CONICAL LIMIT SET AND POINCARÉ EXPONENT FOR ITERATIONS OF RATIONAL FUNCTIONS

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Abstract. We contribute to the dictionary between action of Kleinian groups and iteration of rational functions on the Riemann sphere. We define the Poincaré exponent $\delta(f,z)=\inf\{\alpha\ge 0:\mathcal{P}(z,\alpha)\le 0\}$ where

$$\mathcal{P}(z,\alpha):=\limsup_{n\to\infty}\frac{1}{n}\log\sum_{f^n(x)=z}|(f^n)'(x)|^{-\alpha}.$$

We prove that $\delta(f,z)$ and $\mathcal{P}(z,\alpha)$ do not depend on z, provided z is non-exceptional. $\mathcal{P}$ plays the role of pressure, we prove that it coincides with Denker-Urbański's pressure if $\alpha\le\delta(f)$. Various notions of "conical limit set" are considered. They all have Hausdorff dimension equal to $\delta(f)$ equal to the hyperbolic dimension of Julia set and equal to the exponent of some conformal Patterson-Sullivan measures. In an Appendix we discuss also notions of "conical limit set" introduced recently by Urbański and by Lyubich and Minsky.

INTRODUCTION.

For every Kleinian group G with $\Lambda_c(G)$ the conical limit set, $\delta(G)$ the Poincaré exponent, $\alpha(G)$ the infimum of exponents of conformal measures and HD standing for Hausdorff dimension it holds

$$\mathrm{HD}(\Lambda_c(G))=\delta(G)=\alpha(G) \tag{0.1}$$

and Patterson construction gives conformal measures of exponent equal precisely $\delta(G)$. This is a part of a beautiful theory linking these notions, see [Pa], [S1], [BJ] (also [N]).

In iterations of rational functions conformal measures were introduced by D. Sullivan [S2] and a general theory of conformal measures was developed by M. Denker and M. Urbański in [DU1] and [DU2]. In [DU2] the *dynamical dimension* of Julia set $J=J(f)$ for a rational function f was introduced and defined as follows

$$\mathrm{HD}_{\mathrm{mhyp}}(J)=\sup\{\mathrm{HD}(\mu):\mu \text{ is an ergodic } f\text{-invariant measure of positive Lyapunov exponent}\} \tag{0.2}$$

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where $\mathrm{HD}(\mu)$ is the infimum of Hausdorff dimensions of sets of full measure μ (mhyp abbreviates: *measure hyperbolic*). It happens (see Appendix 2 or [PUbook, Ch.8,9]) that $\mathrm{HD}_{\mathrm{mhyp}}(J) = \mathrm{HD}_{\mathrm{hyp}}(J)$, the *hyperbolic dimension* which is supremum of Hausdorff dimensions of isolated hyperbolic subsets of J, the notion introduced by M. Shishikura [Shi]. (X is called isolated if every trajectory $f^j(x), j = 0, 1, \ldots$ contained in a sufficiently small neighbourhood of X must be contained in X. Compact $X \subset J$ is called hyperbolic if there exists $n > 0$ such that $|(f^n)'| > 1$.)

The main theorem of [DU2]+[P2] (see also Appendix 2) asserts that

$$\mathrm{HD}_{\mathrm{mhyp}}(J) = \alpha(f). \tag{0.3}$$

where $\alpha(f)$ is the infimum of exponents of conformal measures for f. Moreover a conformal measure for which this infimum is attained is constructed in [DU2], but not by the Patterson procedure. Recall that μ is called α-conformal for f (or conformal with exponent α) if for every Borel set $E \in J$ on which f is injective $\mu(f(E)) = \int_E |f'|^\alpha \, d\mu$.

Meanwhile a definition of a "conical Julia set" whose Hausdorff dimension would be equal to $\mathrm{HD}_{\mathrm{mhyp}}(J)$, also a definition, analogous to $\delta(G)$, of the Poincaré exponent and an equality similar to (0.1) were missing. In this paper we try to fill this gap.

SECTION 1. Basic concepts.

Definition 1.1. For each rational function f of degree ≥ 2, for every $z \in \bar{\mathbb{C}}$ and every $\alpha \geq 0$ consider the following *Poincaré sequence*:

$$P(z, \alpha, n) := \sum_{f^n(x)=z} |(f^n)'(x)|^{-\alpha}$$

We call $\alpha = \delta(f, z)$ the *Poincaré exponent* with respect to z if α is the smallest number such that $\limsup_{n\to\infty} \frac{1}{n} \log P(z, \alpha, n) \leq 0$. If this limsup is positive for every α we set $\delta(f, z) = \infty$.

Notice that $\delta(f, z) > 0$. Indeed $P(z, \alpha, n) \geq \deg(f)^n (\sup |f'|)^{-n\alpha}$ hence
$\frac{1}{n} \log P(z, \alpha, n) \geq \log \deg(f) - \alpha \log(\sup |f'|)$ hence
$\delta(f, z) \geq \log \deg(f) / \log(\sup |f'|)$.

Notice also that the smallest α in the Definition 1.1 exists. This is an easy exercise also using $\sup |f'| < \infty$ (see Prop. A2.2.).

The Main Theorem of the paper (Section 3) says that $\delta(f, z)$ as a function of z is constant and attains its minimum everywhere except a "thin" set E (of Hausdorff dimension 0). We call this constant the *Poincaré exponent* and denote by $\delta(f)$. Before we prove this Main Theorem we just write

$$\delta(f) = \inf_z \delta(f, z). \tag{1.1}$$

One can define *Pressure* $\mathcal{P}(z,\alpha)=\mathcal{P}(f,-\alpha\log|f'|,z)$ as $\limsup_{n\to\infty}\frac{1}{n}\log P(z,\alpha,n)$. So $\delta(f,z)$ is the first zero of $\mathcal{P}(z,\alpha)$ as a function of α. $\mathcal{P}(z,\alpha)$ similarly to $\delta(f,z)$ is constant independent of z except $z\in E$ where it is not smaller. Denote this constant by $\mathcal{P}(\alpha)$. Clearly $\mathcal{P}(z,\alpha)$ and $\mathcal{P}(\alpha)$ are continuous (Appendix 2, Prop.A2.1).

In fact for $0\le\alpha\le\delta(f)$, $\mathcal{P}(\alpha)$ coincides with pressure defined in [DU2] and also with $\sup h_\mu(f)-\alpha\int\log|f'|d\mu$ where supremum is taken over all probability ergodic f-invariant measures (or only measures with positive Lyapunov exponent), $h_\mu(f)$ is the entropy. So on the domain $0\le\alpha\le\delta$, $\mathcal{P}(\alpha)$ is strictly decreasing and convex. We explain these facts concerning pressure in Appendix 2. They are not needed in the main course of the paper but they are interesting because they for example generalize Variational Principle [W] to the case $-\log|f'|$ has singularities of value ∞.

