \to +\infty} \frac{1}{n} \ln Z_n(t,p) = \inf \left\{ \frac{1}{n} \ln Z_n(t,p) \mid n \geq 1 \right\},$$

see for example Lemma 2.1.1 and Lemma 2.1.2 of [MU03]. Here m is the function defined in §3.2, that to each point $z \in D$ it associates the least good time of z. The number (3.4) is called the *pressure function of F for the potential* $-\ln |F'| - pm$. It is easy to see that for every $t, p \in \mathbb{R}$ the sequence $(\frac{1}{n} \ln Z_n(t,p))_{n \geq 1}$ is uniformly bounded from below, so that (3.4) does not take the value $-\infty$. Note however that if D has infinitely many connected components, then we have $P(F, 0) = +\infty$.

The function,

$$\begin{array}{cccc} \mathscr{P}: & \mathbb{R}^2 & \to & \mathbb{R} \cup \{+\infty\} \\ & (t,p) & \mapsto & P(F, -t \ln |F'| - pm), \end{array}$$

will be important in what follows. Notice that if $\mathscr{P}$ is finite at $(t_0, p_0) \in \mathbb{R}^2$, then it is finite on the set $\{(t,p) \in \mathbb{R}^2 \mid t \geq t_0, p \geq p_0\}$. Furthermore, restricted to the set where it is finite, the function $\mathscr{P}$ it is strictly decreasing on each of its variables.

In the following property will be important to use the results of [MU03].

(*) There is a constant $C_M > 0$ such that for every $\kappa \in (0,1)$ and every ball B of $\overline{\mathbb{C}}$, the following property holds. Every collection of pairwise disjoint sets of the form $D_{\underline{W}}$, with $\underline{W} \in E^*$, intersecting B and with diameter at least $\kappa \cdot \text{diam}(B)$, has cardinality at most $C_M \kappa^{-2}$.

In fact, F determines a Graph Directed Markov System (GDMS) in the sense of [MU03], except maybe for the "cone property" (4d). But in [MU03] the cone property is only used in [MU03, Lemma 4.2.6] to prove (*). Thus, when property (*) is satisfied all the results of [MU03] apply to F.

In [PRL07, Proposition A.2] we have shown that property (*) holds when the pleasant couple $(\widehat{V}, V)$ is nice.

Lemma 3.5. *Let f be a rational map of degree at least two and let $(\widehat{V}, V)$ be a pleasant couple for f satisfying property (*). Then the function $\mathscr{P}$ defined above satisfies the following properties.*

1. *The function $\mathscr{P}$ is real analytic on the interior of the set where it is finite.*
2. *The function $\mathscr{P}$ is strictly negative on $\{(t,p) \in \mathbb{R}^2 \mid p > P(t)\}$.*

Proof.
1. Since $\ln |F'|$ defines a Hölder function of the symbolic space associated to F, for each $(t,p) \in \mathbb{R}^2$ the function $-\ln |F'| - pm$ defines a Hölder function of the symbolic space associated to F, and in the case $\mathscr{P}(t,p) < +\infty$ the function $-t \ln |F'| - pm$ is summable in the sense of [MU03]. Then the desired result follows from [MU03, Theorem 2.6.12].
2. Let $(t_0, p_0) \in \mathbb{R}^2$ be such that $p_0 > P(t_0)$. Then for each point $z_0 \in V$ for which (2.1) holds, we have

$$\sum_{k \geq 1} \sum_{y \in F^{-k}(z)} \exp(-p_0 m(y)) |(F^k)'(y)|^{-t_0} \leq \sum_{n \geq 1} \exp(-p_0 n) \sum_{y \in f^{-n}(z_0)} |(f^n)'(y)|^{-t_0} < +\infty,$$

which implies that $\mathscr{P}(t_0, p_0) \leq 0$. This shows that the function $\mathscr{P}$ is non-positive on $\{(t,p) \in (0,+\infty) \times \mathbb{R} \mid p > P(t)\}$. That $\mathscr{P}$ is strictly negative on this set follows from the fact that, on this set, $\mathscr{P}$ is strictly decreasing on each of its variables. □

4. From the induced map to the original map

The purpose of this section is to prove the following theorem. We denote by $J_{\mathrm{con}}(f)$ the "conical Julia set" of f, which is defined in §4.1. Recall that conformal measures were defined in §2.3.

Theorem C. *Let f be a rational map of degree at least two, let $(\widehat{V}, V)$ be a pleasant couple for f satisfying property (*), and let $\mathscr{P}$ be the corresponding pressure function defined in §3.4. Then for each $t_0 \in (t_-, +\infty)$, the following properties hold.*

Conformal measure: *If $\mathscr{P}$ vanishes at $(t,p) = (t_0, P(t_0))$, then there is a unique $(t_0, P(t_0))$-conformal probability measure for f. Moreover this measure is non-atomic, ergodic, and it is supported on $J_{\mathrm{con}}(f)$.*

Equilibrium state: *If $\mathscr{P}$ is finite on a neighborhood of $(t,p) = (t_0, P(t_0))$, and vanishes at this point, then there is a unique equilibrium measure of f for the potential $-t_0 \ln |f'|$. Furthermore, this measure is ergodic, absolutely continuous with respect to the unique $(t_0, P(t_0))$-conformal* *probability measure of f, and its density is bounded from below by a positive constant almost everywhere. If furthermore $(\widehat{V}, V)$ satisfies the conclusions of Lemma 3.3, then the equilibrium state is exponentially mixing and it satisfies the central limit theorem.*

Analyticity of the pressure function: *If $\mathscr{P}$ is finite on a neighborhood of $(t, p) = (t_0, P(t_0))$, and for each $t \in \mathbb{R}$ close to t_0 we have $\mathscr{P}(t, P(t)) = 0$, then the pressure function P is real analytic on a neighborhood of $t = t_0$.*

After some general considerations in §4.1, the assertions about the conformal measure are shown in §4.2. The assertions concerning the equilibrium state are shown in §4.3, and the analyticity of the pressure function is shown in §4.4.

Throughout the rest of this section we fix f, $(\widehat{V}, V)$, F, $\mathscr{P}$ as in the statement of the theorem.

4.1. **The conical Julia set and sub-conformal measures.** The *conical Julia set* of f, denoted by $J_{\text{con}}(f)$, is by definition the set of all those points x in $J(f)$ for which there exists $\rho(x) > 0$ and an arbitrarily large positive integer n, such that the pull-back of the ball $B(f^n(x), \rho(x))$ to x by f^n is univalent. This set is also called *radial Julia set.*

We will need the following general result, which is a strengthened version of [McM00, Theorem 5.1], [DMNU98, Theorem 1.2], with the same proof. Given $t, p \in \mathbb{R}$ we will say that a Borel measure μ is (t, p)-*sub-conformal* f, if for every Borel subset U of $\overline{\mathbb{C}} \setminus \text{Crit}(f)$ on which f is injective we have

$$\exp(p) \int_U |f'|^t d\mu \leq \mu(f(U)). \tag{4.1}$$

Proposition 4.1. *Fix $t \in (t_-, +\infty)$ and $p \in [P(t), +\infty)$. If μ is a (t, p)-sub-conformal measure for f supported on $J_{\text{con}}(f)$, then $p = P(t)$, the measure μ is $(t, P(t))$-conformal, and every other $(t, P(t))$-conformal measure is proportional to μ. Moreover, every subset X of $\overline{\mathbb{C}}$ such that $f(X) \subset X$ and $\mu(X) > 0$ has full measure with respect to μ.*

The proof of this proposition depends on the following lemma.

Lemma 4.2. *Let $t, p \in \mathbb{R}$ and let μ be a (t, p)-sub-conformal measure supported on $J_{\text{con}}(f)$. Suppose that for some $p' \leq p$ there exists a non-zero (t, p')-conformal measure ν that is supported on $J(f)$. Then $p' = p$ and μ is absolutely continuous with respect to ν. In particular $\nu(J_{\text{con}}(f)) > 0$.*

Proof. For $\rho > 0$ put $J_{\text{con}}(f, \rho) := \{x \in J_{\text{con}}(f) \mid \rho(x) \geq \rho\}$, so that $J_{\text{con}}(f) = \bigcup_{\rho > 0} J_{\text{con}}(f, \rho)$. For each $\rho_0 > 0$, Koebe Distortion Theorem implies that there is a constant $C > 1$ such that for every $x \in J_{\text{con}}(f, \rho_0)$ there are arbitrarily small $r > 0$, so that for some integer $n \geq 1$ we have,

$$\mu(B(x, 5r)) \leq C \exp(-np) r^t \quad \text{and} \quad \nu(B(x, r)) \geq C^{-1} \exp(-np') r^t. \tag{4.2}$$

Given a subset X of $J_{\rm con}(f, \rho_0)$, by Vitali's covering lemma, for every $r_0 > 0$ we can find a collection of pairwise disjoint balls $(B(x_j, r_j))_{j>0}$ and positive integers $(n_j)_{j>0}$, such that $x_j \in X$, $r_j \in (0, r_0)$, $X \subset \bigcup_{j>0} B(x_j, 5r_j)$ and such that for each $j > 0$ the inequalities (4.2) hold for $x := x_j$ and $r := r_j$ and $n = n_j$. Moreover, for each positive integer n_0 we may choose r_0 sufficiently small so that for each $j > 0$ we have $n_j \geq n_0$. Since by hypothesis $p' \leq p$, we obtain

$$\nu(X) \geq C^{-2} \exp(n_0(p - p'))\mu(X).$$

Suppose by contradiction that $p' < p$. Choose $\rho_0 > 0$ such that $\mu(J_{\rm con}(f, \rho_0)) > 0$ and set $X := J_{\rm con}(f, \rho_0)$. As in the inequality above $n_0 > 0$ can by taken arbitrarily large, we obtain a contradiction. So $p' = p$ and it follows that μ is absolutely continuous with respect to ν. □

Proof of Proposition 4.1. Let ν be a $(t, P(t))$-conformal measure ν for f supported on $J(f)$. By [PRLS04, Theorem A and Theorem A.7] there is at least one such measure, see also [Prz99]. So Lemma 4.2 implies that $p = P(t)$, and that μ is absolutely continuous with respect to ν.

In parts 1 and 2 we show that ν is proportional to μ. It follows in particular that μ is conformal. In part 3 we complete the proof of the proposition by showing the last statement of the proposition.

1. First note that $\nu' := \nu|_{\overline{\mathbb{C}} \setminus J_{\rm con}(f)}$ is a conformal measure for f of the same exponent as ν. Then Lemma 4.2 applied to $\nu = \nu'$ implies that, if ν' is non-zero, then $\nu'(J_{\rm con}(f)) > 0$. This contradiction shows that ν' is the zero measure and that ν is supported on $J_{\rm con}(f)$.

2. Denote by g the density μ with respect to ν. It is easy to see that satisfies $g \circ f \geq g$ on a set of full ν-measure. Let $\delta > 0$ be such that $\nu(\{g \geq \delta\}) > 0$. As ν is supported on $J_{\rm con}(f)$, there is a density point of $\{g \geq \delta\}$ for ν that belongs to $J_{\rm con}(f)$. Going to large scale and using $g \circ f \geq g$, we conclude that $\{g \geq \delta\}$ contains a ball of definite size, up to a set of ν-measure 0. It follows by the locally eventually onto property of f on $J(f)$ that the set $\{g \geq \delta\}$ has full measure with respect to ν. This implies that g is constant ν-almost everywhere and therefore that ν and μ are proportional. In particular μ is conformal

3. Suppose that X is a Borel subset of $\overline{\mathbb{C}}$ of positive measure with respect to μ and such that $f(X) \subset X$. Then the restriction $\mu|_X$ of μ to X is a $(t, P(t))$-sub-conformal measure supported on the conical Julia set. It follows that $\mu|_X$ is proportional to μ, and thus that $\mu|_X = \mu$ and that X has full measure with respect to μ. □

4.2. **Conformal measure.** Given $t, p \in \mathbb{R}$ we will say that a measure μ supported on the maximal invariant set $J(F)$ of F is (t, p)-*conformal for* F if for every Borel subset U of a connected component W of D we have

$$\mu(F(U)) = \exp(pm_W) \int_U |F'|^t \, d\mu.$$

In view of [MU03, Theorem 4.2.9] the hypothesis that

$$P(F, -t_0 \ln |F'| - P(t_0)m) = \mathscr{P}(t_0, P(t_0)) = 0,$$

implies that F admits a non-atomic $(t_0, P(t_0))$-conformal measure supported on $J(F)$. Therefore the assertions in Theorem C about conformal measures are direct consequences of Proposition 4.1, and of the following proposition.

Proposition 4.3. *Let F be the canonical induced map associated to a pleasant couple $(\widehat{V}, V)$ for f that satisfies property (*). Then for every $t \in (t_-, +\infty)$ and $p \in [P(t), +\infty)$, each (t,p)-conformal measure of F is in fact $(t, P(t))$-conformal, and it is the restriction to V of a non-atomic (t,p)-conformal measure of f supported on $J_{\text{con}}(f)$.*

Proof. The proof of this proposition is a straight forward generalization of that of [PRL07, Proposition B.2]. We will only give a sketch of the proof here.

Since $t > t_-$ there is a $(t, P(t))$-conformal measure $\widehat{\mu}$ for f whose topological support is equal to the whole Julia set of f (Proposition 2.2). Let $\mathfrak{D}_V$ be the collection of connected components of $\overline{\mathbb{C}} \setminus K(V)$. Notice that for each $W \in \mathfrak{D}_V$ we have $\widehat{\mu}(W) \sim \exp(-m_W P(t)) \operatorname{diam}(W)^t$, for an implicit constant independent of W.

Let μ be a (t,p)-conformal measure for F. For each $W \in \mathfrak{D}_V$ denote by $\phi_W : \widehat{V}^{c(W)} \to \widehat{W}$ the inverse of $f^{m_W}|_{\widehat{W}}$, and let μ_W be the measure supported on W, defined by

$$\mu_W(X) = \exp(-m_W p) \int_{f^{m_W}(X \cap W)} |\phi_W'|^t d\mu.$$

Clearly the measure $\sum_{W \in \mathfrak{D}_V} \mu_W$ is supported on $J_{\text{con}}(f)$, non-atomic, and for each $W \in \mathfrak{D}_V$ we have $\mu_W(\overline{\mathbb{C}}) \sim \exp(-m_W p) \operatorname{diam}(W)^t$. Since we also have $\widehat{\mu}(W) \sim \exp(-m_W P(t)) \operatorname{diam}(W)^t$, and $p \geq P(t)$, it follows that the measure $\sum_{W \in \mathfrak{D}_V} \mu_W$ is finite. In view of Proposition 4.1, to complete we just need to show that $\sum_{W \in \mathfrak{D}_V} \mu_W$ is (t,p)-sub-conformal for f. The proof of this fact is similar to what was done in [PRL07, Proposition B.2]. □

4.3. **Equilibrium state.** In the following lemma we use the hypothesis that the pressure function $\mathscr{P}$ is finite on a neighborhood of $(t,p) = (t_0, P(t_0))$.

Lemma 4.4. *Let μ be the unique $(t_0, P(t_0))$-conformal measure of F. Then there is $\varepsilon_0 > 0$ such that for every sufficiently large integer n we have*

$$\sum_{\substack{W \text{ connected component of } D \\ m_W \geq n}} \mu(W) \leq \exp(-\varepsilon_0 n),$$

In particular

$$\sum_{W \text{ connected component of } D} m_W \mu(W) < +\infty.$$

Proof. Since the function $\mathscr{P}$ is finite on a neighborhood of $(t,p)=(t_0,P(t_0))$, there is $\varepsilon>0$ such that $\mathscr{P}(t_0,P(t_0)-\varepsilon)<+\infty$. By [MU03, Proposition 2.1.9] this implies that,

$$\sum_W \exp(-(P(t_0)-\varepsilon)m_W)\sup\{|F'(z)|^{-t_0} \mid z\in W\}<+\infty.$$

As for each connected component W of D we have

$$\mu(W)\le C_0\exp(-P(t_0)m_W)\sup\{|F'(z)|^{-t_0} \mid z\in W\},$$

we obtain,

$$C_1:=\sum_{W \text{connected component of } D}\mu(W)\exp(\varepsilon m_W)<+\infty.$$

So for each $n\ge 1$ we have

$$\exp(\varepsilon n)\sum_{\substack{W \text{ connected component of } D\\ m_W\ge n}}\mu(W)\le C_1.$$

This shows that the lemma holds for each $\varepsilon_0\in(0,\varepsilon)$. □

Existence. It follows from standard considerations that F has an invariant measure ρ that is absolutely continuous with respect to the $(t_0,P(t_0))$-conformal measure μ of F, and that the density of ρ with respect to μ is bounded from below by a positive constant almost everywhere. This result can be found for example in [Gou04, §1], by observing that $F|_{J(F)}$ is a "Gibbs-Markov map". For a proof in a setting closer to ours, but that only applies to the case when V is connected, see [MU03, §6].

The measure

$$\widehat{\rho}:=\sum_W\sum_{j=0}^{m_W-1}f_*^j\rho|_W$$

is easily seen to be invariant by f, and Lemma 4.4 implies that it is finite. Furthermore this measure is absolutely continuous with respect to the $(t_0,P(t_0))$-conformal measure $\widehat{\mu}$ of f, and its density is bounded from below by a positive constant on a subset of V of full measure with respect to $\mu=\widehat{\mu}|_V$. It follows from the locally eventually onto property of Julia sets that the density of ρ with respect to $\widehat{\mu}$ is bounded from below by a positive constant almost everywhere; see for example [PRL07, §8] for details. As $\widehat{\mu}$ is ergodic (Proposition 4.1) it follows that $\widehat{\rho}$ is also ergodic.

We will show now that the probability measure $\widetilde{\rho}$ proportional to $\widehat{\rho}$ is an equilibrium state of f for the potential $-t_0\ln|f'|$. We first observe that ρ is an equilibrium state for F for the potential $-t_0\ln|F'|-P(t_0)m$ [MU03, Theorem 4.2.13 and Theorem 4.4.2]; that is we have

$$P(F,-t_0\ln|F'|-P(t_0)m)=h_\rho(F)-\int t_0\ln|F'|+P(t_0)m\,d\rho,$$

which is equal to 0 by hypothesis. By the generalized Abramov's formula [Zwe05, Theorem 5.1], we have $h_\rho(F) = h_{\widetilde{\rho}}(f)\widehat{\rho}(\overline{\mathbb{C}})$, and by definition of $\widehat{\rho}$ we have $\int m\, d\rho = \widehat{\rho}(\overline{\mathbb{C}})$. We thus obtain,

$$\begin{aligned} h_{\widetilde{\rho}}(f) = (\widehat{\rho}(\overline{\mathbb{C}}))^{-1} h_\rho(F) = (\widehat{\rho}(\overline{\mathbb{C}}))^{-1} \int t_0 \ln |F'| + P(t_0) m\, d\rho \\ = (\widehat{\rho}(\overline{\mathbb{C}}))^{-1} t_0 \int \ln |f'| d\widehat{\rho} + P(t_0) = t_0 \int \ln |f'| d\widetilde{\rho} + P(t_0). \end{aligned}$$

This shows that $\widetilde{\rho}$ is an equilibrium state of f for the potential $-t_0 \ln |f'|$.

Uniqueness. In view of [Dob08, Theorem 8], we just need to show that the Lyapunov exponent of each equilibrium state of f for the potential $-t_0 \ln |f'|$ is positive; see also [Led84].

Let $\widetilde{\rho}'$ be an equilibrium state of f for the potential $-t_0 \ln |f'|$. If f satisfies the Topological Collet-Eckmann Condition then it follows that the Lyapunov exponent of $\widetilde{\rho}'$ is positive, as in this case we have $\chi_{\inf} > 0$. Otherwise we have $\chi_{\inf} = 0$, and then $P(t_0) > 0$ by Proposition 2.1. It thus follows that $h_{\widetilde{\rho}'}(f) > 0$, and therefore that the Lyapunov exponent of $\widetilde{\rho}'$ is positive by Ruelle's inequality.

Statistical properties. When F satisfies the conclusions of Lemma 3.3, the statistical properties of $\widetilde{\rho}$ can be deduced from the tail estimate given by Lemma 4.4 above, using Young's results in [You99]. In the case when there is only one critical point in the Julia set one can apply these results directly, and in the general case one needs to consider the first return map of F to the set $V^{\widetilde{c}}$, where $\widetilde{c}$ is the critical point given by the conclusion of Lemma 3.3, as it was done in [PRL07, §8.2]. In the general case one could also apply directly the generalization of Young's result given in [Gou04, Théorème 2.3.6 and Remarque 2.3.7]. We omit the standard details.

4.4. **Analyticity of the pressure function.** By hypothesis for each t close to t_0 we have $\mathscr{P}(t, P(t)) = 0$. Since the function $\mathscr{P}$ is real analytic on a neighborhood of $(t_0, P(t_0))$ (Lemma 3.5), by the implicit function theorem it is enough to check that $\frac{\partial}{\partial p}\mathscr{P}|_{(t_0,P(t_0))} \neq 0$. This last number is equal to the integral of the (strictly negative) function $-m$, against the equilibrium measure of F for the potential $-t_0 \ln |F'| - P(t_0)m$ [MU03, Proposition 2.6.13], and it is therefore strictly negative.

5. Whitney decomposition of a pull-back

The purpose of this section is to introduce a Whitney type decomposition of a given pull-back of a pleasant couple. It is used to prove the key estimates in the next section.

5.1. **Dyadic squares.** Fix a square root i of -1 in $\mathbb{C}$ and identify $\mathbb{C}$ with $\mathbb{R} \oplus i\mathbb{R}$. For integers j, k and ℓ, the set

$$\left\{ x + iy \mid x \in \left[\frac{j}{2^\ell}, \frac{j+1}{2^\ell} \right], y \in \left[\frac{k}{2^\ell}, \frac{k+1}{2^\ell} \right] \right\},$$

will be called *dyadic square*. Note that two dyadic squares are either nested or have disjoint interiors. We define *a quarter* of a dyadic square Q as one of the four dyadic squares contained in Q and whose side length is one half of that of Q.

Given a dyadic square Q, denote by $\widehat{Q}$ the open square having the same center as Q, sides parallel to that of Q, and length twice as that of Q. Note in particular that for each dyadic square Q the set $\widehat{Q} \setminus Q$ is an annulus whose modulus is independent of Q; we denote this number by m_1.

5.2. **Primitive squares.** Let f be a rational map of degree at least two and fix $r_1 > 0$ sufficiently small so that for each critical value v of f in the Julia set of f there is a univalent map $\varphi_v : B(v, 9r_1) \to \mathbb{C}$ whose distortion is bounded by 2.

We say that a subset Q of $\overline{\mathbb{C}}$ is a *primitive square*, if there is $v \in \mathrm{CV}(f) \cap J(f)$ such that Q is contained in the domain of φ_v, such that $\varphi_v(Q)$ is a dyadic square, and such that $\widehat{\varphi_v(Q)}$ is contained in the image of φ_v. In this case we put $v(Q) := v$ and $\widehat{Q} := \varphi_v^{-1}\left(\widehat{\varphi_v(Q)}\right)$. We say that a primitive square Q_0 is *a quarter* of a primitive square Q, if $Q_0 \subset Q$ and if $\varphi_{v(Q)}(Q_0)$ is a quarter of $\varphi_{v(Q)}(Q)$. Note that each primitive square has precisely four quarters. Furthermore, each primitive square Q contained in $B(\mathrm{CV}, r_1)$ is contained in a primitive square Q' such that Q is a quarter of Q'.

Fix $\Delta \in (0, r_1)$. Then the *Whitney decomposition* associated to (the complement of) a subset F of $\overline{\mathbb{C}}$ is the collection $\mathscr{W}(F)$ of all those primitive squares Q such that $\mathrm{diam}(Q) < \Delta$, $\widehat{Q} \cap F = \emptyset$, and that are maximal with these properties. By definition two distinct elements of $\mathscr{W}(F)$ have disjoint interiors, and each point in $B(\mathrm{CV}(f) \cap J(f), 9r_1) \setminus F$ is contained in an element of $\mathscr{W}(F)$.

Lemma 5.1. *Let $\Delta \in (0, r_1)$, and let F be a finite subset of $\overline{\mathbb{C}}$. Then the following properties hold.*

1. *Let Q_0 be a primitive square contained in $B(\mathrm{CV}(f) \cap J(f), r_1)$ and such that $\mathrm{diam}(Q_0) \leq \Delta$. Then either Q_0 is contained in an element of $\mathscr{W}(F)$, or it contains an element Q of $\mathscr{W}(F)$ such that*
$$\mathrm{diam}(Q) \geq \tfrac{1}{4}(2 + 3\sqrt{\#F})^{-1}\,\mathrm{diam}(Q_0).$$
2. *For each $n \geq 2$ the number of those $Q \in \mathscr{W}(F)$ contained in $B(\mathrm{CV}(f) \cap J(f), r_1)$ and such that $\mathrm{diam}(Q) \in [2^{-(n+1)}\Delta, 2^{-n}\Delta]$ is less than $2599(\#F)$.*

Proof.

1. Let $n \geq 2$ be the least integer such that $(2^n - 2)^2 > 9(\#F)$, so that $2^n \leq 2\left(2 + 3\sqrt{\#F}\right)$. Put $Q'_0 := \varphi_{v(Q)}(Q_0)$ and denote by ℓ_0 the side length of Q_0'. For each element a of F in Q_0 choose a dyadic square Q_a whose side length equal to $2^{-n}\ell_0$ and that contains $\varphi_{v(Q)}(a)$. As there are $(2^n-2)^2$ squares of side length equal to $2^{-n}\ell_0$ contained in the interior of Q_0', and at most $9(\#F) < (2^n-2)^2$ of them intersect one of the squares $\bigcup_{a\in F} Q_a$, we conclude that there is at least one square Q' of side length equal to $2^{-n}\ell_0$ that is contained in the interior Q_0' and such that $\varphi_{v(Q)}^{-1}\left(\widehat{Q'}\right)$ is disjoint from F. It follows that the primitive square $Q := \varphi_{v(Q)}^{-1}(Q')$ is contained in an element of $\mathscr{W}(F)$. As,

$$\operatorname{diam}(Q) \geq \tfrac{1}{2}2^{-n}\operatorname{diam}(Q_0) \geq \tfrac{1}{4}(2+3\#\sqrt{F})^{-1}\operatorname{diam}(Q_0),$$

the desired assertion follows.

2. Let Q be an element of $\mathscr{W}(F)$ contained in $B(\mathrm{CV}(f)\cap J(f), r_1)$, and let Q' be a primitive square such that Q is a quarter of Q'. Then either $\operatorname{diam}(Q') > \Delta$ or $\widehat{Q'}$ intersects F. So, if $\operatorname{diam}(Q) \leq \frac{1}{4}\Delta$, then there is $a\in F$ contained in $\widehat{Q'}$, and therefore $\operatorname{diam}(Q) \geq \frac{1}{4}\operatorname{dist}(Q,a)$. So, if we let $n\geq 2$ be an integer such that $\operatorname{diam}(Q)\in[2^{-(n+1)}\Delta, 2^{-n}\Delta]$, then $Q\subset B(a, 5\cdot 2^{-n}\Delta)$. Since the area of Q is greater than or equal to $\frac{1}{8}\operatorname{diam}(Q)^2 \geq \frac{1}{32}4^{-n}\Delta^2$ and the area of $B(a,5\cdot 2^{-n}\Delta)$ is less than $25\pi 4^{-n}\Delta^2$, we conclude that there are at most $25\cdot 32\pi(\#F) < 2599(\#F)$ elements Q of $\mathscr{W}(F)$ satisfying $\operatorname{diam}(Q)\in[2^{-(n+1)}\Delta, 2^{-n}\Delta]$. $\square$

5.3. **Univalent squares.** For an integer $n\geq 0$ we will say that a subset Q of $\overline{\mathbb{C}}$ is *a univalent square of order* n, if there is a primitive square Q' such that Q is a connected component of $f^{-(n+1)}(Q')$, and such that f^{n+1} is univalent on the connected component of $f^{-(n+1)}(\widehat{Q'})$ containing Q. In this case we denote this last set by $\widehat{Q}$, and note that $\widehat{Q}\setminus Q$ is an annulus of modulus equal to m_1. It thus follows that there is a constant $K_0>1$ such that for every univalent square Q of order n and every $j=1,\ldots,n+1$, the distortion of f^j on Q is bounded by K_0.

Let $(\widehat{V},V)$ be a pleasant couple for f such that $f(\widehat{V})\subset B(\mathrm{CV}(f)\cap J(f), r_1)$. For a pull-back $\widetilde{W}$ of $\widehat{V}$, denote by $\ell(\widetilde{W})$ the number of those $j\in\{0,\ldots,m_{\widetilde{W}}\}$ such that $f^j(\widetilde{W})\subset\widehat{V}$. Moreover, let $\mathscr{W}(\widetilde{W})$ be the collection of all those univalent squares Q that are of order $m_{\widetilde{W}}$, such that $\widehat{Q}\subset\widetilde{W}$, such that $f^{m_{\widetilde{W}}}(Q)$ intersects V, and that are maximal with these properties. Note that for $Q\in\mathscr{W}(\widetilde{W})$ we have $v(Q)=f(c(\widetilde{W}))$. By definition every pair of distinct elements of $\mathscr{W}(\widetilde{W})$ have disjoint interiors. On the other hand, every point in $f^{m_{\widetilde{W}}}|_{\widetilde{W}}^{-1}\left(V^{c(\widetilde{W})}\right)\setminus\operatorname{Crit}(f^{m_{\widetilde{W}}+1})$ is contained in a an element of $\mathscr{W}(\widetilde{W})$, and for each $Q\in\mathscr{W}(\widetilde{W})$ the set $\widehat{Q}$ is disjoint from $\operatorname{Crit}(f^{m_{\widetilde{W}}+1})$.

Proposition 5.2. *Let f be a rational map of degree at least two and let $(\widehat{V},V)$ be a pleasant couple for f. Then there is a constant $C_0>0$ such that for* *every* $\xi \in (0,1)$ *the number of those* $Q \in \mathscr{W}(\widetilde{W})$ *such that*

$$\operatorname{diam}\left(f^{m_{\widetilde{W}}+1}(Q)\right) \geq \xi \operatorname{diam}(\widehat{V}^{c(\widetilde{W})})$$

is less than

$$2600 \deg(f)^{\ell(\widetilde{W})} \left(C_0 + \tfrac{1}{2}\ell(\widetilde{W}) \log_2 \ell(\widetilde{W}) + \ell(\widetilde{W}) \log_2(\xi^{-1})\right).$$

Proof. Put $c = c(\widetilde{W})$ and $v = f(c)$, and let $\xi_0 \in (0,1)$ be sufficiently small so that for each $z \in V^c$ the connected component of $f^{-1}\left(B(f(z), \xi_0 \operatorname{diam}(f(\widehat{V}^c)))\right)$ containing z is contained in $\widehat{V}^c$. Put $F = f^{m_{\widetilde{W}}+1}\left(\widetilde{W} \cap \operatorname{Crit}\left(f^{m_{\widetilde{W}}+1}\right)\right)$, $\Delta := \xi_0 \operatorname{diam}(f(\widehat{V}^c))$ and consider the Whitney decomposition $\mathscr{W}(F)$, as defined in §5.2. Note that $\#F \leq \ell(\widetilde{W})$.

1. We prove first that for every $Q \in \mathscr{W}(\widetilde{W})$ the primitive square $f^{m_{\widetilde{W}}+1}(Q) \subset f(\widehat{V}^c) \subset B(v, r_1)$ contains an element Q' of $\mathscr{W}(F)$ such that

$$\operatorname{diam}(Q') \geq \left(80\sqrt{\#\ell(\widetilde{W})}\right)^{-1} \xi_0 \operatorname{diam}\left(f^{m_{\widetilde{W}}+1}(Q)\right).$$

Let Q_0 be a primitive square contained in $f^{m_{\widetilde{W}}+1}(Q)$ such that

$$\tfrac{1}{4}\xi_0 \operatorname{diam}(f^{m_{\widetilde{W}}+1}(Q)) \leq \operatorname{diam}(Q_0) \leq \Delta.$$

By part 1 of Lemma 5.1 there is an element Q' of $\mathscr{W}(F)$ that either contains Q_0, or that it is contained in Q_0 and

$$\begin{aligned}\operatorname{diam}(Q') &\geq \tfrac{1}{4}\left(2 + 3\sqrt{\#F}\right)^{-1} \operatorname{diam}(Q_0) \\ &\geq \tfrac{1}{16}\left(2 + 3\sqrt{\#F}\right)^{-1} \xi_0 \operatorname{diam}\left(f^{m_W+1}(Q)\right).\end{aligned}$$

As $\#F \leq \ell(\widetilde{W})$ and $\ell(\widetilde{W}) \geq 1$, we just need to show that Q' is in fact contained in $f^{m_{\widetilde{W}}+1}(Q)$. Suppose by contradiction that this is not the case. Then it follows that Q' contains $f^{m_{\widetilde{W}}+1}(Q)$ strictly. Let $\widetilde{Q'}$ be the connected component of $f^{-(m_{\widetilde{W}}+1)}(Q')$ containing Q. By definition of ξ_0 we have that $f^{m_{\widetilde{W}}}(\widetilde{Q'})$ is contained in $\widehat{V}^c$, so $\widehat{\widetilde{Q'}}$ is contained in $\widetilde{W}$. On the other hand $f^{m_{\widetilde{W}}}(\widetilde{Q'})$ intersects V^c, because it contains $f^{m_{\widetilde{W}}}(Q)$ and this set intersects V^c. As by definition of $\mathscr{W}(F)$ the set $\widehat{Q'}$ is disjoint from F, it follows that $f^{m_{\widetilde{W}}+1}$ is univalent on $\widehat{\widetilde{Q'}}$. Thus, by definition of $\mathscr{W}(\widetilde{W})$, the univalent square $\widetilde{Q'}$ is contained in an element of $\mathscr{W}(\widetilde{W})$. But $Q \in \mathscr{W}(\widetilde{W})$ is strictly contained in $\widetilde{Q'}$, so we get a contradiction. This shows that Q' is in fact contained in $f^{m_{\widetilde{W}}+1}(Q)$ and completes the proof of the assertion.