Definition 1.2. For each rational function f and every non-exceptional z (here it means: not a superattracting periodic point for f) for every $K>0$ write

$$\Lambda_{\mathrm{c}}(f,z,K):=\{x\in\bar{\mathbb{C}}:\exists x_i\in f^{-n_i}(z), n_i\to\infty, i=1,2,\dots,\ (\forall i)\ \text{ the properties (i) and (ii) hold }\}$$

where

(i) $$|(f^{n_i})'(x_i)|\ge(1+K^{-1})^{n_i}$$

(ii) $$\operatorname{dist}(x,x_i)\le K|(f^{n_i})'(x_i)|^{-1},$$

(dist is the Riemann distance). Write

$$\Lambda_{\mathrm{c}}(f,z):=\bigcup_{K>0}\Lambda_{\mathrm{c}}(f,z,K).$$

Finally define *the conical limit set*

$$\Lambda_{\mathrm{c}}(f):=\bigcap_{z}\Lambda_{\mathrm{c}}(f,z).$$

We do not know how $\Lambda_{\mathrm{c}}(f,z)$ depends on z. We conclude only *a posteriori* that except for $z\in E$ its dimension is constant equal to $\delta(f)$. Therefore we think that the following concept is of interest:

Definition 1.3.

$$\Lambda_{\mathrm{cw}}(f,z,K):=\{x\in\bar{\mathbb{C}}:\exists x_i\in f^{-n_i}(z), n_i\to\infty, i=1,2,\dots,\ (\forall i)\ \text{the properties(i) and (iii) hold }\}$$

where

(iii) $$\limsup_{i\to\infty}\frac{1}{n_i}\log\big(\operatorname{dist}(x,x_i)\cdot|(f^{n_i})'(x_i)|\big)\le 0.$$

As before we write

$$\Lambda_{\mathrm{cw}}(f,z):=\bigcup_{K>0}\Lambda_{\mathrm{cw}}(f,z,K).$$

Of course $\Lambda_{\rm cw}(f,z) \supset \Lambda_{\rm c}(f,z)$. In Section 3. we prove that $\Lambda_{\rm cw}(f,z)$ does not depend of z except $z \in E$. We write then $\Lambda_{\rm cw}(f)$ and call this the *weak conical limit set.* We show in Section 3. that for every z which has a backward trajectory omitting Crit, the set of points where f' is zero, $\Lambda_{\rm cw}(f) \subset \Lambda_{\rm cw}(f,z)$.

Notice that already (i), together with $x_i \to x$, implies for every z, $\Lambda_{\rm cw}(f,z) \subset J(f)$.

Remark finally that if two points z_1, z_2 belong to a connected open set U which is disjoint from $O(\mathrm{Crit}) := \bigcup_{n=1}^{\infty} f^n(\mathrm{Crit}(f))$ then by bounded distortion for all branches of f^{-n} on a connected open set with closure in U containing z_1 and z_2 we have $\Lambda_{\rm c}(f,z_1) = \Lambda_{\rm c}(f,z_2)$ and the Poincaré exponents also coincide. This is the case for z_1, z_2 belonging to the same Fatou component, in particular for the basin A_∞ of ∞ for f a polynomial. So for polynomials one can define $\Lambda_{\rm c}(f) := \Lambda_{\rm c}(f,z), \quad z \in A_\infty$.

It is easy to see that (iii) is equivalent to x being *non-deep* for the filled-in Julia set, the notion introduced by C. McMullen [McM].

In the next section two more concepts of the limit set will appear in a natural way in relation to an application of the Pesin Theory. We call them *regular* and *tree conical* and denote $\Lambda_{\rm reg}(f)$ and $\Lambda_{\rm tc}(f)$ respectively. In the case of polynomials we define also *radial conical* which is equivalent to *tree conical* (in the case the basin of ∞ is simply-connected). We will have

$$\Lambda_{\rm reg}(f) \subset \Lambda_{\rm tc}(f) \subset \Lambda_{\rm c}(f) \subset \Lambda_{\rm cw}(f)$$

and all these sets have the same Hausdorff dimension.

SECTION 2. Equalities of dimensions and exponents.

Theorem 2.1. For every z

$$\mathrm{HD}(\Lambda_{\rm c}(f,z)) \le \mathrm{HD}(\Lambda_{\rm cw}(f,z)) \le \delta(f,z)$$

So $\mathrm{HD}(\Lambda_{\rm c}(f)) \le \delta(f)$.

Proof. By the conditions (iii) and (i)

$$\Lambda_{\rm cw}(f,z,K) \subset \bigcap_{\varepsilon>0} \bigcap_{N\ge 0} \bigcup_{f^n(y)=z, n>N} B(y, r(n,y)^{1-\varepsilon})$$

where $r(n,y) := |(f^n)'(y)|^{-1}$ and the union consists only from terms satisfying $r(n,y) \le (1+K^{-1})^{-n}$. By the definition of $\delta(f,z)$ and the condition (i) for every $\alpha > \delta(f,z)$ the series $\sum_{n,y} r(n,y)^\alpha$ is convergent, even exponentially fast with n. Indeed, for $\eta > 0$ arbitrarily small and all n large enough we have

$$\sum_y r(n,y)^\alpha = \sum_y |(f^n)'(y)|^{-\delta(f,z)} |(f^n)'(y)|^{\delta(f,z)-\alpha}$$

$$\le (\exp \eta n)(1+K^{-1})^{-n(\alpha-\delta(f,z))}.$$

Hence $\sum_{n,y} r(n,y)^{\alpha(1-\varepsilon)} < \infty$ for $\varepsilon > 0$ small enough. So Hausdorff measure $\mathrm{H}_\alpha(\Lambda_{\mathrm{cw}}(f,z,K)) = 0$.

(This is similar to the Kleinian groups case. The discs $B(y, Kr(n,y))$ or $B(y, r(n,y)^{1-\varepsilon})$ correspond to "shadows".) $\square$

Theorem 2.2. $\mathrm{HD}_{\mathrm{mhyp}}(J) \leq \mathrm{HD}(\Lambda_{\mathrm{c}}(f))$.