2. For each $Q \in \mathscr{W}(\widetilde{W})$ choose an element Q' of $\mathscr{W}(F)$ satisfying the property described in part 1. Note that for each $Q'_0 \in \mathscr{W}(F)$ the number of those $Q \in \mathscr{W}(\widetilde{W})$ such that $Q' = Q'_0$ is less than or equal to $\deg(f)^{\ell(\widetilde{W})}$. As the area of a primitive square Q' is greater than or equal to $\frac{1}{8}\operatorname{diam}(Q')^2$, it follows that for each $\xi \in (0,1)$ the number of those $Q \in \mathscr{W}(\widetilde{W})$ satisfying $\operatorname{diam}(Q') \geq \xi \operatorname{diam}(f(\widehat{V}^c))$ is less than or equal to $8\pi\xi^{-2}\deg(f)^{\ell(\widetilde{W})}$.

Let $\xi \in (0, \frac{1}{4}\xi_0)$ be given and let n_0 be the least integer $n \geq 2$ such that $\xi \geq 2^{-n}80\sqrt{\ell(\widetilde{W})}$, so that $\xi < 2^{-(n_0-1)}80\sqrt{\ell(\widetilde{W})}$. If $Q \in \mathscr{W}(\widetilde{W})$ is such that $\operatorname{diam}\left(f^{m_{\widetilde{W}}+1}(Q)\right) \geq \xi \operatorname{diam}(\widehat{V}^c)$, then we have

$$\operatorname{diam}(Q') \geq \left(80\sqrt{\ell(\widetilde{W})}\right)^{-1} \xi_0 \operatorname{diam}\left(f^{m_{\widetilde{W}}+1}(Q)\right) \geq 2^{-n_0}\xi_0 \operatorname{diam}(\widehat{V}^c).$$

So part 2 of Lemma 5.1 implies that for each $n \geq 2$ the number of those $Q \in \mathscr{W}(\widetilde{W})$ such that

$$\operatorname{diam}(Q') \in \left[2^{-(n+1)}\xi_0 \operatorname{diam}(f(\widehat{V}^c)), 2^{-n}\xi_0 \operatorname{diam}(f(\widehat{V}^c))\right],$$

is less than $2599(\#F)\deg(f)^{\ell(\widetilde{W})} \leq 2599\ell(\widetilde{W})\deg(f)^{\ell(\widetilde{W})}$. So we conclude that the number of those $Q \in \mathscr{W}(\widetilde{W})$ such that $\operatorname{diam}\left(f^{m_{\widetilde{W}}+1}(Q)\right) \geq \xi \operatorname{diam}(\widehat{V}^c)$ is less than

$$\begin{aligned} &\deg(f)^{\ell(\widetilde{W})}\left(8\pi(\tfrac{1}{4}\xi_0)^{-2} + (n_0-2)2599\ell(\widetilde{W})\right) \\ &\leq \deg(f)^{\ell(\widetilde{W})}\left(8\pi(\tfrac{1}{4}\xi_0)^{-2} + 2599\ell(\widetilde{W})\left(\log_2(\xi^{-1}) + \log_2(80) + \tfrac{1}{2}\log_2(\ell(\widetilde{W}))\right)\right). \end{aligned}$$

This completes the proof of the lemma. □

6. The contribution of a pull-back

Fix a rational map f of degree at least two, and a pleasant couple $(\widehat{V}, V)$ for f. Recall that $\mathfrak{D}_V$ is the collection of all the connected components of $\overline{\mathbb{C}} \setminus K(V)$, and that for a pull-back $\widetilde{W}$ of $\widehat{V}$ we denote by $\mathfrak{D}_{\widetilde{W}}$ the collection of all the pull-backs W of V that are contained in $\widetilde{W}$, such that $f^{m_{\widetilde{W}}+1}$ is univalent on $\widehat{W}$, and such that $f^{m_{\widetilde{W}}+1}(W) \in \mathfrak{D}_V$; see §3.3. Furthermore, we denote by $\ell(\widetilde{W})$ the number of those $j \in \{0, \ldots, m_{\widetilde{W}}\}$ such that $f^j(\widetilde{W}) \subset \widehat{V}$.

The purpose of this section is to prove the following.

Proposition 6.1 (Key estimates)**.** *Let f be a rational map of degree at least two that is expanding away from critical points. Then for each sufficiently small pleasant couple $(\widehat{V}, V)$ for f the following properties hold.*

1. *For every $t_0 \in \mathbb{R}$, and every $t, p \in \mathbb{R}$ sufficiently close to t_0 and $P(t_0)$, respectively, we have*

$$\sum_{W \in \mathfrak{D}_V} \exp(-pm_W) \operatorname{diam}(W)^t < +\infty. \tag{6.1}$$

2. *Let $t, p \in \mathbb{R}$ be such that* (6.1) *holds and such that*

$$p > \max\{-t\chi_{\inf}, -t\chi_{\sup}\}.$$

Then for every $\varepsilon > 0$ such that

$$|t|\varepsilon < p - \max\{-t\chi_{\inf}, -t\chi_{\sup}\},$$

there is a constant $C_1 > 0$ such that for each pull-back $\widetilde{W}$ of $\widehat{V}$ we have

$$\sum_{W \in \mathfrak{D}_{\widetilde{W}}} \exp(-pm_W) \operatorname{diam}(W)^t$$
$$\leq C_1(\deg(f)+1)^{\ell(\widetilde{W})} \exp\left(-m_{\widetilde{W}}(p - \max\{-t\chi_{\inf}, -t\chi_{\sup}\} - |t|\varepsilon)\right).$$

The prove this proposition we start with the following lemma.

Lemma 6.2. *Let f be a rational map that is expanding away from critical points. Then for every compact and forward invariant subset K of the Julia set of f that is disjoint from the critical points of f and every $t > 0$ we have*

$$P(f|_K, -t\ln|f'|) < P(t).$$

Proof. By hypothesis f is uniformly expanding on K. Enlarging K if necessary we may assume that the restriction of f to K admits a Markov partition, see [PU02, Theorem 3.5.2 and Remark 3.5.3],[5] so that there is at least one equilibrium state μ for $f|_K$ with potential $-t\ln|f'|$.

We enlarge K with more cylinders to obtain a compact forward invariant subset K' of $J(f)$, so that f restricted to K' admits a Markov partition and so that the relative interior of K in K' is empty. It follows that μ cannot be an equilibrium measure for $f|_{K'}$ for the potential $-t\ln|f'|$, so we have

$$P(f|_K, -t\ln|f'|) = h_\mu(f) - t\int_K \ln|f'|d\mu < P(f|_{K'}, -t\ln|f'|) \leq P(t).$$

□

To prove Proposition 6.1, let f be a rational map of degree at least two, and let $(\widehat{V}, V)$ be a pleasant couple for f. We will define a constant $r_0 > 0$ as follows. If $\chi_{\inf} = 0$ we put $r_0 = \operatorname{dist}(\partial V, \operatorname{Crit}(f) \cap J(f))$. Suppose that $\chi_{\inf} > 0$. Then by [PRLS03, Main Theorem] there exists $r_0' > 0$ such that for every z_0 in $J(f)$, every $\varepsilon > 0$, every sufficiently large integer n, and every connected component W of $f^{-n}(B(z_0, r_0'))$, we have

$$\operatorname{diam}(W) \leq \exp(-n(\chi_{\inf} - \varepsilon)).$$

Then we put $r_0 = \min\{r_0', \operatorname{dist}(\partial V, \operatorname{Crit}(f) \cap J(f))\}$.

Given a subset Q of $\overline{\mathbb{C}}$ we define $n_Q \in \{0, 1, \ldots, +\infty\}$ as follows. If there are infinitely many integers n such that $\operatorname{diam}(f^n(Q)) < r_0$, then we put $n_Q = +\infty$. Otherwise we let n_Q be the largest integer $n \geq 0$ such that $\operatorname{diam}(f^n(Q)) < r_0$.

Lemma 6.3. *Let f be a rational map of degree at least two. Then for every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 1$ such that for each connected subset Q of $\overline{\mathbb{C}}$ that intersects the Julia set of f we have,*

$$C(\varepsilon)^{-1}\exp(-n_Q(\chi_{\sup} + \varepsilon)) \leq \operatorname{diam}(Q) \leq C(\varepsilon)\exp(-n_Q(\chi_{\inf} - \varepsilon))$$

[5] An analogous result in the case of diffeomorphisms is shown in [Fis06].

Proof. The inequality on the right holds trivially when $\chi_{\inf} = 0$, and when $\chi_{\inf} > 0$ it is given by the definition of $r_0 > 0$. The inequality on the left is a direct consequence of part 2 of Proposition 2.3. □

Proof of Proposition 6.1. Let $r_1 > 0$ be as in the definition of primitive squares in §5.2, and let $(\widehat{V}, V)$ be a pleasant couple for f such that $f(\widehat{V}) \subset B(\mathrm{CV}(f) \cap J(f), r_1)$. Furthermore, let $A_1 > 0$ and $K_1 > 1$, given by Koebe Distortion Theorem, such that for each pull-back W of V such that f^{m_W} is univalent on $\widehat{W}$ we have $\mathrm{diam}(W) \le A_1 \,\mathrm{dist}(W, \partial \widehat{W})$, and such that for each $j = 1, \ldots, m_W$ the distortion of f^j on W is bounded by K_1.

1. Note that it is enough to show that there are $t < t_0$ and $p < P(t_0)$ for which (6.1) holds.

Let V' be a sufficiently small neighborhood of $\mathrm{Crit}(f) \cap J(f)$ contained in V, so that for each $c \in \mathrm{Crit}(f) \cap J(f)$ the set

$$K' = \{z \in J(f) \mid \text{ for every } n \ge 0,\ f^n(z) \notin V'\}$$

intersects V^c. By Lemma 6.2 we have $P(f|_{K'}, -t_0 \ln |f'|) < P(t_0)$. Let $t < t_0$ and $p < P(t_0)$ be sufficiently close to t_0 and $P(t_0)$, respectively, so that $p > P(f|_{K'}, -t \ln |f'|)$.

For each $c \in \mathrm{Crit}(f) \cap J(f)$ choose a point $z(c)$ in $K' \cap V^c$ and for each univalent pull-back W of V let z_W be the unique point in $f^{-m_W}(z(c(W)))$ contained in W. Note that $z_W \in K'$ and that there is a distortion constant $C > 0$ independent of W such that $\mathrm{diam}(W) \le C |(f^{m_W})'(z_W)|^{-1}$. On the other hand, when $W \in \mathfrak{D}_V$ we have $z_W \in K'$.

Since by hypothesis the restriction of f to K' is uniformly expanding, we have

$$\begin{aligned}\limsup_{n \to +\infty} \tfrac{1}{n} \ln \sum_{\substack{W \in \mathfrak{D}_V \\ m_W = n}} |(f^n)'(z_W)|^{-t} \\ \le \limsup_{n \to +\infty} \tfrac{1}{n} \ln \sum_{c \in \mathrm{Crit}(f) \cap J(f)} \sum_{z \in K' \cap f^{-n}(z(c))} |(f^n)'(z)|^{-t} \\ \le P(f|_{K'}, -t \ln |f'|),\end{aligned}$$

hence

$$\begin{aligned}C_2 := \sum_{W \in \mathfrak{D}_V} \exp(-m_W p) \,\mathrm{diam}(W)^t \\ \le C^{|t|} \sum_{W \in \mathfrak{D}_V} \exp(-m_W p) |(f^{m_W})'(z_W)|^{-t} < +\infty.\end{aligned}$$

2.1. Put $C_3 := \min\{c \in \mathrm{Crit}(f) \cap J(f) \mid \mathrm{dist}(z(c), \partial V^c)/\,\mathrm{diam}(V^c)\}$, and observe that for each pull-back W' of V such that $\widehat{W'}$ is a univalent pull-back of $\widehat{V}$, we have $\mathrm{dist}(z_{W'}, \partial W') \ge C_3 K_1^{-1} \,\mathrm{diam}(W')$. We will show that for each pull-back $\widetilde{W}$ of $\widehat{V}$, for each $Q \in \mathscr{W}(\widetilde{W})$, and each $W \in \mathfrak{D}_{\widetilde{W}}$ such that $z_W \in Q$, we have

$$\operatorname{diam}(f^{m_{\widetilde{W}}+1}(W)) \leq 8C_3^{-1}K_1 \operatorname{diam}\left(f^{m_{\widetilde{W}}+1}(Q)\right).$$

Put $Q' = f^{m_{\widetilde{W}}+1}(Q)$ and $W' = f^{m_{\widetilde{W}}+1}(W)$, and suppose by contradiction that $\operatorname{diam}(W') > 8C_3^{-1}K_1 \operatorname{diam}(Q')$. Observe that Q' is a primitive square contained in $B(f(c(\widetilde{W})), r_1)$ and that $W' \in \mathfrak{D}_V$. So there is a primitive square Q'_0 such that Q' is a quarter of Q'_0. We have

$$\operatorname{diam}(\widehat{Q'}_0) \leq 8 \operatorname{diam}(Q') < C_3 K_1^{-1} \operatorname{diam}(W') \leq \operatorname{dist}(z_{W'}, \partial W').$$

Since by hypothesis $z_W \in Q$, we have $z_{W'} = f^{m_{\widetilde{W}}+1}(z_W) \in Q' \subset \widehat{Q'}_0$, so the last inequality implies that $\widehat{Q'}_0 \subset W'$. But $f^{m_{\widetilde{W}}+1}$ is univalent on W, so the connected component Q_0 of $f^{-(m_{\widetilde{W}}+1)}(Q'_0)$ containing Q is a univalent square of order $m_{\widetilde{W}}$ satisfying $\widehat{Q}_0 \subset \widetilde{W}$, that contains Q strictly. This contradicts the hypothesis that $Q \in \mathscr{W}(\widetilde{W})$.

2.2. We will now show that there is a constant $C_4 > 0$ such that for each pull-back $\widetilde{W}$ of $\widehat{V}$ and each $Q \in \mathscr{W}(\widetilde{W})$ we have

$$\sum_{\substack{W \in \mathfrak{D}_{\widetilde{W}} \\ z_W \in Q}} \exp(-pm_W) \operatorname{diam}(W)^t \leq C_4 \exp(-pn_Q) \operatorname{diam}(Q)^t. \tag{6.2}$$

Let $\widetilde{W}$ be a pull-back of $\widehat{V}$ and let $Q \in \mathscr{W}(\widetilde{W})$. Put $Q' = f^{m_{\widetilde{W}}+1}(Q)$, and let B be a ball whose center belongs to Q' and of radius equal to $(8C_3^{-1}K_1 + 1)\operatorname{diam}(Q')$. By part 2.1, for each $W \in \mathfrak{D}_{\widetilde{W}}$ such that $z_W \in Q$ we have $f^{m_{\widetilde{W}}+1}(W) \subset B$. Since the distortion of $f^{m_{\widetilde{W}}+1}$ is bounded by K_0 on Q, and by K_1 on each element of $\mathfrak{D}_{\widetilde{W}}$, we obtain,

$$\begin{aligned}\sum_{\substack{W \in \mathfrak{D}_{\widetilde{W}} \\ z_W \in Q}} &\exp(-pm_W) \operatorname{diam}(W)^t \\ &\leq \exp(-p(m_{\widetilde{W}}+1))(K_0K_1)^{|t|} \left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^t \cdot \\ &\qquad \cdot \sum_{\substack{W' \in \mathfrak{D}_V \\ W' \subset B}} \exp(-pm_{W'}) \operatorname{diam}(W')^t.\end{aligned}$$

If there is no $W \in \mathfrak{D}_{\widetilde{W}}$ such that $z_W \in Q$, then there is nothing to prove. So we assume that there is an element W_0 of $\mathfrak{D}_{\widetilde{W}}$ such that $z_{W_0} \in Q$. Then Q', and hence B, intersects K', as it contains the point $z_{f^{m_{\widetilde{W}}+1}(W_0)}$. Since by hypothesis the restriction of f to K' is uniformly expanding, there is $n_0 \geq 0$ independent of $\widetilde{W}$, such that $n_{Q'} \leq n_B + n_0$ and such that there is an integer $n'_B \geq 0$ satisfying $|n'_B - n_B| \leq n_0$, such that $f^{n'_B}$ is univalent on B and has distortion bounded by 2 on this set. We have $|n_{Q'} - n'_B| \leq 2n_0$, so there is a constant $C_5 > 0$ independent of B such that $\operatorname{diam}(f^{n'_B}(B)) > C_5$. So, if we put

$$C_6 := \exp(|p| 2n_0) \left(2C_5^{-1} 2(8C_3^{-1} K_1 + 1)\right)^{|t|} C_2,$$

then we have

$$\sum_{\substack{W' \in \mathfrak{D}_V \\ W' \subset B}} \exp(-p m_{W'}) \operatorname{diam}(W')^t$$
$$\leq \exp(-p n'_B) 2^{|t|} \left(\frac{\operatorname{diam}(B)}{\operatorname{diam}(f^{n'_B}(B))} \right)^t \cdot \sum_{\substack{W'' \in \mathfrak{D}_V \\ W'' \subset f^{n'_B}(B)}} \exp(-p m_{W''}) \operatorname{diam}(W'')^t$$
$$\leq \exp(-p n_{Q'}) \exp(-|p| 2n_0) (2C_5^{-1})^{|t|} \operatorname{diam}(B)^t C_2$$
$$\leq C_6 \exp(-p n_{Q'}) \operatorname{diam}(Q')^t.$$

Inequality (6.2) with constant $C_4 := C_6 (K_0 K_1)^{|t|}$, is then a direct consequence of the last two displayed (chains of) inequalities.

2.3. We will now complete the proof of the proposition. For each $Q \in \mathscr{W}(\widetilde{W})$ put $Q' := f^{m_{\widetilde{W}}+1}(Q)$. Let $Q \in \mathscr{W}(\widetilde{W})$ such that there is $W \in \mathfrak{D}_{\widetilde{W}}$ satisfying $z_W \in Q$. As this last point is in the Julia set of f, by Lemma 6.3 we have

$$\operatorname{diam}(Q)^t \leq C(\varepsilon)^{|t|} \exp\left(n_Q (\max\{-t\chi_{\sup}, -t\chi_{\inf}\} + |t|\varepsilon)\right).$$

Since the elements of $\mathscr{W}(\widetilde{W})$ cover $\widetilde{W} \setminus \operatorname{Crit}\left(f^{m_{\widetilde{W}}+1}\right)$, if we put

$$\gamma := \exp(-p + \max\{-t\chi_{\sup}, -t\chi_{\inf}\} + |t|\varepsilon) \in (0, 1),$$

then by summing over $Q \in \mathscr{W}(\widetilde{W})$ in (6.2) we obtain

$$\sum_{W \in \mathfrak{D}_{\widetilde{W}}} \exp(-p m_W) \operatorname{diam}(W)^t$$
$$\leq C_4 C(\varepsilon) \sum_{\substack{Q \in \mathscr{W}(\widetilde{W}) \\ Q \cap J(f) \neq \emptyset}} \gamma^{n_Q} = C_4 C(\varepsilon) \gamma^{m_{\widetilde{W}}+1} \sum_{\substack{Q \in \mathscr{W}(\widetilde{W}) \\ Q \cap J(f) \neq \emptyset}} \gamma^{n_{Q'}}.$$

To estimate this last number, observe that by Lemma 6.3, for each $Q \in \mathscr{W}(\widetilde{W})$ intersecting the Julia set of f we have

$$\operatorname{diam}(Q') \geq C(\varepsilon)^{-1} \exp(-n_{Q'}(\chi_{\sup} + \varepsilon)).$$

So, if we put $\widetilde{\gamma} = \gamma^{\frac{\ln 2}{\chi_{\sup}+\varepsilon}}$, $C_7 = \widetilde{\gamma}^{-\log_2 C(\varepsilon) - \log_2 \operatorname{diam}(\widehat{V}^{c(\widetilde{W})})}$ and for each $Q \in \mathscr{W}(\widetilde{W})$ we put $\xi(Q') = \operatorname{diam}(Q') / \operatorname{diam}(\widehat{V}^{c(\widetilde{W})})$, then we have $\gamma^{n_{Q'}} \leq$

$C_7 \widetilde{\gamma}^{-\log_2 \xi(Q')}$. So Proposition 5.2 implies that,

$$\sum_{\substack{Q \in \mathscr{W}(\widetilde{W}) \\ Q \cap J(f) \neq \emptyset}} \gamma^{n_{Q'}} \leq C_7 \sum_{\substack{Q \in \mathscr{W}(\widetilde{W}) \\ Q \cap J(f) \neq \emptyset}} \widetilde{\gamma}^{-\log_2 \xi(Q')} \leq$$

$$\leq 2600 C_7 \deg(f)^{\ell(\widetilde{W})} \left(C_0 + \tfrac{1}{2} \ell(\widetilde{W}) \log_2 \ell(\widetilde{W}) + \ell(\widetilde{W}) \sum_{n \geq 0} \widetilde{\gamma}^n \right).$$

This completes the proof of the proposition. □

7. Proof of Theorem A

The purpose of this section is to prove the following stronger version of Theorem A. Recall that each nice couple is pleasant and satisfies property (*), see §3.4.

Theorem A'. *Let f be a rational map of degree at least two that is expanding away from critical points, and that has arbitrarily small pleasant couples having property (*). Then following properties hold.*

- **Analyticity of the pressure function:** *The pressure function of f is real analytic on (t_-, t_+), and linear with slope $-\chi_{\sup}(f)$ (resp. $-\chi_{\inf}(f)$) on $(-\infty, t_-]$ (resp. $[t_+, +\infty)$).*
- **Equilibrium states:** *For each $t_0 \in (t_-, t_+)$ there is a unique equilibrium state of f for the potential $-t_0 \ln |f'|$. Furthermore this measure is ergodic and mixing.*

Throughout the rest of this section we fix a rational map f and $t_0 \in (t_-, t_+)$ as in the statement of this theorem. Recall that by Proposition 2.1 we have $P(t_0) > \max\{-t_0 \chi_{\inf}, -t_0 \chi_{\sup}\}$. Put

$$\gamma_0 = \exp\left(-\tfrac{1}{2}(P(t_0) - \max\{-t_0 \chi_{\inf}, -t_0 \chi_{\sup}\})\right) \in (0, 1),$$

and choose $L \geq 0$ sufficiently large so that

$$(2L(\# \operatorname{Crit}(f) \cap J(f)))^{2/L} (\deg(f) + 1)^{1/L} \gamma_0 < 1. \tag{7.1}$$

Let $(\widehat{V}, V)$ be a pleasant couple for f that is sufficiently small so that for each $\ell \in \{1, \ldots, L\}$ the set $f^\ell(\operatorname{Crit}(f) \cap J(f))$ is disjoint from $\widehat{V}$ (recall that our standing convention is that no critical point of f in its Julia set is mapped to a critical point under forward iteration.) We assume furthermore that $(\widehat{V}, V)$ has property (*).

We show in §7.1 that the pressure function $\mathscr{P}$ defined in §3.4 is finite on a neighborhood of $(t, p) = (t_0, p_0)$, and we show in §7.2 that for each t close to t_0 the function $\mathscr{P}$ vanishes at $(t, p) = (t, P(t))$. Then Theorem A' follows from Theorem C.

7.1. **The function $\mathscr{P}$ is finite on a neighborhood of $(t,p)=(t_0,P(t_0))$.** We will use the following lemma form [PRL07].

Lemma 7.1 ([PRL07], Lemma 7.1)**.** *Let f be a rational map, let $(\widehat{V},V)$ be a pleasant couple for f, and let $L\geq 1$ be the least integer such that $f^L(\mathrm{Crit}(f)\cap J(f))$ intersects $\widehat{V}$. Then for each positive integer n, there are at most $(2L(\#\,\mathrm{Crit}(f)\cap J(f)))^{2n/L}$ bad pull-backs of $\widehat{V}$ of order n.*

By the considerations in §3.4, to show that $\mathscr{P}$ is finite on a neighborhood of $(t,p)=(t_0,p_0)$ we just need to show that there are $t<t_0$ and $p<P(t_0)$ such that

$$\sum_{W\in\mathfrak{D}}\exp(-pm_W)\,\mathrm{diam}(W)^t<+\infty. \tag{7.2}$$

Let $t<t_0$ and $p<P(t_0)$ be given by part 1 of Proposition 6.1. Taking t and p closer to t_0 and $P(t_0)$, respectively, we assume that there is $\varepsilon>0$ sufficiently small so that

$$p-\max\{-t\chi_{\inf},-t\chi_{\sup}\}-|t|\varepsilon>\tfrac{1}{2}(P(t_0)-\max\{-t_0\chi_{\inf},-t\chi_{\sup}\}),$$

and put

$$\gamma:=\exp(-p+\max\{-t\chi_{\inf},-t\chi_{\sup}\}+|t|\varepsilon)\in(0,\gamma_0).$$

For each $c\in\mathrm{Crit}(f)\cap J(f)$ we have, by applying part 2 of Proposition 6.1 to $\widetilde{W}=\widehat{V}^c$,

$$\sum_{W\in\mathfrak{D}_{\widehat{V}^c}}\exp(-pm_W)\,\mathrm{diam}(W)^t\leq C_1(\deg(f)+1). \tag{7.3}$$

Since for each pull-back $\widetilde{W}$ of $\widehat{V}$ we have $\ell(\widetilde{W})\leq 1+\frac{n}{L}$, using part 2 of Proposition 6.1 again we obtain

$$\begin{aligned}\sum_{\widetilde{W}\text{ bad pull-back of }\widehat{V}}\ \sum_{W\in\mathfrak{D}_{\widetilde{W}}}&\exp(-pm_W)\,\mathrm{diam}(W)^t\\ &\leq C_1\sum_{\widetilde{W}\text{ bad pull-back of }\widehat{V}}(\deg(f)+1)^{\ell(\widetilde{W})}\gamma^{m_{\widetilde{W}}}\\ &\leq C_1(\deg(f)+1)\sum_{n\geq 1}\Big((2L(\#\,\mathrm{Crit}(f)\cap J(f)))^{2/L}(\deg(f)+1)^{1/L}\gamma\Big)^n\\ &<+\infty.\end{aligned}$$

As $\gamma\in(0,\gamma_0)$, we have by (7.1) that the sum above is finite. Then (7.2) follows from (7.3) and Lemma 3.4.

7.2. **For each t close to t_0 we have $\mathscr{P}(t,P(t))=0$.** In view of Lemma 3.5 we just need to show that for each t close to t_0 we have $\mathscr{P}(t,P(t))\geq 0$. Suppose by contradiction that in each neighborhood of t_0 we can find t such that $\mathscr{P}(t,P(t))<0$. As $\mathscr{P}$ is finite on a neighborhood of $(t_0,P(t_0))$, it follows that $\mathscr{P}$ is continuous at this point (Lemma 3.5), so we can find t close to t_0 and

$$p \in (\max\{-t\chi_{\inf}, -t\chi_{\sup}\}, P(t)),$$

such that $\mathscr{P}(t,p) < 0$, and such that the conclusion of part 1 of Proposition 6.1 holds for this values of t and p.

We show below that for $z_0 \in V$ for which all (2.1), (2.2), and (2.3) hold, the double sum

$$T_f(p, z_0) := 1 + \sum_{n \geq 1} \exp(-pn) \sum_{y \in f^{-n}(z_0)} |(f^n)'(y)|^{-t}$$

is finite. This contradicts the fact that $p < P(t)$, and shows that $\mathscr{P}(t, P(t)) = 0$.

1. Given $z_0 \in V$ and an integer n we will say that an element y of $f^{-n}(z_0)$ is a *univalent* (resp. *bad*) *iterated preimage of* z_0 *of order* n, if the pull-back of $\widehat{V}$ by f^n containing y is univalent (resp. bad). For $y \in f^{-n}(z_0)$ there are three cases: y is univalent, bad, or there is $m \in \{1, \ldots, n-1\}$ such that $f^m(y)$ is a bad iterated preimage of z_0 of order $n - m$ and y is a univalent iterated preimage of $f^m(y)$ of order m. Therefore, if for each $p \in \mathbb{R}$ and $w \in V$ we put

$$U(p, w) := 1 + \sum_{n \geq 1} \exp(-pn) \sum_{y \in f^{-n}(w), \text{ univalent}} |(f^n)'(y)|^{-t},$$

then we have

$$T_f(p, z_0) = U(p, z_0) + \sum_{n \geq 1} \exp(-pn) \sum_{w \in f^{-n}(z_0), \text{ bad}} |(f^n)'(w)|^{-t} U(p, w). \tag{7.4}$$

2. We denote by L_V the first entry map to V, which is defined on the set of points $y \in \overline{\mathbb{C}} \setminus V$ having a good time, by $L_V(y) = f^{m(y)}(y)$. Note that for each $w_0 \in V$, each positive integer n and each univalent iterated preimage $y \in f^{-n}(w_0)$ of w_0 of order n, we have that $m(y) \leq n$ and that $L_V(y) \in V$ is a univalent iterated preimage of w_0 of order $n - m(y)$. Moreover note for each $k \geq 1$, each element of $F^{-k}(w_0)$ is a univalent iterated preimage of w_0, and conversely, that for each univalent iterated preimage y of w_0 there is a positive integer k such that F^k is defined at y and $F^k(y) = w_0$ (see §3.2). Therefore, if for $z \in V$ and $p \in \mathbb{R}$ we put

$$L(p, z) := 1 + \sum_{y \in L_V^{-1}(z_0)} \exp(-pm(y)) |(f^{m(y)})'(y)|^{-t},$$

then we have,

$$U(p, w_0) = L(p, w_0) + \sum_{k \geq 1} \sum_{y \in F^{-k}(w_0)} \exp(-pm(y)) |(F^k)'(y)|^{-t} L(p, y). \tag{7.5}$$

3. Since $\mathscr{P}(t,p) < 0$, for each $w \in V$ the double sum

$$T_F(p,w) := \sum_{k\geq 1} \sum_{y\in F^{-k}(w)} \exp(-pm(y))|(F^k)'(y)|^{-t},$$

is finite.

Fix $z_0 \in V$ such that all, (2.1), (2.2), and (2.3) hold. Notice that a bad pull-back of $\widehat{V}$ of order n contains at most $\deg(f)^{n/L}$ bad iterated preimages of z_0 of order n. Thus Lemma 7.1 implies that for each positive integer n and each $z_0 \in V$ there are at most

$$\left((2L\# \operatorname{Crit}(f) \cap J(f))^2 \deg(f)\right)^{n/L}$$

bad iterated preimages of z_0 order n. As $p > \max\{-t\chi_{\inf}, -t\chi_{\sup}\}$, by (7.1), it follows that

$$\sum_{n\geq 1} \exp(-pn) \sum_{w\in f^{-n}(z_0), \text{ bad}} |(f^n)'(w)|^{-t} < +\infty.$$

So by (7.4), to prove that $T_f(p,z_0)$ is finite it is enough to prove that the supremum $\sup_{w\in V} U(p,w)$ is finite. Note that there is a distortion constant $C > 0$, such that for each $c \in \operatorname{Crit}(f) \cap J(f)$ and each $w, w' \in V$, we have $U(p,w) \leq C^t U(p,w')$. Thus, to prove that $T_f(p,z_0)$ is finite it is enough to prove that for each $w \in V$ we have $U(p,w) < +\infty$.