Proof. This Theorem follows from the following property true for every f-invariant ergodic probability measure μ on J with characteristic Lyapunov exponent $\chi_\mu(f) = \int \log|f'| d\mu > 0$ and μ-a.e. x, (see [PUbook]):

There exists $\eta > 0$ and a sequence of integers $n_j \to \infty$ such that $\limsup_{j\to\infty} n_j/j \leq 2$, each f^{n_j} is injective on the component B'_j of $f^{-n_j}(B(f^{n_j}(x),\eta))$ which contains x, has distortion bounded by 2 (i.e. $(\forall y_1, y_2 \in B'_j) |(f^{n_j})'(y_1)|/|(f^{n_j})'(y_2)| \leq 2$) and $|(f^{n_j})'(x)| \geq \exp(n_j \chi_\mu(f)/2)$. (2.1)

For every (except at most two) z there exists $t > 0$ depending only on η and z such that for every $w \in J$ we have $f^t(B(w,\eta)) \ni z$. So for every x as above we find $x_j \in B'_j$ such that $f^{n_j+t}(x_j) = z$ and $|(f^{n_j})'(x_j)|^{-1} \geq \mathrm{dist}(x, x_j)/2\eta$. So $|(f^{n_j+t})'(x_j)|^{-1} \geq \mathrm{dist}(x,x_j)/(2\eta \sup|(f^t)'|)$, i.e. condition (ii) holds. Condition (i) also holds, with, say, $1 + K^{-1} = \exp(\chi/2)$. We conclude that μ-a.e. x belongs to $\Lambda_c(f)$, which by the definition of $\mathrm{HD}_{\mathrm{hyp}}$ proves the Theorem $\square$

(We have not made use here the property $\limsup_{j\to\infty} n_j/j \leq 2$. Taking this into account would suggest another definition of Λ_{c} and Λ_{cw}.)

The proof above and the definition (0.2) justify the following

Definition 2.3. $x \in \bar{\mathbb{C}}$ is called *regular* if it satisfies the property (2.1) with a number $\chi > 0$ (we need not link χ to any measure). We denote the set of regular points by $\Lambda_{\mathrm{reg}}(f)$.

We immediately obtain $\Lambda_{\mathrm{reg}}(f) \subset \Lambda_{\mathrm{c}}(f)$ and $\mathrm{HD}_{\mathrm{hyp}}(J) \leq \mathrm{HD}(\Lambda_{\mathrm{reg}}(f))$.

(Remark that if the property $\limsup n_j/j \leq 2$ is omitted in (1.2) then the inclusion above still holds. We need this property later, to obtain $\Lambda_{\mathrm{reg}}(f) \subset \Lambda_{\mathrm{tc}}(f)$.)

Recall that our aim is to prove

$$\underline{\alpha(f) = \mathrm{HD}_{\mathrm{mhyp}}(J) = \mathrm{HD}_{\mathrm{hyp}}(J) = \mathrm{HD}(\Lambda_{\mathrm{c}}(f)) = \delta(f).} \tag{2.2}$$

To this end it is left only to prove

Theorem 2.4. For every α-conformal measure μ there exists z such that $\delta(f,z) \leq \alpha$.

Proof. Write $O_n(\mathrm{Crit}) := \bigcup_{j=1}^n (\mathrm{Crit})$. We have $\mu(O_n(\mathrm{Crit})) = 0$, otherwise $\mu(\mathrm{Crit}) = \infty$. Notice that for every $n \geq 0$

$$\int_{J\setminus O_n(\mathrm{Crit})} P(z,\alpha,n)\, d\mu(z) \leq 1. \tag{2.3}$$

We obtain this by cutting a neighbourhood of J into a finite number of topological discs of boundaries of measure μ equal 0 containing $O_n(\mathrm{Crit})$. We consider all the branches of f^{-n} on each such disc U. For each such branch g we have by the definition of conformal measure $\mu(g(U)) = \int_U |g'|^\alpha d\mu$. Finally we sum these equalities up over all the branches and U's. (Notice that we cannot assert the equality in (2.3) because of possible atoms of μ at critical points.)

For every $\varepsilon > 0$ and n by (2.3)

$$\mu\{z : P(z,\alpha,n) \geq \exp n\varepsilon\} \leq \exp -n\varepsilon.$$

So for μ-a.e. z we have $\limsup_{n\to\infty} \frac{1}{n}\log P(z,\alpha,n) \leq \varepsilon$. Hence there exists z (even μ-a.e.) such that $\limsup_{n\to\infty} \frac{1}{n}\log P(z,\alpha,n) \leq 0$. Hence $\delta(f) \leq \alpha$ by (1.1), the definition of $\delta(f)$. $\square$

Remark 2.5. The proof of (2.2) is over in the general case. However in the case where f is a polynomial for example (or more generally if there exists a completely invariant basin of attraction to a sink, (for very non-polynomial examples see [P4]), one wants to consider $\delta(f,z)$ for z in the basin, whereas z produced in the above proof belongs to J. We succeeded due to our *ad hoc* definition (1.1), but if we want to consider only z in the basin, the Main Theorem in the next section becomes crucial.

Remark 2.6. The results of this Section rely on the equality (0.3), namely on the quite sophisticated Denker and Urbański construction of a conformal measure with the exponent $\alpha = \mathrm{HD}_{\mathrm{mhyp}}(J)$. Now *a posteriori* due to $\delta(f,z) = \mathrm{HD}_{\mathrm{mhyp}}(J)$ we obtain such a measure below, just by modifying slightly the Patterson-Sullivan construction (see also [DU1]). Unfortunately without any additional assumptions on f we do not know where this measure is supported. (We cannot for example exclude the possibility that this is supported in a Siegel disc S and its pre-images for iterates of f if $z \in S$.)

Fix z and write $\delta = \delta(f,z)$. Assume $\delta < \infty$. For all $0 < t \leq 1$ construct a sequence of positive numbers $\varphi_t(n)$, $n = 0,1,2,...$ such that $\lim_{t\nearrow 1}\varphi_t(n) = \varphi_1(n)$,

$$\lim_{t\nearrow 1}\limsup_{n\to\infty}\sup_{t\leq\tau\leq 1}\max\{\varphi_\tau(n)/\varphi_\tau(n+1), \varphi_\tau(n+1)/\varphi_\tau(n)\} = 1,$$

moreover

$$P(t) := \sum_n \varphi_t(n)P(z,\delta,n) < \infty \text{ for } t<1 \text{ and } \sum_n \varphi_1(n)P(z,\delta,n) = \infty.$$

To find such sequences proceed for example as follows: Write $a_n = P(z,\delta,n)$. Note that $\limsup_{n\to\infty}\frac{1}{n}\log a_n = 0$ (see Appendix 2, Prop. A2.2). It is not difficult to find b_n such that $b_n \geq a_n$, $\limsup_{n\to\infty}\frac{1}{n}\log b_n = 0$, $b_n/b_{n+1} \to 0$ and there exists a sequence of integers $n_i \to \infty$ such that $b_{n_i} = a_{n_i}$ for every i. Next define

$$\varphi_t(n) = b_n^{-1}P(z,\delta,n)\cdot t^n.$$

For $t < 1$ set

$$\mu_t = \sum_n \sum_{f^n(y)=z} D_y \cdot \varphi_t(n)|(f^n)'|^{-\delta}/P(t).$$

where D_y is the Dirac delta measure at y. Finally define μ as a weak* limit of μ_t as $t \nearrow 1$. As $P(1) = \infty$ the measure μ is δ-conformal.

We end this Section with the promised definition of $\Lambda_{\rm tc}$ the set of *tree conical* limit points. Recall first, [P3] or [PUZ], that all points of $\bigcup_{n\geq 0} f^{-n}(z)$ can be organized in a *geometric coding tree*. Briefly: we define a graph $\mathcal{T}$ by joining z to its f-preimages with curves $\gamma^1, ..., \gamma^d$, next consider all the curves $f^{-n}(\gamma_j)$. These curves are the edges of $\mathcal{T}$, whereas the points of $\bigcup_{n\geq 0} f^{-n}(z)$ are the vertices. Each sequence of symbols $\beta = (\beta_0, \beta_1, ...)$ for $1 \leq \beta_n \leq d = \deg f$ corresponds to a line of an infinite sequence of edges and vertices in $\mathcal{T}$, we call this an infinite branch and denote by $b(\beta)$. Now we are in the position to write:

Definition 2.7. x is *tree conical limit point*, i.e. $x \in \Lambda_{\rm tc}(f,z)$, iff there exists a branch $b(\beta)$ converging to x and a sequence of vertices $x_i \in b(\beta)$ such that x_i and integers n_i satisfy (i), (ii) (from the definition of $\Lambda_{\rm c}$).

Now recall that the main theorem of [P3], Theorem B, implies that if $\lim_{n\to\infty} \sup_{\gamma\in\mathcal{T}_n} \operatorname{diam}\gamma \to 0$ where $\mathcal{T}_n$ is the set of all the edges of the n-th generation (i.e. in $f^{-n}(\bigcup_{j=1,\dots,d} \gamma^j)$) then for every μ an f-invariant measure of positive Lyapunov exponent, μ-a.e. x is tree conical. (Formally [P3, Th.B] gives only the accessibility along $b(\beta)$; verifying (i), (ii) needs looking in the Proof to make use of the positive Lyapunov exponent.) This concerns in fact all the points satisfying (2.1). Thus for $\Lambda_{\rm tc}(f) = \bigcap_z \Lambda_{\rm tc}(f,z)$

$$\Lambda_{\rm reg}(f) \subset \Lambda_{\rm tc}(f).$$

In the case f is a polynomial and $z \in A_\infty$ we can replace $b(\alpha)$ by an *external ray* r and write $x_i \in r$ in Definition 2.7. This concerns simply connected A_∞ as well as non simply-connected, see [P3, Section 3].

SECTION 3. On the independence of δ and Λ of z.

We rely on the following combinatorial

Lemma 3.1 . There exists $C > 0$ such that for every set W of $n \geq 0$ points in $\bar{\mathbb{C}}$ and $0 < r < 1/2$, for every $z_1, z_2 \in \bar{\mathbb{C}} \setminus B(W,r)$ there exists a sequence of discs $B_1 = B(q_1, \rho_1), ..., B_k = B(q_k, \rho_k)$ such that for every $j = 1, ..., k$ each $2B_j := B(q_j, 2\rho_j)$ is disjoint from W, $z_1 \in B_1, z_2 \in B_k$, $\bigcup_{j=1}^k B_j$ is connected and else

$$k \leq C\sqrt{n}\sqrt{\log 1/r} \qquad \text{if} \quad n \geq \log 1/r$$
$$k \leq C \log 1/r \qquad \text{if} \quad n < \log 1/r.$$

Remark 3.2 Another formulation is to replace the number k of discs by the number of squares in the Whitney covering [Stein] (Our proof is in this spirit).

Notice that k is often much larger than $d_h(z_1, z_2)$ the distance in the hyperbolic metric d_h on $\bar{\mathbb{C}} \setminus W$ (suppose $\#W \geq 3$). If z_2 is fixed and the Euclidean distance of z_1 to W is r very close to 0, then k is of order $\log 1/r$ whereas $d_h(z_1, z_2)$ is of order $\log\log 1/r$.

One can replace k by the comparable quantity: the *quasi-hyperbolic* distance between z_1 and z_2 in $\bar{\mathbb{C}} \setminus W$, see [Po].

Proof of Lemma 3.1. See Appendix 1.

Theorem 3.3. There exists $E \subset \bar{\mathbb{C}}$ of Hausdorff dimension 0 such that for every $z_1, z_2 \in \bar{\mathbb{C}} \setminus E$ and every $\alpha \geq 0$ it holds $\mathcal{P}(\alpha) := \mathcal{P}(z_1, \alpha) = \mathcal{P}(z_2, \alpha)$ and $\delta := \delta(f, z_1) = \delta(f, z_2)$. Moreover for every $z \in \bar{\mathbb{C}}$ it holds $\mathcal{P}(\alpha) \leq \mathcal{P}(z, \alpha)$ and $\delta \leq \delta(f, z)$.

Proof. For every $n \geq 1$ set $r_n = \exp -\sqrt{n}$. Set $b_n = B(f^n(\mathrm{Crit}), r_n)$ $E' = \bigcap_N \bigcup_{n>N} b_n$ and finally $E = E' \cup O(\mathrm{Crit})$, where $O(\mathrm{Crit}) = \bigcup_{j=1}^\infty f^j(\mathrm{Crit})$.

$\mathrm{HD}(E) = 0$ because $\sum_{n=1}^\infty r_n^\varepsilon < \infty$ for every $\varepsilon > 0$ and Crit is finite. Consider now arbitrary $z_1, z_2 \in \bar{\mathbb{C}} \setminus E$. Then there exists N such that for every $n > N$, $z_\nu \notin b_n$ for $\nu = 1, 2$. Let $a = \min_{\nu=1,2, j=1,\dots,N} \mathrm{dist}(f^j(\mathrm{Crit}), z_\nu)$. Fix an arbitrary $n > N$ large enough that $r = r_n < a$. Set $W = \bigcup_{j=1,\dots,n} f^j(\mathrm{Crit})$. We apply now Lemma 3.1. and consider the discs $B_1, ..., B_k$.

We can assume that the diameters of B_j are smaller than a constant κ, depending only on f, so that for all the components of $f^{-n}(2B_j)$ the diameters of the complements in $\bar{\mathbb{C}}$ are larger than a constant (for this it is sufficient to have κ smaller than the minimal distance between each two distinct points of a periodic orbit of period at least 3). This influences the constant C in Lemma 3.1. We conclude that there exists a constant $\Delta > 0$ (not depending on n) such that the distortion (see the definition in (2.1)) of all the branches of f^{-n} on each B_j is bounded by Δ. (This is a variant of Koebe's distortion theorem, see for example [P5].)

Thus, for every $\alpha > 0$, using Lemma 3.1 the case $n \geq \log 1/r$, moreover, using $n \geq \log 1/\sqrt{r}$, for n large enough

$$P(z_1, \alpha, n)/P(z_2, \alpha, n) \leq \Delta^{\alpha k} \leq \Delta^{\alpha C n^{3/4}}$$

hence $\limsup_{n\to\infty} \frac{1}{n} \log P(z_1, \alpha, n) = \limsup_{n\to\infty} \frac{1}{n} \log P(z_2, \alpha, n)$, hence $\mathcal{P}(z_1, \alpha) = \mathcal{P}(z_2, \alpha)$ and $\delta(z_1) = \delta(z_2)$. The first part of the proof is over.