By the conclusion of part 1 of Proposition 6.1, for each $z \in V$ the sum $L(p,z)$ is finite. By bounded distortion it follows that

$$C' := \sup_{z\in V} L(p,z) < +\infty,$$

and by (7.5) in follows that for each $w \in V$ we have $U(p,w) \leq C' T_F(p,w) < +\infty$. This shows that $T_f(p,z_0)$ is finite, and completes the proof that for each t close to t_0 we have $\mathscr{P}(t, P(t)) = 0$.

8. On equilibrium states after the freezing point

The purpose of this section is to prove Theorem B. We begin with the following general lemma.

Lemma 8.1. *Let f be a rational map whose freezing point t_+ is finite, and let $t \in (t_+, +\infty)$. Then an invariant probability measure μ supported on the Julia set of f is an equilibrium state of f for the potential $-t\ln|f'|$ if, and only if, $\int \ln|f'|d\mu = \chi_{\inf}$. Moreover, for such μ we have $h_\mu(f) = 0$.*

Proof. By definition of P and t_+, for each $t \in [t_+, +\infty)$ we have

$$h_\mu - t\chi_\mu \leq P(t) = -t\chi_{\inf}.$$

Thus, if μ satisfies $\chi_\mu = \chi_{\inf}$, then we have $h_\mu = 0$ and μ is an equilibrium state of f for the potential $-t\ln|f'|$.

On the other hand, suppose that for some $t_0 \in (t_+, +\infty)$ the measure μ is an equilibrium state of f for the potential $-t_0 \ln|f'|$. That is, $h_\mu - t_0\chi_\mu =$

$P(t_0) = -t_0 \chi_{\inf}$. As for each $t \in (t_+, +\infty)$ we have $h_\mu - t\chi_\mu \le P(t) = -t\chi_{\inf}$, we conclude that $\chi_\mu = \chi_{\inf}$ and $h_\mu = 0$. □

Lemma 8.2. *Let f be a rational map of degree at least two and let μ be an ergodic invariant measure supported on $J(f)$ and such that the Lyapunov exponent χ_μ of μ is positive. Then for each $t > 0$ we have $P(t) > -t\chi_\mu$.*

We will now give the proof of Theorem B assuming this lemma.

Proof of Theorem B given Lemma 8.2. Let f be a rational map satisfying the TCE condition and whose freezing point t_+ is finite. To prove part 1 of the theorem, observe that if μ is an invariant probability measure supported on $J(f)$, then we have $\chi_\mu > 0$ [PRLS03, Main Theorem], so Lemma 8.2 implies that $-t_+\chi_{\inf} = P(t_+) > -t_+\chi_\mu$. That is, we have $\chi_\mu > \chi_{\inf}$ as wanted.

In view of Lemma 8.1 part 2 follows from part 1. So it remains to prove part 3. Since the Lyapunov exponent of each invariant measure supported on the Julia set is positive [PRLS03, Main Theorem], by [Dob08, Corollary 11] there is at most one equilibrium measure of f for the potential $-t_+ \ln |f'|$, see also [Led84]. If such a measure μ exists, then we have

$$h_\mu(f) = P(t_+) + t_+\chi_\mu = t_+(\chi_\mu - \chi_{\inf}) > 0,$$

and on the other hand,

$$\lim_{t \to (t_+)^-} P'(t) \le -\chi_\mu < -\chi_{\inf} = \lim_{t \to (t_+)^+} P'(t).$$

□

The proof of Lemma 8.2 occupies the rest of this section. We first consider the following general fact.

Lemma 8.3. *Let $(Z, \mathcal{F}, \nu)$ be a measure space and let $T : Z \to Z$ be an ergodic measure preserving map. Then for each integrable function $\varphi : Z \to \mathbb{R}$ such that $\int \varphi d\mu = 0$ there is a set of full measure of $x \in Z$ such that*

$$\liminf_{n \to +\infty} \sum_{j=0,\ldots,n-1} \varphi \circ T^j(x) \le 0.$$

Proof. It is enough to show that for each $k \ge 1$ and $\varepsilon > 0$ the set

$$E := \left\{ x \in Z \mid \text{for each } n \ge k \text{ we have } \sum_{j=0,\ldots,n-1} \varphi \circ T^j(x) > \varepsilon \right\}$$

has measure 0 with respect to ν. Suppose by contradiction that there is $k \ge 1$ and $\varepsilon > 0$ for which the set E has positive measure with respect to ν. As T is ergodic it follows that there $x \in E$ such that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0,\ldots,n-1} \varphi \circ T^j(x) = 0,$$

and such that there exist $P > 1$ and an increasing sequence of positive integers $(n_\ell)_{\ell \geq 1}$, such that for each $\ell \geq 1$ we have $n_\ell \leq P\ell$ and $T^{n_\ell}(x) \in E$. Putting $n_0 = 0$ it follows that for all $m \geq 0$ we have $n_{(m+1)k} \geq n_{mk} + k$, and therefore for each $m_0 \geq 1$ we have

$$\sum_{j=0,\ldots,n_{m_0 k}-1} \varphi \circ T^j(x) = \sum_{m=0,\ldots,m_0-1} \sum_{j=n_{mk},\ldots,n_{(m+1)k}-1} \varphi \circ T^j(T^{n_{mk}}(x)) \geq m_0 \varepsilon \geq (Pk)^{-1} n_{m_0 k} \varepsilon.$$

But this implies that $0 = \int \varphi d\mu \geq (Pk)^{-1}\varepsilon$. This contradiction finishes the proof of the lemma. □

We will say that a sequence of pairwise distinct holomorphic maps $(\phi_\ell)_{\ell \geq 1}$ is an *Iterated Function System* or an *IFS*, if there are $z_0 \in \overline{\mathbb{C}}$ and $\rho > 0$ such that for each $\ell \geq 1$ the map ϕ_ℓ is defined on $B(z_0, \rho)$ and takes images in $B(z_0, \rho/2)$. We will say that such an iterated function system is *free* if for every pair of distinct sequences $\ell_1, \ldots, \ell_k$ and $\ell'_1, \ldots, \ell'_{k'}$ the maps $\phi_{\ell_1} \circ \cdots \circ \phi_{\ell_k}$ and $\phi_{\ell'_1} \circ \cdots \circ \phi_{\ell'_{k'}}$ are distinct. Moreover, we will say that such an IFS is *generated by* f, if there is a sequence of positive integers $(m_\ell)_{\ell \geq 1}$ such that $f^{m_\ell} \circ \phi_\ell$ is the identity on $B(z_0, \rho)$. As by definition these maps are pairwise distinct, in this case we have $m_\ell \to +\infty$ as $\ell \to +\infty$.

Let (Z, T) be the natural extension of $(\overline{\mathbb{C}}, f)$ and denote by $\pi : Z \to X$ the corresponding projection, so that $f \circ \pi = \pi \circ F$. We identify each point x of Z with the sequence $(\pi \circ F^n(x))_{n \in \mathbb{Z}}$.

Lemma 8.4. *Let $z_0 \in J(f)$, $\rho > 0$ and let $(\phi_\ell)_{\ell \geq 1}$ be an IFS generated by f that is defined on $B(z_0, \rho)$ and such that the sequence $(\phi_\ell(z_0))_{\ell \geq 1}$ has an accumulation point that is not in the forward orbit of a critical point of f. Then there is $\widehat{C} > 0$, a sequence $(\ell_k)_{k \geq 1}$, and a free IFS $(\widehat{\phi}_\ell)_\ell$ generated by f that is defined on $B(z_0, \rho/2)$ and such that for every $k \geq 1$ we have*

$$|\widehat{\phi}'_k(z_0)| \geq \widehat{C} |\phi'_{\ell_k}(z_0)|.$$

Proof. Suppose first that the sequence $(\phi_\ell(z_0))_{\ell \geq 1}$ is not contained in a finite number of elements of Z. Then we can choose a sequence $(\ell_k)_{k \geq 1}$ such that for every $k, k', n \geq 1$ we have $f^n(\phi_{\ell_k}(z_0)) \neq \phi_{\ell_{k'}}(z_0)$. It follows that $(\phi_{\ell_K})_{k \geq 1}$ is an IFS generated by f that is free. So we are reduced to the case when the sequence $(\phi_\ell(z_0))_{\ell \geq 1}$ is contained in a finite number of elements of Z. Replacing the IFS by a subsequence if necessary we assume that this sequence is contained in a single element $(z_n)_{n \in \mathbb{Z}}$ of Z and that it converges to a point $\widehat{z}_0$ that is not in the forward orbit of a critical point of f. It follows that for every $n \geq 1$ there is an inverse $\widetilde{\phi}_n$ of f^n defined on $B(z_0, \rho)$ and such that $\widetilde{\phi}_n(z_0) = z_n$. So for every $\ell \geq 1$ we have $\phi_\ell = \widetilde{\phi}_{m_\ell}$.

By the locally eventually onto property of Julia set we can find an integer M and distinct points $y_0, y_1 \in B(z_0, \rho/4)$ such that $f^M(y_0) = f^M(y_1) = \widehat{z}_0$. As $\widehat{z}_0$ is not in the forward orbit of critical points, it follows that f^M is locally injective at y_0 and y_1. Let $\widehat{\rho} > 0$ be sufficiently small so that the local inverse ψ_0 (resp. ψ_1) of f^M such that $\psi_0(z_0) = y_0$ (resp. $\psi_0(z_0) = y_1$) is defined on $B(\widehat{z}_0, \widehat{\rho})$. The classical Fatou argument implies that for every sufficiently large ℓ we have $\phi_\ell(B(z_0, 2\rho/3)) \subset B(\widehat{z}_0, \widehat{\rho})$ and that

$$\psi_0 \circ \phi_\ell(B(z_0, \rho/2)) \subset B(z_0, \rho/4) \text{ and}$$
$$\psi_1 \circ \phi_\ell(B(z_0, \rho/2)) \subset B(z_0, \rho/4).$$

Notice that for each ℓ we have that $\psi_0 \circ \phi_\ell$ or $\psi_1 \circ \phi_\ell$ is different from $\widetilde{\phi}_{m_\ell + M}$ on $B(z_0, \rho/2)$. Interchanging y_0 and y_1 if necessary we assume that for infinitely many ℓ we have that $\psi_0 \circ \phi_\ell$ is different from $\widetilde{\phi}_{m_\ell+M}$ on $B(z_0, \rho/2)$. Replacing $(\phi_\ell)_{\ell \ge 1}$ by a subsequence if necessary we assume that for every $\ell \ge 1$ we have $m_{\ell+1} - m_\ell > M$, $\psi_0 \circ \phi_\ell(B(z_0, \rho/2)) \subset B(z_0, \rho/4)$ and that

$$\widehat{\phi}_\ell := \psi_0 \circ \phi_\ell|_{B(z_0, \rho/2)}$$

is different from $\widetilde{\phi}_{m_\ell+M}$ on $B(z_0, \rho/2)$. It follows that $(\widehat{\phi}_\ell)_{\ell \ge 1}$ is an IFS generated by f that is defined on $B(z_0, \rho/2)$, that clearly satisfies the last property stated in the lemma. It remains to prove that this IFS is free. Let $\ell_1, \ldots, \ell_k$ and $\ell'_1, \ldots, \ell'_{k'}$ be sequences of positive integers such that the maps $\widehat{\phi}_{\ell_1} \circ \cdots \circ \widehat{\phi}_{\ell_k}$ and $\widehat{\phi}_{\ell'_1} \circ \cdots \circ \widehat{\phi}_{\ell'_{k'}}$ coincide. We assume without loss of generality that $\ell_k \ne \ell'_{k'}$ and that $m_{\ell_k} < m_{\ell'_{k'}}$. As f is of degree at least two it follows that there is N such that each of these maps is an inverse of f^N. We thus have

$$f^{N - m_{\ell_k} - M} \circ \widehat{\phi}_{\ell_1} \circ \cdots \circ \widehat{\phi}_{\ell_k} = \widehat{\phi}_{\ell_k} = \psi_0 \circ \widetilde{\phi}_{m_{\ell_k}}|_{B(z_0, \rho/2)}.$$

Since for every $\ell \ge 1$ we have $m_{\ell+1} - m_\ell > M$, we have $m_{\ell'_{k'}} > m_{\ell_k} + M$ and

$$f^{N - m_{\ell_k} - M} \circ \widehat{\phi}_{\ell'_1} \circ \cdots \circ \widehat{\phi}_{\ell'_{k'}} = \widetilde{\phi}_{m_{\ell_k}+M}|_{B(z_0, \rho/2)}.$$

But by construction $\psi_0 \circ \widetilde{\phi}_{m_{\ell_k}}$ and $\widetilde{\phi}_{m_{\ell_k}+M}$ are different on $B(z_0, \rho/2)$. This contradiction proves that $(\widehat{\phi}_\ell)_{\ell \ge 1}$ is free and finishes the proof of the lemma. □

Lemma 8.5. *Let f be a rational map of degree at least two and let μ be an ergodic invariant measure whose Lyapunov exponent χ_μ is positive. Then there is $C > 0$ and a free IFS $(\phi_\ell)_{\ell \ge 1}$ generated by f with the following property. Let $z_0 \in J(f)$, $\rho > 0$ and $(m_\ell)_{\ell \ge 1}$ be such that $(\phi_\ell)_{\ell \ge 1}$ is defined on $B(z_0, \rho)$ and such that for each $\ell \ge 1$ the map $f^{m_\ell} \circ \phi_\ell$ is the identity on $B(z_0, \rho)$. Then for each $\ell \ge 1$ we have $\phi'_\ell(z_0) \ge C \exp(-m_\ell \chi_\mu)$.*

Proof. Let ν be the unique measure on Z such that $\pi_* \nu = \mu$. As μ is ergodic, it follows that ν is. By [PU02, Theorem 9.2.3] there is a set of full measure of points $(x_n)_{n \in \mathbb{Z}}$ in Z for which there is $r > 0$ such that for each $n \ge 1$ the pull-back of $B(x, r)$ by f^n to x_{-n} is univalent. Fix such a $(x_n)_{n \in \mathbb{Z}}$ with $x_0 \in J(f)$ and that also satisfies the conclusions of Lemma 8.3 for T^{-1} instead of T, and for $\varphi := \ln |f'| \circ \pi - \chi_\mu$. Then there is an increasing sequence of positive integers $(n_\ell)_{\ell \ge 1}$ such that for all $\ell \ge 1$ we have $|(f^{n_\ell})'(x_{-n_\ell})| \le$

$2\exp(n_\ell \chi_\mu)$. Replacing $(n_\ell)_{\ell\geq 1}$ by a subsequence if necessary we assume that $((x_{-n_\ell+j})_{j\in\mathbb{Z}})_{\ell\geq 1}$ converges in Z to some point $(y_j)_{j\in\mathbb{Z}}$.

In each of the following cases we will construct and IFS generated by f that satisfies the derivative estimate and that also satisfies the hypothesis of Lemma 8.4.

Case 1. $(y_j)_{j\in\mathbb{Z}}$ is periodic. Then y_0 is a periodic point of f whose Lyapunov exponent is less than or equal to χ_μ. If y_0 is not repelling, then y_0 is parabolic and we can find a repelling periodic point $p_0 \in J(f)$ whose Lyapunov exponent is less than χ_μ. Let $(p_j)_{j\in\mathbb{Z}}$ be the periodic orbit of F that projects to p_0. Suppose now that p_0 is in the forward orbit of a critical point c of f, and denote by $\deg_f(c)$ the local degree of f at c. Then it is easy to see that for each $\varepsilon > 0$ there is a repelling periodic point $p \in J(f)$ such that $\chi(p) \leq \chi(p_0)/\deg_f(c) + \varepsilon$. Replacing $(p_j)_{j\in\mathbb{Z}}$ by the periodic orbit of F that projects to p, if necessary, we assume that p_0 is not in the forward orbit of a critical point of f. Denote by n the period of p_0 and let ϕ be a local inverse of f^n that fixes p_0. Let $\rho > 0$ be sufficiently small so that ϕ is defined on $B(p_0, \rho)$ and so that the closure of $\phi(B(p_0, \rho))$ is contained in $B(p_0, \rho)$. It follows that there is an integer L so that $\phi^L(B(p_0, \rho)) \subset B(p_0, \rho)$. Then $(\phi^{\ell L})_{\ell\geq 1}$ is an IFS generated by f that satisfies the derivative estimate and the hypothesis of Lemma 8.4 with $\widehat{z}_0 = p_0$.

Case 2. $(y_j)_{j\in\mathbb{Z}}$ is not periodic. Then there is a positive integer N such that y_{-N} is not in the forward orbit of a critical point of f. By the locally eventually onto property of Julia sets it follows that there is a point $z_0 \in B(x_0, r/4)$ and a positive integer M such that $f^M(z_0) = y_{-N}$. As y_{-N} is not in the forward orbit of a critical point of f it follows there is $\rho \in (0, r/4)$ such that f^M is univalent on $B(z_0, \rho)$. By the choice of $(x_n)_{n\in\mathbb{Z}}$ the pull-back of $B(x_0, r)$ by $f^{n_\ell+N}$ containing $x_{-n_\ell-N}$ is univalent. The classical Fatou argument implies then that there is $L \geq 1$ such that for all $\ell \geq L$ the pull-back of $B(x_0, r/2)$ by $f^{n_\ell+N}$ containing $x_{-n_\ell-N}$ is contained in $f^M(B(z_0, \rho/2))$. So, if for each $\ell \geq 1$ we put $m_\ell = n_{L+\ell}+N+M$ and denote by z_ℓ the unique point in $B(z_0, \rho/2)$ such that $f^M(z_\ell) = x_{-m_\ell+M}$, then the pull-back of $B(z_0, \rho)$ by f^{m_ℓ} containing z_ℓ in univalent and contained in $B(z_0, \rho/2)$. So, if we denote by ϕ_ℓ the inverse branch of f^{m_ℓ} such that $\phi(z_0) = z_\ell$, then $(\phi_\ell)_{\ell\geq 1}$ is an IFS generated by f. To prove that this IFS satisfies the hypothesis of Lemma 8.4 notice first that the sequence $(\phi_\ell(z_0))_{\ell\geq 1} = (z_\ell)_{\ell\geq 1}$ accumulates on z_0. As $f^M(z_0) = y_{-N}$ and y_{-N} is not in the forward orbit of a critical point of f, it follows that z_0 is not in the forward orbit of a critical point of f.

To verify the derivative estimate, let $K > 1$ be such that for each $\ell \geq 1$ the distortion of f^{n_ℓ} on the pull-back of $B(z_0, r/2)$ by f^{n_ℓ} containing x_{-n_ℓ} is bounded by K. Then it follows that for each ℓ we have

$$|(f^{m_\ell})'(z_\ell)| \leq 2K \exp(\chi_\mu m_\ell) \left(\sup_{\overline{\mathbb{C}}} |f'| \right)^{M+N}.$$

This shows the desired estimate with $C := 2K(\sup_{\overline{\mathbb{C}}} |f'|)^{M+N}$. $\square$

Proof of Lemma 8.2. Let $t > 0$ be given and let z_0, $\rho > 0, (\phi_\ell)_{\ell \geq 1}, \ldots$ be given by Lemma 8.5. Choose a non-periodic point $\zeta \in B(z_0, \rho/2)$ such that the sequence $(\Lambda_n)_{n\geq 1}$ defined by $\Lambda_n := \sum_{w \in f^{-n}(\zeta)} |(f^n)'(w)|^{-t}$ satisfies,

$$\limsup_{n\to+\infty} \tfrac{1}{n} \ln \Lambda_n = P(t).$$

Then the radius of convergence of the series

$$\Pi(z) := \sum_{n\geq 1} \Lambda_n z^n$$

is precisely $\exp(-P(t))$. So to prove the lemma we have to show that the radius of convergence of this series is strictly less than $\exp(t\chi_\mu)$.

Let $K > 1$ be such that for each $\ell \geq 1$ the distortion of ϕ_ℓ on $B(z_0, \rho/2)$ is bounded by K. For each $k \geq 1$ and each sequence of positive integers $\ell_1, \ldots, \ell_k$ put

$$w_{\ell_1,\ldots,\ell_k} = \phi_{\ell_K} \circ \ldots \circ \phi_{\ell_1}(\zeta) \in f^{-(m_{\ell_1}+\ldots+m_{\ell_k})}(\zeta),$$

and note that by Lemma 8.5 we have

$$|(f^{m_{\ell_1}+\ldots+m_{\ell_k}})'(w_{\ell_1,\ldots,\ell_k})| \leq KC^k \exp((m_{\ell_1} + \ldots + m_{\ell_k})\chi_\mu).$$

Since the IFS $(\phi_\ell)_{\ell\geq 1}$ is free and ζ is non-periodic the point $w_{\ell_1,\ldots,\ell_k}$ uniquely determines k and the sequence $\ell_1, \ldots, \ell_k$. Therefore, if we put

$$\Phi(z) := \sum_{\ell\geq 1} (KC \exp(m_\ell \chi_\mu))^{-t} z^{m_\ell},$$

then the coefficients of the series in the variable z

$$\Phi(z) + \Phi(z)^2 + \Phi(z)^3 + \ldots$$

are less than or equal to the corresponding coefficients of the series $\Pi(z)$. But the radius of convergence of Φ is equal to $\exp(t\chi_\mu)$ and we have

$$\lim_{s\to\exp(t\chi_\mu)^-} \Phi(s) = +\infty.$$

So there is $s \in (0, \exp(t\chi_\mu))$ such that $\Phi(s) \geq 1$, and therefore the radius of convergence of Π is less than or equal to s. $\square$

Appendix A. Puzzles and nice couples

This appendix is devoted to show that several classes of polynomials satisfy the conclusions of Theorem A. In §A.1 we consider the case of at most finitely renormalizable polynomials without indifferent periodic points, in §A.2 we consider the case of some infinitely renormalizable quadratic polynomials, and finally in §A.3 we consider the case of real quadratic polynomials, by giving the proof of Corollary 1.2.

A.1. **At most finitely renormalizable polynomials.** The purpose of this section is to prove the following result.

Theorem D. *Every at most finitely renormalizable complex polynomial or polynomial-like map without indifferent periodic points, has arbitrarily small nice couples.*

Remark A.1. It is not clear to us if the nice couples can be made of a finite union of puzzles.

The proof relies on the fundamental result that diameters of puzzles tend to 0 as their depth tends to ∞ [KvS06]; see also [QY06] for the case when the Julia set is totally disconnected. We use this fact to show that a "backward contraction" property similar to that in [RL07], and then follow the construction of nice couples given in [RL07, §6].

Let f an at most finitely renormalizable polynomial, and consider the puzzle constriction described in [KvS06, §2.1]. Given an integer $n \geq 0$ we denote by Υ_n the collection of all puzzles of depth n. For $P \in \Upsilon_n$ and $p \in P$ we put $P_n(p) := P$, and for $p \in J(f)$ that is not contained in an element of Υ_n we put

$$P_n(p) := \text{interior}\left(\bigcup_{P \in \Upsilon_n, p \in \overline{P}} \overline{P} \right).$$

Given $m \geq 0$ we define $P_n(p,m)$ inductively by $P_n(p,0) := P_n(p)$ and

$$P_n(p, m+1) = \text{interior}\left(\bigcup_{P \in \Upsilon_n, \overline{P} \cap \overline{P_n(p,m)} \neq \emptyset} \overline{P} \right).$$

The set $P_n(p,m)$ is connected, but it is not simply-connected in general. Clearly $P_n(p,m) \subset P_n(p,m+1)$. On the other hand, $P_{n+1}(p,m)$ is compactly contained in $P_n(p,m+1)$. We will use several times the following facts.

1. If $P_n(p,m)$ and $P_n(p',m')$ intersect, then $P_n(p,m) \subset P_n(p', m' + 2m+1)$.
2. If $k \geq 1$ is such that f^k is univalent on $P_{n+k}(n,m)$, then this set is equal to the connected component of $f^{-k}(P_n(f^k(p),m))$ containing p.

Since puzzles shrink to 0 as the depth grows to infinity [KvS06], there is an integer $k_0 \geq 1$ such that for each $p \in J(f)$ the set $P_{k_0}(p,12)$ is contained in $P_0(p,1)$. It thus follows that for each $n \geq 1$ such that f^n is univalent on $P_{n+k_0}(p,12)$, the set $P_{n+k_0}(p,12)$ is contained in $P_n(p,1)$. We assume furthermore that k_0 is sufficiently large so that for each $n \geq k_0$, $m \in \{1,\ldots,27\}$ and $c \in \text{Crit}(f) \cap J(f)$ the connected component of $f^{-1}(P_n(f(c),m))$ containing c is equal to $P_{n+1}(c,m)$.

Part 3 of the following lemma is analogous to [RL07, Lemma 6.2].

Lemma A.2. *For each $n \geq 1$ and $m \geq 0$ put*

$$V_n(m) = \bigcup_{c \in \mathrm{Crit}(f) \cap J(f)} P_n(c, m).$$

Let k_0 be as above, and let $n \geq k_0 + 1$ be a sufficiently large integer such that for each $k = 1, \ldots, k_0$ the set $f^k(V_n(34))$ is disjoint from $V_n(34)$.

Then we have the following properties.

1. *Each pull-back of $V_n(12)$ is either contained in $V_n(12)$ or it is contained in a univalent pull-back of $V_n(12)$.*
2. *Each pull-back of $V_n(12)$ intersecting $V_n(9)$ is contained in $V_n(12)$.*
3. *For each $c \in \mathrm{Crit}(f) \cap J(f)$ the connected component of $\mathbb{C} \setminus K(V_n(9))$ containing c is contained in $P_n(c, 12)$.*

Proof.

1. We will show by induction that for every integer $k \geq 0$, each connected component of $f^{-k}(V_n(12))$ satisfies the desired assertion. The case $k = 0$ being trivial we assume the desired assertion holds for some $k \geq 0$ and let W be a pull-back of $V_n(12)$ by f^{k+1}.

Suppose first that f^k is univalent on $f(W)$. If W is does not contain a critical point, then f^{k+1} is univalent on W and there is nothing to prove. If W contains a critical point c and if we denote by w the unique element of $f(W)$ such that $f^k(w) \in \mathrm{Crit}(f)$, then we have $f(W) = P_{n+k}(w, 12) \subset P_{n+k}(f(c), 25)$. Therefore we have $W \subset P_{n+k+1}(c, 25) \subset P_n(c, 25) \subset V_n(34)$, and our choice of n implies that $k \geq k_0$. By the definition of k_0 we have $f(W) = P_{n+k}(w, 12) \subset P_n(w, 1) \subset P_n(f(c), 3)$. Hence

$$W \subset P_{n+1}(c, 3) \subset P_n(c, 3) \subset V_n(12).$$

Suppose now that f^k is not univalent on $f(W)$ so there is an integer $j \in \{1, \ldots, k\}$ such that $f^j(W)$ contains a critical point. By the induction hypothesis applied to $f^j(W)$ we have $f^j(W) \subset V_n(12)$. So there is a pull-back $\widetilde{W}$ of $V_n(12) \supset V_n(12)$ by f^j containing W. Then the desired assertion follows from the induction hypothesis applied to $\widetilde{W}$.

2. In view of part 1 we just need to prove the assertion for univalent pull-backs. Let $k \geq 1$ be a given integer, and let W be a univalent pull-back of $V_n(12)$ by f^k such that for some $c \in \mathrm{Crit}(f) \cap J(f)$ the set W intersects $P_n(c, 9)$. If we denote by $w \in W$ the point determined by $f^k(w) \in \mathrm{Crit}(f)$, then

$$W = P_{n+k}(w, 12) \subset P_n(w, 12) \subset P_n(c, 34) \subset V_n(34).$$

So, by the definition of n we have $k \geq k_0 + 1$, and by the definition of k_0 we have that $W = P_{n+k}(w, 12) \subset P_n(w, 1)$. Since W intersects $P_n(c, 9)$, it follows that $W \subset P_n(w, 1) \subset P_n(c, 12) \subset V_n(12)$.

3. Given $c \in \mathrm{Crit}(f) \cap J(f)$ and an integer $m \geq 0$ let $\widetilde{V}_m^c$ be the connected component of $\bigcup_{j \in \{0, \ldots, m\}} f^{-j}(V_n(9))$ containing c. By definition $\widetilde{V}_0^c =$

$P_n(c, 9)$, and the set $\bigcup_{m \geq 0} \widetilde{V}_m^c$ is equal to the connected component of $\overline{\mathbb{C}} \setminus K(V_n(9))$ containing c. So we just need to show that for each $m \geq 0$ the set $\widetilde{V}_m^c$ is contained in $P_n(c, 12)$.

We will proceed by induction. The case $m = 0$ being trivial, assume that for each $j \in \{0, \ldots, m\}$ we have $\widetilde{V}_j^c \subset P_n(c, 12)$. For each $z \in \widetilde{V}_{m+1}^c$ denote by $k(z)$ the least integer $k \geq 0$ such that $f^k(z) \in V_n(9)$; we have $k(z) \in \{0, \ldots, m+1\}$. Given a connected component X of $\widetilde{V}_{m+1}^c \setminus P_n(c, 9)$ let $z \in X$ minimizing $k(z)$. Then $k(z) > 0$ and for some $c_0 \in \mathrm{Crit}(f) \cap J(f)$ we have $f^{k(z)}(X) \subset \widetilde{V}_{m+1-k(z)}^{c_0} \subset P_n(c_0, 12)$. Then part 2 implies that $X \subset P_n(c, 12)$. This completes the induction step, and the proof of the lemma. □

Proof of Theorem D. Let n be sufficiently large so that it satisfies the condition in the statement Lemma A.2, but with k_0 replaced by $k_0 + 2$. Furthermore, we assume that n is sufficiently large so that $V_n(12)$ is disjoint from the forward orbits of critical points that are not in the Julia set.

For $c \in \mathrm{Crit}(f) \cap J(f)$ let $\widetilde{V}^c$ be the connected component of $\mathbb{C} \setminus K(V_n(9))$ containing c. By part 3 of Lemma A.2 we have $\widetilde{V}^c \subset P_n(c, 12)$. Furthermore, by construction the set $\widetilde{V} := \bigcup_{c \in \mathrm{Crit}(f) \cap J(f)} \widetilde{V}^c$ satisfies $\partial \widetilde{V} \subset K(V_n(9)) = K(\widetilde{V})$, so for every $m \geq 1$ we have $f^m(\partial \widetilde{V}) \cap \widetilde{V} = \emptyset$.

1. Fix $c \in \mathrm{Crit}(f) \cap J(f)$ and put $v = f(c)$. Let $\widetilde{U}_1^v$ (resp. $\widetilde{U}_2^v$) be the union of $P_{n+1}(v, 5)$ (resp. $P_{n+2}(v, 1)$) and all the connected components of $\overline{\mathbb{C}} \setminus K(\widetilde{V})$ that intersect this set. We will show that $\widetilde{U}_1^v$ (resp. $\widetilde{U}_2^v$) is contained in $P_{n+1}(v, 8)$ (resp. $P_{n+2}(v, 4)$).

Let W be a connected component of $\mathbb{C} \setminus K(\widetilde{V})$ intersecting $P_{n+2}(v, 1)$. By part 1 of Lemma A.2, the set W is contained in a univalent pull-back $\widetilde{W}$ of $V_n(12)$. Let $c_0 \in \mathrm{Crit}(f) \cap J(f)$ and $k \geq 1$ be such that $f^k(\widetilde{W}) = P_n(c_0, 12)$, and let $w \in \widetilde{W}$ be determined by $f^k(w) = c_0$. Since W intersects $P_{n+2}(v, 1)$ we have

$$W \subset \widetilde{W} = P_{n+k}(w, 12) \subset P_{n+2}(v, 26) \subset P_n(v, 34).$$

Then, by the definition of n we must have $k \geq k_0 + 2$. So by the definition of k_0 we conclude that W is contained in $P_{n+2}(w, 1)$. Since W intersects $P_{n+2}(v, 1)$ we have $W \subset P_{n+2}(v, 4)$. This shows that $\widetilde{U}_2^v \subset P_{n+2}(v, 4)$.

The proof that $\widetilde{U}_1^v \subset P_{n+1}(v, 8)$ is analogous.

2. We will show now that for each $j = 1, 2$ we have $\partial \widetilde{U}_j^v \subset K(\widetilde{V})$. To do this it is enough to show that each connected component of $\overline{\mathbb{C}} \setminus K(\widetilde{V})$ is disjoint from the boundary of $\widetilde{U}_j^v$. Just notice that, by the definition of $\widetilde{U}_j^v$, each connected component W of $\overline{\mathbb{C}} \setminus K(\widetilde{V})$ is either contained in the interior of $\widetilde{U}_j^v$, or it is disjoint from (the closure of) $\widetilde{U}_j^v$. In both cases we conclude that W is disjoint from the boundary of $\widetilde{U}_j^v$.