Consider now an arbitrary $z \in \bar{\mathbb{C}} \setminus O(\mathrm{Crit})$. The following holds, see [P1,Lemma 3]:

$\forall(0 < \varepsilon_1 < 1)$ $\exists(C > 0, \varepsilon_2 > 0)$ $\forall(0 < \varepsilon \leq \varepsilon_2)$ $\forall(m > 0)$ the set $f^{-m}(z)$ contains at least the number $C(\deg f)^{\varepsilon_1 m}$ of (m, ε)-separated points. (Recall that x, y are called (m, ε)-separated if $\max_{j=0,\dots,m} \mathrm{dist}(f^j(x), f^j(y)) \geq \varepsilon$.)

Fix now $\varepsilon_1 = 3/4$ and suppose that m is small enough that $2L^m r_n \leq \varepsilon_2$, where $L = \sup |f'|$. To be concrete we set

$$m = \log(\varepsilon_2/2r_n)/\log L = \log(\varepsilon_2/2)/\log L + n^{1/2}/\log L \tag{3.1}$$

We shall calculate how large m need be that $f^{-m}(\{z\}) \not\subset \bigcup_{j=1}^n B(f^j(\mathrm{Crit}), r_n)$. It is sufficient to have

$$C(\deg f)^{3m/4} > n\sharp(\mathrm{Crit}). \tag{3.2}$$

Indeed if two distinct points $x, y \in f^{-m}(z)$ are in the same disc $B(f^i(c), r_n)$ for $c \in \mathrm{Crit}$, then their $f^j, j \leq m$, images are not more than $L^m 2r_n$ apart, so x and y are not (m, ε_2)-separated.

Observe finally that m defined in (3.1) satisfies (3.2) if n is large enough.

So we have found $y \in f^{-m}(z) \setminus \bigcup_{j=1}^{n} B(f^j(\mathrm{Crit}), r_n)$. Fix $z_2 \in \bar{\mathbb{C}} \setminus E$. For an arbitrary $\varepsilon > 0$ if n is large enough $P(y, \alpha, n) \geq (\exp -n\varepsilon) P(z_2, \alpha, n)$ by the first part of the proof. So

$$P(z, \alpha, n+m) \geq L^{-m}(\exp -n\varepsilon) P(z_2, \alpha, n) \geq (\exp -2n\varepsilon) P(z_2, \alpha, n).$$

Again using (3.1) that m grows much slower than n we obtain $\mathcal{P}(z_2, \alpha) \leq \mathcal{P}(z, \alpha)$ and $\delta(z) \geq \delta(z_2)$. The Theorem is proved. $\square$

Theorem 3.4. There exists $E \subset \bar{\mathbb{C}}$ of Hausdorff dimension 0 such that for every $z_1, z_2 \in \bar{\mathbb{C}} \setminus E$

$$\Lambda_{\mathrm{cw}}(f) := \Lambda_{\mathrm{cw}}(f, z_1) = \Lambda_{\mathrm{cw}}(f, z_2)$$

and for every $z \in \bar{\mathbb{C}}$ which has a backward trajectory omitting Crit, $\Lambda_{\mathrm{cw}}(f, z) \supset \Lambda_{\mathrm{cw}}(f)$.

Proof. We set E the same as in Proof of Theorem 3.3. Let $x \in \Lambda_{\mathrm{cw}}(f, z_1, K)$ and x_i be a sequence of f^{n_i}-preimages of z_1 converging to x satisfying (i) and (iii) in the definition of Λ_{cw}. For each n large enough there is a 1-to-1 correspondence between $f^{-n}(z_1)$ and $f^{-n}(z_2)$. Namely each branch of f^{-n} in a neighbourhood of z_1 extends along the chain $B_1, ..., B_k$ (see Proof of Theorem 3.1) to z_2. So let x_i corresponds to $x_i' \in f^{-n_i}(z_2)$. We obtain for n_i large enough

$$\frac{|(f^{n_i})'(x_i')|}{|(f^{n_i})'(x_i)|} \geq (1 - (2K)^{-1})^{n_i} \quad \text{hence} \quad |(f^{n_i})'(x_i')| \geq (1 + (3K)^{-1})^{n_i}$$

if K is large enough. We obtain also the growth of $|\mathrm{dist}(x_i, x_i')(f^{n_i})'(x_i')|$ slower than every exponential as $n_i \to \infty$. So

$$\limsup_{i\to\infty} \log\bigl(\mathrm{dist}(x, x_i') \cdot |(f^{n_i})'(x_i')|\bigr) \leq$$
$$\leq \limsup_{i\to\infty} \log\bigl(\mathrm{dist}(x, x_i) \cdot |(f^{n_i})'(x_i')|\bigr) \leq 0.$$

So $x \in \Lambda_{\mathrm{cw}}(f, z_2, 3K)$.

For an arbitrary z which has a backward trajectory τ omitting Crit we find $z_1 \in \tau$ such that $z_1 \notin O(\mathrm{Crit})$ and next for every n we find $y \in f^{-m}(z_1)$ as in Proof of Theorem 3.3. (i.e. not too close to $\bigcup_{j=1}^{n} f^j(\mathrm{Crit})$). We can have additionally $|(f^m)'(y)| \geq 1$ because most of points in $f^{-m}(z_1)$ satisfy this, see for example [FLM]. So we can repeat the above estimates, using $m/n \to 0$. $\square$

Appendix 1. Proof of Lemma 3.1.

We can assume $z_1 = -1, z_2 = 1$ in $\mathbb{C}$. Next change the coordinates on the union of the triangles Δ_1 with the vertices $-1, i, -i$ and Δ_2 with the vertices $1, i, -i$ by a map Φ to the strip $T := \{0 \leq \Im(z) \leq 1\}$ as follows: First deform Δ_1 to Δ_1' the domain between the straight rays from -1 through i and $-i$, and the arc (containing

0) of the circle with origin at -1 and radius 1. Next map Δ_1' to $T \cap \{\Re z \leq 0\}$ by $z \mapsto \frac{i}{2} + \frac{2}{\pi}\log(z+1)$. On Δ_2 write $\Phi = S \circ \Phi \circ S^{-1}$ where S is the symmetry with respect to the imaginary axis.

The Proof of the Lemma reduces now to a construction of a "chain of squares" in T i.e. a family of squares with a smallest possible number of elements, not intersecting $\Phi(W)$, which union joins the interval $\Re z = -m$ to $\Re z = m$ for $m = 2\log 1/r$, with every two adjacent squares in the chain of a comparable size. We shall use certain *tryadic squares*, see below.

We can assume $2m = 3^l, n = 3^t$ for certain non-negative even integers l, t. Write $W' = \Phi(W)$. We can adjust our construction by changing Φ slightly so that no $\Re z, \Im z$ for $z \in W'$ is rational tryadic.