3. By part 1, for every $m \geq 0$ the set $f^m(\partial \widetilde{U}_j^v)$ is disjoint from $\widetilde{V}$. We denote by U_j^v the union of $\widetilde{U}_j^v$ and all bounded connected components of $\overline{\mathbb{C}} \setminus \widetilde{U}_j^v$. Then U_j^v is simply connected, the set ∂U_j^v is contained in $\partial \widetilde{U}_j^v$, and for every $m \geq 0$ the set $f^m(\partial U_j^v)$ is disjoint from $\widetilde{V}$. By part 1 we have $U_2^v \subset P_{n+2}(v, 4)$ and $U_1^v \subset P_{n+1}(v, 8)$. Therefore, the connected component $\widehat{V}^c$ (resp. V^c) of $f^{-1}(U_1^v)$ (resp. $f^{-1}(U_2^v)$) containing c is compactly contained in $\widetilde{V}^c$ (resp. $\widehat{V}^c$). Then $\widehat{V} := \bigcup_{c \in \mathrm{Crit}(f) \cap J(f)} \widehat{V}^c$ is a nice set for f, the set $V := \bigcup_{c \in \mathrm{Crit}(f) \cap J(f)} V^c$ is compactly contained in $\widehat{V}$, and for every $m \geq 1$ the set $f^m(\partial V)$ is disjoint from $\widehat{V}$. This shows that $(\widehat{V}, V)$ is a nice couple for f. As n can be taken arbitrarily large, we conclude that f has arbitrarily small nice couples. □

A.2. **Infinitely renormalizable quadratic maps.** The purpose of this section is to show that each infinitely renormalizable polynomial or polynomial-like map whose small critical Julia sets converge to 0 satisfy the hypotheses of Theorem A'. This includes the case of infinitely renormalizable quadratic maps with *a priori* bounds; see [KL08, McM94] and references therein for results on *a priori* bounds.

The *post-critical set* of a rational map f is by definition

$$P(f) := \overline{\bigcup_{n \geq 1} f^n(\mathrm{Crit}(f))}.$$

Lemma A.3. *Let f be a rational map and let V be a nice set for f such that ∂V is disjoint from the post-critical set of f. Then for every neighborhood $\widetilde{V}$ of $\overline{V}$ there is $\widehat{V} \subset \widetilde{V}$ such that $(\widehat{V}, V)$ is a pleasant couple.*

Proof. We will assume that $P(f)$ contains at least three points; otherwise f is conjugated to a power map [McM94, Theorem 3.4] and then the assertion is vacuously true. We will denote by $\|f'\|$ the derivative of f with respect to the hyperbolic metric on $\overline{\mathbb{C}} \setminus P(f)$. Then by Schwarz lemma we have $\|f'\| \geq 1$ on $\overline{\mathbb{C}} \setminus f^{-1}(P(f))$ (*cf.*, [McM94, Theorem 3.5]). Furthermore, for $z \in \overline{\mathbb{C}} \setminus P(f)$ and $r > 0$ we denote by $B_{\mathrm{hyp}}(z, r)$ the ball corresponding to the hyperbolic metric on $\overline{\mathbb{C}} \setminus P(f)$.

Let $\varepsilon > 0$ be sufficiently small such that $B_{\mathrm{hyp}}(\partial V, 2\varepsilon)$ is disjoint from $P(f)$, and for each $c \in \mathrm{Crit}(f) \cap J(f)$ put

$$\widehat{V}^c := B_{\mathrm{hyp}}(V^c, \varepsilon) \text{ and } \widehat{V} := B_{\mathrm{hyp}}(V, \varepsilon) = \bigcup_{c \in \mathrm{Crit}(f) \cap J(f)} \widehat{V}^c.$$

By construction $\widehat{V}$ is a neighborhood of $\overline{V}$ and the set $\widehat{V} \setminus V$ is disjoint from $P(f)$. So for each pull-back W of V the set $\widehat{W} \setminus W$ is disjoint from $\mathrm{Crit}(f)$. We thus have $\widehat{W} \cap \mathrm{Crit}(f) = \emptyset$ when $W \cap V = \emptyset$. On the other hand, since $\|f'\| \geq 1$ on $f^{-1}(P(f))$, when $W \subset V$ we have $\widehat{W} \subset \widehat{V}$. This shows that $(\widehat{V}, V)$ is a pleasant couple for f. □

In what follows we shall use some terminology of [McM94] and [AL08, §2.4, Appendix A].

Definition A.4. Let $\mathcal{U}, \mathcal{V} \subset \mathbb{C}$ be connected and simply-connected domains satisfying $\overline{\mathcal{U}} \subset \mathcal{V}$, and let $f : \mathcal{U} \to \mathcal{V}$ be a quadratic-like map. Let $f^m : \mathcal{U}' \to \mathcal{V}'$ be a simple renormalization of f. We call the renormalization *immediate* if some *small Julia sets* touch each other (at a β-fixed point). Otherwise, if they are all pairwise disjoint, we call the renormalization *primitive.*

Proposition A.5. *Let f be an infinitely renormalizable quadratic-like map for which the diameters of small critical Julia sets converge to* 0*. Then f has arbitrarily small pleasant couples having property (*). In particular the conclusions of Theorem A' hold for f.*

Proof. We will show that there are arbitrarily small puzzles containing the critical point in the Julia set whose boundaries are disjoint from the post-critical set. Then Lemma A.3 implies that there are arbitrarily small pleasant couples. That each of these pleasant couples satisfy property (*) is a repetition of the proof of [MU03, Lemma 4.2.6], using the fact that puzzles are John domains (*i.e.*, that they have the "cone property" for "twisted angles").

Let $\mathcal{SR}(f)$ be the set of all integers $n \geq 2$ such that f^n is simply renormalizable and let J_n be the corresponding critical small Julia set. Then J_n is decreasing with n. For each $k \geq 1$ we denote by $m(k)$ the k-th element of $\mathcal{SR}(f)$.

We consider the usual puzzle construction with the α-fixed point of f. Then for each $\ell \geq 1$ there is a puzzle of depth ℓ, that we denote by P_ℓ, whose closure contains $J_{m(1)}$. We have $\bigcap_{\ell \geq 1} \overline{P_\ell} = J_{m(1)}$. More generally, by induction it can be show that, if for some $s \geq 1$ we consider the puzzle construction with the α-fixed points of the renormalizations of $f^{m(1)}$, $f^{m(2)}$, $\ldots$, $f^{m(s)}$, then for each $\ell \geq 1$ there is a puzzle of depth ℓ, that we denote by $\widetilde{P}_\ell$, that contains $J_{m(s)}$ (So that $\widetilde{P}_\ell$ is bounded by a finite number of arcs in an equipotential line and by the closure of some preimages of external rays landing at the α-fixed points of the renormalizations of $f^{m(1)}$, $f^{m(2)}$, $\ldots$, $f^{m(s)}$.) Furthermore we have

$$\bigcap_{\ell \geq 1} \widetilde{P}_\ell = J_{m(s)}.$$

In particular the diameter of $\widetilde{P}_\ell$ can be made arbitrarily close to that of $J_{m(s)}$. Note that for each $\ell \geq 1$ the set $\partial \widetilde{P}_\ell$ only intersects $J(f)$ at preperiodic points, and it is thus disjoint from the post-critical set of f. Furthermore for each $n \geq 1$ we have $f^n(\partial \widetilde{P}_\ell) \cap \widetilde{P}_\ell = \emptyset$, because $\widetilde{P}_\ell$ is a puzzle. To show that f is expanding away from critical points we just need to show that f is uniformly expanding on $K(\widetilde{P}_\ell) \cap J_{m(s)}$. As this set is compactly contained in $\mathbb{C} \setminus P(f)$, it is enough to show that the derivative $\|f'\|$ of f with respect of the hyperbolic metric on this set is strictly larger than 1 on $\mathbb{C} \setminus f^{-1}(P(f))$.

Since $f^{-1}(P(f))$ contains $P(f)$ strictly, this is a consequence of Schwarz lemma. □

A.3. **Real quadratic maps: The proof of Corollary 1.2.** Let f be a real quadratic polynomial. If f is at most finitely renormalizable without indifferent periodic points, then by Theorem D the map f satisfies the hypotheses of Theorem A. If f is infinitely renormalizable, then f has *a priori* bounds by [McM94], so the diameters of the small Julia set converge to 0 and then the assertion follows from Proposition A.5.

It remains to consider the case when f has an indifferent periodic point. Fix $t_0 \in (t_-, t_+)$. Since f is real it follows that f has a parabolic periodic point, and since f is quadratic it follows that f does not have critical points in the Julia set. Therefore the function $\ln|f'|$ is bounded and continuous on $J(f)$, and since the measure theoretic entropy of f is upper semi-continuous [FLM83, Lju83], there is an equilibrium state ρ of f for the potential $-t_0 \ln|f'|$. Since f has a parabolic periodic point it follows that t_+ is the first zero of P, so we have $P(t_0) > 0$ and therefore the Lyapunov exponent of ρ is positive. Since by [PRLS04, Theorem A and Theorem A.7] there is a $(t_0, P(t_0))$-conformal measure of f (see also [Prz99]), [Dob08, Theorem 8] implies that μ is in fact the unique equilibrium state of f for the potential $-t_0 \ln|f'|$. The analyticity of P at $t = t_0$ is given by [MS00] when $t_0 < 0$ and when $t_0 \geq 0$ the fact that P is analytic at $t = t_0$ can be shown in an analogous way as in [SU03], using and induced map defined with puzzles.

Appendix B. Rigidity, multifractal analysis, and level-1 large deviations

The purpose of this appendix is to prove that, besides some well-known exceptional maps, the pressure function of each of the maps considered in this paper is strictly convex on (t_-, t_+) (Theorem E below). We derive consequences for the dimension spectrum of Lyapunov exponents (§B.1) and pointwise dimensions (§B.2), as well as some level-1 large deviations results (§B.3). See [Pes97, Mak98] for background in multifractal analysis, and [DZ98] for background in large deviation theory.

Theorem E. *Let f be a rational map satisfying the hypotheses of Theorem A'. If f is not conjugated to a power, Tchebyshev or Lattès map, then for every $t \in (t_-, t_+)$ we have $P''(t) > 0$. In particular*

$$\chi^*_{\inf} := \inf\{-P'(t) \mid t \in (t_-, t_+)\} < \chi^*_{\sup} := \sup\{-P'(t) \mid t \in (t_-, t_+)\}.$$

It is well known that for a power, Tchebyshev or Lattès map, $t_+ = +\infty$ and the pressure function P is affine on $(t_-, +\infty)$; in particular in this case we have $\chi^*_{\inf} = \chi^*_{\sup}$. For a general rational map f and for $t_0 \in (t_-, 0)$, a result analogous to Theorem E was shown by Makarov and Smirnov in [MS00, §3.8] .

Proof. Suppose that for some $t_0 \in (t_-, t_+)$ we have $P''(t_0) = 0$. Let $(\widehat{V}, V)$ be a pleasant couple as in §7, so that the corresponding pressure function $\mathscr{P}$ is finite on a neighborhood of $(t, p) = (t_0, P(t_0))$, and such that for each $t \in \mathbb{R}$ close to t_0 we have $\mathscr{P}(t, P(t)) = 0$, see §3.4 for the definition of $\mathscr{P}$. Then the implicit function theorem implies that the function,

$$\begin{aligned} p_0(\tau) := \mathscr{P}(t_0 + \tau, P(t_0) + \tau P'(t_0)) \\ = P(F, -t_0 \ln |F'| - P(t_0)m - \tau(\ln |F'| + P'(t_0)m)), \end{aligned}$$

defined for $\tau \in \mathbb{R}$ in a neighborhood of $t = 0$, satisfies $p_0''(0) = 0$.

Let μ be the equilibrium measure of F for the potential $-t_0 \ln |F'| - P(t_0)m$ and put $\psi = -\ln |F'| - P'(t_0)m$. Since for each t close to t_0 we have $\mathscr{P}(t, P(t)) = 0$, the implicit function theorem gives $p_0'(0) = 0$. Thus, by [MU03, Proposition 2.6.13] we have

$$\int \psi d\mu = \int -\ln |F'| - P'(t_0) m d\mu = p_0'(0) = 0.$$

On the other hand, by [MU03, Proposition 2.6.14]

$$p_0''(0) = \sum_{k \geq 0} \left(\int \psi \circ F^k \cdot \psi d\mu - \left(\int \psi d\mu \right)^2 \right),$$

is the asymptotic variance of ψ with respect to μ; which by hypothesis is equal to 0. In view of Lemma 4.4 the function ψ is such that for each $p > 0$ we have $\int |\psi|^p d\mu < +\infty$, so [MU03, Lemma 4.8.8] implies that there is a measurable function $u : J(F) \to \mathbb{R}$ such that $\psi = u \circ F - u$. Put

$$\widetilde{J} := \{ z \in \overline{\mathbb{C}} \setminus K(V) \mid f^{m(z)}(z) \in J(F) \}$$

and extend u to a function defined on $\widetilde{J}$, that for each $z \in \widetilde{J} \setminus J(F)$ it is given by,

$$u(z) = u(f^{m(z)}(z)) - \sum_{j=0}^{m(z)-1} (-\ln |f'(f^j(z))| - P'(t_0)).$$

An argument similar to the construction of the conformal measure given in the proof of Proposition 4.3, shows that we have $\ln |f'| = -P'(t_0) + u \circ f - u$ on $\widetilde{J}$; see also [PRL07, Proposition B.2]. By construction this last set has full measure with respect to the equilibrium state of f for the potential $-t_0 \ln |f'|$, *cf.* §4.3. Thus, an argument similar to the proof of [Zdu90, §§5–8] (see also [MS00, §3.8] or [May02, Theorem 3.1]) implies that f is a power, Tchebyshev or Lattès map. □

B.1. **Dimension spectrum for Lyapunov exponents.** Let f be a rational map of degree at least two. For $z \in \overline{\mathbb{C}}$ we define

$$\chi(z) = \lim_{n \to +\infty} \frac{1}{n} \ln |(f^n)'(z)|,$$

whenever the limit exists; it is called the *Lyapunov exponent of f at z*. The *dimension spectrum of Lyapunov exponents* is the function $L : (0, +\infty) \to \mathbb{R}$ defined by,

$$L(\alpha) := \mathrm{HD}(\{z \in J(f) \mid \chi(z) = \alpha\}).$$

Following [MS00, §1.3] we will say that f is *exceptional* if there is a finite subset Σ of $\overline{\mathbb{C}}$ such that

$$f^{-1}(\Sigma) \setminus \mathrm{Crit}(f) = \Sigma. \tag{B.1}$$

A rational map f is exceptional if and only if $t_- > -\infty$. Furthermore, in this case there is a set Σ_f containing at most four points such that (B.1) is satisfied with $\Sigma = \Sigma_f$, and such that each finite set Σ satisfying (B.1) is contained in Σ_f. Power, Tchebyshev and Lattès maps are all exceptional.

It has been recently shown in [GPR08, Theorem 2] that if f is not exceptional, or if f is exceptional and $\Sigma_f \cap J(f) = \emptyset$, then for each $\alpha \in (0, +\infty)$ we have

$$L(\alpha) = \frac{1}{\alpha} \inf\{P(t) + \alpha t \mid t \in \mathbb{R}\}.$$

Equivalently, the functions $\alpha \mapsto -\alpha L(\alpha)$ and $s \mapsto P(-s)$ form a Legendre pair. Note that a Tchebyshev or a Lattès map f is exceptional and Σ_f intersects $J(f)$.

The following is a direct consequence of Theorem E.

Corollary B.1. *Let f be a rational map satisfying the hypotheses of Theorem A'. Suppose furthermore that f is not conjugated to a power map, and that either f is not exceptional, or that f is exceptional and Σ_f is disjoint from $J(f)$. Then the dimension spectrum for Lyapunov exponents of f is real analytic on $(\chi^*_{\inf}, \chi^*_{\sup})$.*

B.2. **Dimension spectrum for pointwise dimension.** Let μ_0 be the unique measure of maximal entropy of f. Then for $z \in J(f)$ we define

$$\alpha(z) := \lim_{r \to 0^+} \frac{\ln \mu_0(B(z,r))}{\ln r},$$

whenever the limit exists; it is called the *pointwise dimension of μ_0 at z*. The *dimension spectrum for pointwise dimensions* is defined as the function

$$D(\alpha) := \mathrm{HD}(\{z \in J(f) \mid \alpha(z) = \alpha\}).$$

When f is a polynomial with connected Julia set it was shown in [MS00, §5] that

$$D(\alpha) = \inf\left\{t + \alpha \frac{P(t)}{\ln \deg(f)} \mid t \in \mathbb{R}\right\}.$$

Equivalently, the functions $\beta \mapsto -\beta D(\frac{1}{\beta})$ and $s \mapsto (\ln \deg(f))^{-1} P(-s)$ form a Legendre pair. So the following is a direct consequence of Theorem A' and Theorem E.

Corollary B.2. *Let f be a polynomial with connected Julia set satisfying the hypotheses of Theorem A'. If f is not a power or Tchebyshev map, then the dimension spectrum for pointwise dimensions of f is real analytic on $(-(\chi^*_{\inf})^{-1}, -(\chi^*_{\sup})^{-1})$.*

Remark B.3. In the uniformly hyperbolic case one has $D(\alpha) = L(\ln \deg(f)/\alpha)$. This also holds when the set of those $z \in J(f)$ for which $\chi(z)$ exists and satisfies $\chi(z) \leq 0$ has Hausdorff dimension equal to 0, like for rational maps satisfying the TCE condition [PRL07, §1.4]. In fact, it is easy to see that for $z \in J(f)$ belonging to the "conical Julia set" and for which both $\alpha(z)$ and $\chi(z)$ exists, and $\chi(z) > 0$, we have $\alpha(z) = \log \deg(f)/\chi(z)$. Then the assertion follows from [GPR08, Proposition 3], that the set of those $z \in J(f)$ that are not in the conical Julia set and $\chi(z) > 0$ has Hausdorff dimension equal to 1.

B.3. **Large deviations.** The purpose of this section is to present a sample application of Theorem E to level-1 large deviations, using the characterizations of the pressure function given in [PRLS04].

Corollary B.4. *Let f be a rational map satisfying the hypotheses of Theorem A', and that is not conjugated to a power, Tchebyshev, or Lattès map. Fix $t_0 \in (t_-, t_+)$ and let ρ_{t_0} be the equilibrium state of f for the potential $-t_0 \ln |f'|$. Fix $x_0 \in J(f)$ such that (2.1) holds, and for each $n \geq 1$ put*

$$\omega_n := \sum_{x \in f^{-n}(x_0)} \frac{|(f^n)'(x)|^{-t_0}}{\sum_{y \in f^{-n}(x_0)} |(f^n)'(y)|^{-t_0}} \delta_x.$$

Given $\varepsilon \in (0, -P'(t_0) - \chi^*_{\inf})$, let $t(\varepsilon) \in (t_-, t_0)$ be determined by $P'(t(\varepsilon)) = P'(t_0) - \varepsilon$. Then we have,

$$\lim_{n \to +\infty} \frac{1}{n} \ln \omega_n \left\{ x \in J(f) \mid \frac{1}{n} \ln |(f^j)'(x)| > \int \ln |f'| d\rho_{t_0} + \varepsilon \right\}$$
$$= P(t(\varepsilon)) - P(t_0) - (t(\varepsilon) - t_0) P'(t(\varepsilon)) < 0.$$

Similarly, given $\varepsilon \in (0, \chi^*_{\sup} + P'(t_0))$ let $\tilde{t}(\varepsilon) \in (t_0, t_+)$ be determined by $P'(\tilde{t}(\varepsilon)) = P'(t_0) + \varepsilon$. Then we have,

$$\lim_{n \to +\infty} \frac{1}{n} \ln \omega_n \left\{ x \in J(f) \mid \frac{1}{n} \ln |(f^j)'(x)| < \int \ln |f'| d\rho_{t_0} - \varepsilon \right\}$$
$$= P(\tilde{t}(\varepsilon)) - P(t_0) - (\tilde{t}(\varepsilon) - t_0) P'(\tilde{t}(\varepsilon)) < 0.$$

For a rational map satisfying the TCE condition, or the weaker "Hypothesis H" of [PRLS04], a similar result can be obtained for periodic points. See [Com08] and references therein for analogous statements in the case of uniformly hyperbolic rational maps, and [KN92] for similar results in the case of Collet-Eckmann unimodal maps and t_0 near 1.

Proof. First observe that by the choice of x_0, for each $s \in \mathbb{R}$ we have

$$\lim_{n\to+\infty} \frac{1}{n} \ln \int \exp(s \ln |(f^n)'|) d\omega_n = \lim_{n\to+\infty} \frac{1}{n} \ln \frac{\sum_{x\in f^{-n}(x_0)} |(f^n)'(x)|^{-t_0+s}}{\sum_{y\in f^{-n}(x_0)} |(f^n)'(y)|^{-t_0}} = P(t_0 - s) - P(t_0).$$

We will apply [PS75, Theorem] to the space $J := \prod_{n\geq 1} J(f)$ endowed with the probability measure $\mathsf{P} := \bigotimes_{n\geq 1} \omega_n$. Furthermore for each $n \geq 1$ we take the random variable $W_n : J \to \mathbb{R}$ as $W_n(\prod_{j\geq 1} x_j) := \ln |(f^n)'(x_n)|$. So for each $s \in \mathbb{R}$ we have

$$\int \exp(sW_n) d\mathsf{P} = \int \exp(s \ln |(f^n)'|) d\omega_n,$$

and by the computation above,

$$\lim_{n\to+\infty} \int \exp(sW_n) d\mathsf{P} = P(t_0 - s) - P(t_0).$$

Using that $\int \ln |(f^n)'| d\rho_{t_0} = -P'(t_0)$, and that the function $s \mapsto P(t_0 - s) - P(t_0)$ is real analytic and strictly convex on $(t_0 - t_+, t_0 - t_-)$ (Theorem E), [PS75, Theorem] gives

$$\lim_{n\to+\infty} \frac{1}{n} \ln \omega_n \left\{ \tfrac{1}{n} \ln |(f^n)'| > \int \ln |(f^n)'| d\rho_{t_0} + \varepsilon \right\} = P(t(\varepsilon)) - P(t_0) - (t(\varepsilon) - t_0) P'(t(\varepsilon)).$$

The second assertion is obtained analogously with $W_n(\prod_{j\geq 1} x_j) := -\ln |(f^n)'(x_n)|$. □

References

[AL08] Artur Avila and Mikhail Lyubich. Hausdorff dimension and conformal measures of Feigenbaum Julia sets. *J. Amer. Math. Soc.*, 21(2):305–363, 2008.

[BJ03] I. Binder and P. Jones. -. In preparation, 2003.

[BK98] Henk Bruin and Gerhard Keller. Equilibrium states for S-unimodal maps. *Ergodic Theory Dynam. Systems*, 18(4):765–789, 1998.

[BMS03] I. Binder, N. Makarov, and S. Smirnov. Harmonic measure and polynomial Julia sets. *Duke Math. J.*, 117(2):343–365, 2003.

[Bru03] H. Bruin. Minimal Cantor systems and unimodal maps. *J. Difference Equ. Appl.*, 9(3-4):305–318, 2003. Dedicated to Professor Alexander N. Sharkovsky on the occasion of his 65th birthday.

[BS96] I. N. Baker and G. M. Stallard. Error estimates in a calculation of Ruelle. *Complex Variables Theory Appl.*, 29(2):141–159, 1996.

[BS05] D. Beliaev and S. Smirnov. Harmonic measure on fractal sets. In *European Congress of Mathematics*, pages 41–59. Eur. Math. Soc., Zürich, 2005.

[BT06] Henk Bruin and Mike Todd. Complex maps without invariant densities. *Nonlinearity*, 19(12):2929–2945, 2006.

[BT07] Henk Bruin and Mike Todd. Equilibrium states for interval maps: the potential $-t \log |df|$. arXiv:0704.2199v1, 2007.

[BT08] Henk Bruin and Mike Todd. Equilibrium states for interval maps: potentials with $\sup \phi - \inf \phi < h_{\mathrm{top}}(f)$. *Comm. Math. Phys.*, 283(3):579–611, 2008.

[Com08] Henri Comman. Strengthened large deviations for rational maps and Gibbs fields, with unified proof. 2008. arXiv:0808.0327v1.

[CRL08] María Isabel Cortez and Juan Rivera-Letelier. Invariant measures of minimal post-critical sets of logistic maps. arXiv:0804.4550v1, 2008.

[DMNU98] M. Denker, R. D. Mauldin, Z. Nitecki, and M. Urbański. Conformal measures for rational functions revisited. *Fund. Math.*, 157:161–173, 1998. Dedicated to the memory of Wiesław Szlenk.

[Dob07] Neil Dobbs. Renormalisation-induced phase transitions for unimodal maps. arXiv:0712.3023v1, to appear in Comm. Math. Phys., 2007.

[Dob08] Neil Dobbs. Measures with positive lyapunov exponent and conformal measures in rational dynamics. arXiv:0804.3753v1, 2008.

[DU91] M. Denker and M. Urbański. Ergodic theory of equilibrium states for rational maps. *Nonlinearity*, 4(1):103–134, 1991.

[DZ98] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 1998.

[Erë91] A. È. Erëmenko. Lower estimate in Littlewood's conjecture on the mean spherical derivative of a polynomial and iteration theory. *Proc. Amer. Math. Soc.*, 112(3):713–715, 1991.

[Fis06] Todd Fisher. Hyperbolic sets that are not locally maximal. *Ergodic Theory Dynam. Systems*, 26(5):1491–1509, 2006.

[FLM83] Alexandre Freire, Artur Lopes, and Ricardo Mañé. An invariant measure for rational maps. *Bol. Soc. Brasil. Mat.*, 14(1):45–62, 1983.

[Gou04] Sebastien Gouezël. *Vitesse de décorrélation et théorèmes limites pour les applications non uniformément dilatantes*. PhD thesis, 2004.

[GPR08] Katrin Gelfert, Feliks Przytycki, and Michal Rams. Lyapunov spectrum for rational maps. 2008. arXiv:0809.3363v1.

[GW07] Katrin Gelfert and Christian Wolf. Topological pressure for one-dimensional holomorphic dynamical systems. *Bull. Pol. Acad. Sci. Math.*, 55(1):53–62, 2007.

[Jon05] Peter W. Jones. On scaling properties of harmonic measure. In *Perspectives in analysis*, volume 27 of *Math. Phys. Stud.*, pages 73–81. Springer, Berlin, 2005.

[KL08] Jeremy Kahn and Mikhail Lyubich. A priori bounds for some infinitely renormalizable quadratics. II. Decorations. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(1):57–84, 2008.

[KN92] Gerhard Keller and Tomasz Nowicki. Spectral theory, zeta functions and the distribution of periodic points for Collet-Eckmann maps. *Comm. Math. Phys.*, 149(1):31–69, 1992.

[KvS06] Oleg Kozlovski and Sebastian van Strien. Local connectivity and quasiconformal rigidity of non-renormalizable polynomials. arXiv:math/0609710v2, 2006.

[Led84] François Ledrappier. Quelques propriétés ergodiques des applications rationnelles. *C. R. Acad. Sci. Paris Sér. I Math.*, 299(1):37–40, 1984.

[Lju83] M. Ju. Ljubich. Entropy properties of rational endomorphisms of the riemann sphere. *Ergodic Theory Dynam. Systems*, 3(3):351–385, 1983.

[Mak98] N. G. Makarov. Fine structure of harmonic measure. *Algebra i Analiz*, 10(2):1–62, 1998.

[May02] Volker Mayer. Comparing measures and invariant line fields. *Ergodic Theory Dynam. Systems*, 22(2):555–570, 2002.

[McM94] Curtis T. McMullen. *Complex dynamics and renormalization*, volume 135 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1994.

[McM00] Curtis T. McMullen. Hausdorff dimension and conformal dynamics. II. geometrically finite rational maps. *Comment. Math. Helv.*, 75(4):535–593, 2000.

[MN05] Ian Melbourne and Matthew Nicol. Almost sure invariance principle for nonuniformly hyperbolic systems. *Comm. Math. Phys.*, 260(1):131–146, 2005.

[MN06] Ian Melbourne and Matthew Nicol. Large deviations for nonuniformly hyperbolic systems. To appear in Trans. Amer. Math. Soc., 2006.

[MS96] N. Makarov and S. Smirnov. Phase transition in subhyperbolic Julia sets. *Ergodic Theory Dynam. Systems*, 16(1):125–157, 1996.

[MS00] N. Makarov and S. Smirnov. On "thermodynamics" of rational maps. I. Negative spectrum. *Comm. Math. Phys.*, 211(3):705–743, 2000.

[MS03] N. Makarov and S. Smirnov. On thermodynamics of rational maps. II. Non-recurrent maps. *J. London Math. Soc. (2)*, 67(2):417–432, 2003.

[MU03] R. Daniel Mauldin and Mariusz Urbański. *Graph directed Markov systems*, volume 148 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2003. Geometry and dynamics of limit sets.

[Pes97] Yakov B. Pesin. *Dimension theory in dynamical systems*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1997. Contemporary views and applications.

[PR98] Feliks Przytycki and Steffen Rohde. Porosity of Collet-Eckmann Julia sets. *Fund. Math.*, 155(2):189–199, 1998.

[PRL07] Feliks Przytycki and Juan Rivera-Letelier. Statistical properties of topological Collet-Eckmann maps. *Ann. Sci. École Norm. Sup. (4)*, 40(1):135–178, 2007.

[PRLS03] Feliks Przytycki, Juan Rivera-Letelier, and Stanislav Smirnov. Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps. *Invent. Math.*, 151(1):29–63, 2003.

[PRLS04] Feliks Przytycki, Juan Rivera-Letelier, and Stanislav Smirnov. Equality of pressures for rational functions. *Ergodic Theory Dynam. Systems*, 24(3):891–914, 2004.

[Prz90] Feliks Przytycki. On the Perron-Frobenius-Ruelle operator for rational maps on the Riemann sphere and for Hölder continuous functions. *Bol. Soc. Brasil. Mat. (N.S.)*, 20(2):95–125, 1990.

[Prz98] Feliks Przytycki. Iterations of holomorphic Collet-Eckmann maps: conformal and invariant measures. appendix: on non-renormalizable quadratic polynomials. *Trans. Amer. Math. Soc.*, 350(2):717–742, 1998.

[Prz99] Feliks Przytycki. Conical limit set and Poincaré exponent for iterations of rational functions. *Trans. Amer. Math. Soc.*, 351(5):2081–2099, 1999.

[PS75] D. Plachky and J. Steinebach. A theorem about probabilities of large deviations with an application to queuing theory. *Period. Math. Hungar.*, 6(4):343–345, 1975.

[PS08] Yakov Pesin and Samuel Senti. Equilibrium measures for maps with inducing schemes. *J. Mod. Dyn.*, 2(3):397–430, 2008.

[PU02] F. Przytycki and M. Urbański. Fractals in the plane, ergodic theory methods. to appear in Cambridge University Press. Available on `http://www.math.unt.edu/~urbanski` and `http://www.impan.edu.pl/~feliksp`, 2002.

[QY06] Weiyuan Qiu and Yongcheng Yin. Proof of the Branner-Hubbard conjecture on Cantor Julia sets. arXiv:math/0608045v1, 2006.

[RL07] Juan Rivera-Letelier. A connecting lemma for rational maps satisfying a no-growth condition. *Ergodic Theory Dynam. Systems*, 27(2):595–636, 2007.

[Rue92] David Ruelle. Spectral properties of a class of operators associated with conformal maps in two dimensions. *Comm. Math. Phys.*, 144(3):537–556, 1992.

[Rue04] David Ruelle. *Thermodynamic formalism*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2004. The mathematical structures of equilibrium statistical mechanics.

[SU03] Bernd O. Stratmann and Mariusz Urbanski. Real analyticity of topological pressure for parabolically semihyperbolic generalized polynomial-like maps. *Indag. Math. (N.S.)*, 14(1):119–134, 2003.

[TK05] Marta Tyran-Kamińska. An invariance principle for maps with polynomial decay of correlations. *Comm. Math. Phys.*, 260(1):1–15, 2005.

[Urb03] Mariusz Urbański. Thermodynamic formalism, topological pressure, and escape rates for critically non-recurrent conformal dynamics. *Fund. Math.*, 176(2):97–125, 2003.

[You99] Lai-Sang Young. Recurrence times and rates of mixing. *Israel J. Math.*, 110:153–188, 1999.

[Zdu90] Anna Zdunik. Parabolic orbifolds and the dimension of the maximal measure for rational maps. *Invent. Math.*, 99(3):627–649, 1990.

[Zin96] Michel Zinsmeister. *Formalisme thermodynamique et systèmes dynamiques holomorphes*, volume 4 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 1996.

[Zwe05] Roland Zweimüller. Invariant measures for general(ized) induced transformations. *Proc. Amer. Math. Soc.*, 133(8):2283–2295 (electronic), 2005.

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