Define inductively a sequence of horizontal strips

$$T_j = \{z : a_j \cdot 3^{-j} \leq \Im z \leq (a_j + 1) \cdot 3^{-j}\}$$

where $T_0 = T$, such that $T_{j+1} \subset T_j$ and $\sharp(W' \cap T_j) \leq 3^{t-j}$ for every j. In particular $W' \cap T_{t+1} = \emptyset$.

Call every interval in $[-m, m]$ of the form $[b \cdot 3^{-v}, (b+1) \cdot 3^{-v}]$ for an integer b and non-negative integer v, tryadic or v-tryadic. For every v-tryadic I denote by $K(I)$ the square $I \times T_v$. These are the squares we shall use to construct our chain.

For every v-tryadic I we define

$$\varphi(I) := \inf\{j \geq 0 : W' \cap T_j \cap \{\Re z \in I\} = \emptyset\}.$$

We shall define now by induction certain families of tryadic intervals. Let $\mathcal{I}_0$ consists of three $(t+1)$-tryadic intervals with maximal possible φ.

Suppose that a family $\mathcal{I}_j$ is already defined and it consists of $(t+1-j)$-tryadic intervals. Suppose $\#\mathcal{I}_j = 3^{j+1}$. Suppose also that $j < t+1$ and $2j - t < l$. The first inequality means that $K(I)$'s for $I \in \mathcal{I}_j$ are not yet of the side 1 (the maximal possible size). The latter means that $3^{-(t+1-j)}3^{j+1} < 3^l$ i.e. $\mathcal{I}_j$ does not cover yet the whole $[-m, m]$.

We construct $\mathcal{I}_{j+1}$:

Every $I \in \mathcal{I}_j$ is contained in a $(t-j)$-tryadic $\hat{I}$. We denote the set of such intervals $\hat{I}$ by $\mathcal{I}^1_{j+1}$ and include in $\mathcal{I}_{j+1}$. If I is not adjacent to $-m$ or m or is not a middle interval of $\hat{I}$ we include also in $\mathcal{I}_{j+1}$ the $(t-j)$-tryadic interval adjacent to I. We have up to now in $\mathcal{I}_{j+1}$ a family $\mathcal{I}^2_{j+1}$ of at most $2 \cdot 3^{j+1}$ elements. Complete it to $\mathcal{I}_{j+1}$ so that $\#\mathcal{I}_{j+1} = 3^{j+2}$ by $(t-j)$-tryadic intervals with maximal possible values of φ.

For every j write $\mathcal{I}^{\mathrm{ad}}_j$ for the family of all the $(t+1-j)$-tryadic intervals in $\bigcup\{\hat{I} : I \in \mathcal{I}_j\} \setminus \bigcup\{I \in \mathcal{I}_j\}$

We include in our chain of squares joining $\Re z = -m$ to $\Re z = m$ all the squares $K(I)$ for

$$I \in \mathcal{I} := \mathcal{I}_0 \cup \bigcup_j \mathcal{I}^{\mathrm{ad}}_{j-1} \cup \bigcup_j (\mathcal{I}_j \setminus \mathcal{I}^1_j)$$

. The unions are over $j = 1, ...(l+t)/2$ if $(l+t)/2 \leq t+1$ i.e. $l \leq t+2$, in particular $\log 1/r < n$. In this case $\mathcal{I}$ covers $[-m, m]$. The number of squares in the chain is bounded by $\sum_{j=0}^{(l+t)/2} 3^{j+2} \leq \mathrm{Const}\sqrt{m}\sqrt{n}$ as asserted in the Lemma.

In the case $l > t+2$ the union is over $j = 1, ..., t+1$ and $[-m, m]$ is not yet covered by the intervals in $\mathcal{I}$. Fortunately all the points of W' lie over $I \in \mathcal{I}$, so we just add to the chain an appropriate family of unit squares (0-tryadic). The total number of squares in the chain does not exceed $2m + \text{Const}\; n \geq \text{Const} \log 1/r$ as asserted in the Lemma.

We end the Proof by checking that all our squares are indeed disjoint from W'. By induction we prove that for every j and interval I with interior not intersecting $\bigcup \mathcal{I}_j$ we have $\varphi(I) \leq t - j$. In particular for $I \in \mathcal{I}_{j+1} \setminus I^1_{j+1}$ we have $K(I) \cap W' = \emptyset$. This will end the Proof, by the definition of $\mathcal{I}$.

For $j = 0$, outside $\bigcup\{I \in \mathcal{I}_0\}$ we have $\varphi \leq t-1$ which is even better than demanded. So the same estimate is sufficient outside $\bigcup\{I \in \mathcal{I}_1\}$. To build $\mathcal{I}_2$ we added to $\mathcal{I}^2_2$ at least nine $(t-1)$-tryadic intervals, so we have exhausted all for which $\varphi > t-2$.

In general: To build $\mathcal{I}_j$ we added to $\mathcal{I}^2_j$ at least 3^j number of $(t+1-j)$-tryadic intervals, so among them all for which $\varphi > t-j$. $\square$

Appendix 2. Pressure.

Proposition A2.1. For all $\alpha \geq 0$ $\mathcal{P}(\alpha) \geq \log \deg f - \alpha \log \sup |f'|$. $\square$

Proposition A2.2. $\mathcal{P}(z, \alpha)$ and $\mathcal{P}(\alpha)$ are Lipschitz continuous functions of $\alpha \geq 0$.

Proof. If there exists $\beta > 0$ such that $\mathcal{P}(z, \beta) < \infty$ then for every $\varepsilon_1 > 0$ one has $P(z, \beta, n) \leq \exp n(\mathcal{P}(z, \beta) + \varepsilon_1)$ for n large enough. Hence for every $y \in f^{-n}(z)$ one has $|(f^n)'(y)|^{-\beta} \leq \exp n(\mathcal{P}(z, \alpha) + \varepsilon_1)$, hence $|(f^n)'(y)| \geq \exp -n(\mathcal{P}(z, \beta) + \varepsilon_1)/\beta$. One has also $|(f^n)'(y)| \leq \sup |f'|^n$. So for every $\alpha \geq 0$, ε such that $\alpha + \varepsilon \geq 0$ and n large and $|\varepsilon| \leq \varepsilon_0$

$$P(z, \alpha, n) \exp\big(-|\varepsilon| n D(\varepsilon_1, z, \beta)\big) \leq P(z, \alpha + \varepsilon, n) \leq P(z, \alpha, n)\big(\exp |\varepsilon| n D(\varepsilon_1, z, \beta)\big),$$

where $D(\varepsilon_1, z, \beta) := \max\{(\mathcal{P}(z, \beta) + \varepsilon_1)/\beta, \log \sup |f'|\}$. We conclude that $|\mathcal{P}(z, \alpha + \varepsilon) - \mathcal{P}(z, \alpha)| \leq |\varepsilon| D(z)$ where $D(z) := \inf\{D(0, z, \beta) : \mathcal{P}(z, \beta) < \infty\}$ and $|\mathcal{P}(\alpha + \varepsilon) - \mathcal{P}(\alpha)| \leq D|\varepsilon_0| D(\beta)$, where $D = \inf_z D(z)$. Thus we have proved Lipschitz continuity $\forall \alpha \geq 0$.

We have proved also that either $(\forall \alpha > 0) \mathcal{P}(z, \alpha) = \infty$ or $(\forall \alpha > 0) \mathcal{P}(z, \alpha) < \infty$ $\square$

Proposition A2.3. $\mathcal{P}(\alpha)$ is monotone decreasing.

Proof. For $z \notin E$ (see Proof of Th.3.3) for every n large enough there exist all the branches of f^{-n} on $B(z, r_n)$. Hence, by $r_n = \exp -\sqrt{n}$ and by Koebe's distortion theorem, there exists $C > 0$ such that for every $y \in f^{-n}(z)$ one has $|(f^n)'(y)|^{-1} \leq C \operatorname{diam} \operatorname{Comp}_y f^{-n}(B(z, r_n/2)/r_n \leq C \exp \sqrt{n}$. Therefore for all $\alpha \geq 0, \varepsilon \geq 0$ one has

$$P(z, \alpha + \varepsilon, n) \leq P(z, \alpha, n) \sup_{f^n(y) = z} |(f^n)'(y)|^{-\varepsilon} \leq P(z, \alpha, n) C^\alpha \exp \varepsilon \sqrt{n}$$

and applying $\limsup_{n\to\infty}\frac{1}{n}\log$ to these expresions one gets
$\mathcal{P}(z,\alpha+\varepsilon)\le\mathcal{P}(z,\alpha)$. $\square$

Recall now the definition of pressure by Denker and Urbański [DU2]:

Definition A2.4. Let V be an open set in J such that $V\supset \mathrm{Crit}\cap J$. Define $K(V):=J\setminus\bigcup_{n\ge 0}f^{-n}(V)$. As $K(V)\cap\mathrm{Crit}=\emptyset$ we can consider the standard topological pressure $\mathcal{P}(f|_{K(V)},-\alpha\log|f'|)$ for the map $f|_{K(V)}$ and the real continuous function $-\alpha\log|f'|$ on the compact set $K(V)$, see [W]. Define finally

$$\mathcal{P}_{\mathrm{DU}}(\alpha)=\sup_V \mathcal{P}(f|_{K(V)},-\alpha\log|f'|)$$

supremum over all V considered above.

Two other definitions are of interest:

Definition A2.5. *Hyperbolic pressure*

$$\mathcal{P}_{\mathrm{hyp}}(\alpha):=\sup_X \mathcal{P}(f|_X,-\alpha\log|f'|),$$

supremum taken over all isolated hyperbolic subsets of J.

Definition A2.6. *Hyperbolic variational pressure*

$$\mathcal{P}_{\mathrm{hypvar}}(\alpha):=\sup_\mu\{\mathrm{h}_\mu(f)-\alpha\int\log|f'|d\mu\},$$

supremum taken over all ergodic f-invariant measures of positive Lyapunov exponent, i.e. $\chi_\mu(f)=\int\log|f'|d\mu>0$.

Definition A2.7. *Variational pressure*

$$\mathcal{P}_{\mathrm{var}}(\alpha):=\sup_\mu\{\mathrm{h}_\mu(f)-\alpha\int\log|f'|d\mu\},$$

supremum taken over all ergodic f-invariant measures on J.

Definition A2.8. *minimal conformal eigenvalue*

$$\lambda(\alpha)=\inf\{\lambda>0:\exists\mu \text{ a probability measure on } J \text{ with Jacobian } \lambda|f'|^\alpha\},$$

Here we say $\varphi:J\to\mathbb{R},\varphi\ge 0$ is Jacobian for μ and $f|_J$ if φ is μ-integrable and for every Borel set $E\in J$ on which f is injective $\mu(f(E))=\int_E|f'|^\alpha d\mu$. We write $\varphi=\mathrm{Jac}_\mu(f|_J)$. (Compare the definition of α-conformal at the end of Introduction, there Jacobian was equal to $|f'|^\alpha$.)

Theorem A2.9. For every $0\le\alpha\le\delta$

$$\log\lambda(\alpha)=\mathcal{P}(\alpha)=\mathcal{P}_{\mathrm{var}}(\alpha)=\mathcal{P}_{\mathrm{hypvar}}(\alpha)=\mathcal{P}_{\mathrm{hyp}}(\alpha)=\mathcal{P}_{DU}(\alpha). \tag{A2.1}$$

All these functions are Lipschitz continuous and monotone decreasing on $\alpha\ge 0$.

Sketch of Proof. (It virtually repeats [DU2] and Section 2.)

1. We prove $(\forall\alpha\geq 0)$ $\mathcal{P}(\alpha)\geq\mathcal{P}_{\rm hypvar}(\alpha)$: For every μ as in the definition of $\mathcal{P}_{\rm hypvar}$ for every $\varepsilon>0$ arbitrarily small and n large enough, one constructs an (n,ε)-separated set S_n such that $\sum_{y\in S_n}|(f^n)'(y)|^{-\alpha}\geq\exp n\big(h_\mu(f)-\alpha\int\log|f'|d\mu-\varepsilon\big)$. (This is Katok's construction, see for example [PUbook].) One can assume also that 2.1 holds (for a sequence of n's) and replace $y\in S_n$ by $y\in f^{-n-t}(z)$ as in Proof of Theorem 2.2.

2. $\forall\alpha\geq 0$ $\mathcal{P}_{\rm hypvar}(\alpha)=\mathcal{P}_{\rm hyp}(\alpha)$: The $\geq$ inequality follows from Variational Principle, see [W], and the obvious fact that every probability f-invariant measure on a hyperbolic set X has positive Lyapunov exponent. The opposite inequality results from Katok's construction (the sets S_n above are in fact constructed in respective hyperbolic sets).

3. If $\mathcal{P}_{\rm DU}(\alpha)>0$ then $\mathcal{P}_{\rm hypvar}(\alpha)\geq\mathcal{P}_{\rm DU}(\alpha)$: For every $\varepsilon>0$ one can find, by Variational Principle, an ergodic f-invariant μ on $K(V)$ such that $h_\mu(f)-\alpha\int\log|f'|d\mu>P(f|_{K(V)},-\alpha\log|f'|)-\varepsilon$. For ε small enough the latter expression is positive and
$\int\log|f'|d\mu\geq 0$ by [P2] or [DU2, Cor. 4.2]. Hence $h_\mu(f)$ and therefore $\int\log|f'|d\mu$ are strictly positive, see [R] or [PUbook, Ch.8.1].

4. If $\mathcal{P}_{\rm var}(\alpha)>0$, then $\mathcal{P}_{\rm var}(\alpha)=\mathcal{P}_{\rm hypvar}(\alpha)$: Indeed, as in 3. if $h_\mu(f)-\alpha\int\log|f'|d\mu>\mathcal{P}_{\rm var}(\alpha)-\varepsilon$ then for ε small enough this is positive. As $\int\log|f'|d\mu\geq 0$, by [P2], we obtain $h_\mu(f)$ and therefore $\int\log|f'|d\mu$ strictly positive. Hence $\mathcal{P}_{\rm hypvar}(\alpha)\geq\mathcal{P}_{\rm var}(\alpha)-\varepsilon$ for every $\varepsilon>0$.

5. We prove that for every $0\leq\alpha<\delta(f)$ there exists a sequence of decreasing V_n's such that $\mathcal{P}(f|_{K(V_n)},-\alpha\log|f'|)>0$ and $\lim_{n\to\infty}\mathcal{P}(f|_{K(V_n)},-\alpha\log|f'|)\geq\log\lambda(\alpha)$. In particular $\mathcal{P}_{\rm DU}(\alpha)>0$ and else $\mathcal{P}_{\rm DU}(\alpha)\geq\log\lambda(\alpha)$:

Take V_n of the form $\bigcup_{j\geq 0}f^{-j}(\hat V_n)$ where $\hat V_n$ is a union of small discs $B(x_c,r_n)$ for a distinguished point x_c in the ω-limit set for each $c\in\operatorname{Crit}\cap J$. One can choose x_c so that there exists $C>0$ such that for a sequence $j_k\to\infty$, $|(f^{j_k})'(x_c)|\geq C$, [P2]. Hence, [DU2, Lemma 5.4], for $r_n\searrow 0$ there exists a sequence of measures μ_n on $K(V_n)$ with Jacobians
$\operatorname{Jac}_{\mu_n}(f|_{K(V_n)})=\lambda_n|f'|^\alpha$ for $1\leq\lambda_n\leq\exp\mathcal{P}(f|_{K(V_n)},-\alpha\log|f'|)$ on $K(V_n)\setminus\partial\hat V_n$
Here by Jacobian we understand, analogously as in Definition A2.8, a nonnegative μ_n-integrable function φ on $K(V)$, such that for every Borel $E\subset K(V_n)$ on which f is injective $\mu(f(E))=\int_E\varphi d\mu$. Moreover for every $E\in\partial\hat V_n$ on which f is injective

$$\mu_n(f(E))\geq\lambda_n\int_E|f'|^\alpha d\mu_n. \tag{A2.2}$$

A weak* limit $\mu=\lim_{n_k\to\infty}\mu_{n_k}$ has Jacobian satisfying

$$\operatorname{Jac}_\mu(f|_J)=\lambda|f'|^\alpha.$$

for $\lambda=\lim\lambda_n$. We have made use here of the fact that μ has no atoms at $f(x_c)$, because $\liminf\lambda_n\geq 1$ and $\sum_j|(f^j)'(x_c)|^\alpha=\infty$. Such atoms would cause troubles with an estimate of Jacobian, see [DU2], because $\partial\hat V_n$ accumulate at x_c and we have only an inequality in (A2.2) resulting with only an inequality for μ for $E=\{x_c\}$.)

We have applied here $\mathcal{P}(f|_{K(V_n)},-\alpha\log|f'|)>0$. If it were not true, we would find $\alpha_n\leq\alpha$ such that $1/n>\mathcal{P}(f|_{K(V_n)},-\alpha_n\log|f'|)>0$ and the above construction would give a β-conformal measure with exponent $\beta\leq\alpha$. Then however, by (2.2) (Th. 2.4), $\delta(f)\leq\beta\leq\alpha<\delta(f)$, contradiction.

We have also made use here of $\lambda<\infty$ true because $\mathcal{P}(f|_{K(V_n)},-\alpha\log|f'|)$ were upper bounded by $\mathcal{P}(\alpha)$ due to the already proved **1.-3.**.

6. $\forall\alpha\geq 0\ \log\lambda(\alpha)=\mathcal{P}(\alpha)$: Indeed, if $\mathrm{Jac}_\mu(f|_J)=\lambda|f'|^\alpha$ then we find z for which $\mathcal{P}(z,\alpha)\leq\log\lambda$ as in Proof of Theorem 2.4. Hence $\mathcal{P}(\alpha)\leq\log\lambda(\alpha)$. The opposite inequality follows already from **1.-5.** for $\alpha<\delta(f)$. For an arbitrary $\alpha\geq 0$ one can construct μ with Jacobian $\exp\mathcal{P}(z,\alpha)$ by Patterson-Sullivan's method as in Remark 2.6.

7. The Lipschitz continuity and monotone decreasing of $\mathcal{P}_{\mathrm{var}}(\alpha)$, $\mathcal{P}_{\mathrm{hypvar}}(\alpha)$, $\mathcal{P}_{\mathrm{hyp}}(\alpha)$ and $\mathcal{P}_{\mathrm{DU}}(\alpha)$ follows from the definitions (variational principle) and from $0\leq\int\log|f'|\,d\mu\leq\sup|f'|$ for every probability invariant measure μ on J [P2]. Therefore (A2.1) extends from $\alpha<\delta$ to $\alpha=\delta$. $\square$

Corollary A2.10. $\mathcal{P}(\alpha)$ is a strictly decreasing, convex function on $0\leq\alpha\leq\delta(f)$.

Proof. This is so for the affine function $h_\mu(f)-\alpha\int\log|f'|d\mu$ for each μ of positive Lyapunov exponent, so supremum over μ's, namely $\mathcal{P}_{\mathrm{hypvar}}(\alpha)$ is monotone decreasing, convex. As this attains 0 at $\delta(f)$ the convexity implies this is strictly decreasing. $\square$

Remark A2.11. The equalities (A2.1) immediately yield the equalities (2.2). Namely the least zeros of $\log\lambda(\alpha),\mathcal{P}(\alpha),\mathcal{P}_{\mathrm{hypvar}}(\alpha),\mathcal{P}_{\mathrm{hyp}}(\alpha)$ are equal to $\alpha(f),\delta(f),\mathrm{HD}_{\mathrm{mhyp}}(J)$ and $\mathrm{HD}_{\mathrm{hyp}}(J)$ respectively.

Appendix 3. Some properties of $\Lambda_{\mathrm{c}},\Lambda_{\mathrm{cw}}$ and other definitions of "conical".

After distributing the first version of this paper I was asked about a relation between the definitions of "conical limit set" in the recent preprints [LM], [U2], [DMNU] and my definitions. In [U2] and [DMNU] x is called conical if there exists $\eta>0$ and a sequence of integers $n_j\to\infty$ such that each f^{n_j} is injective on $\mathrm{Comp}_x f^{-n_j}(B(f^{n_j}(x),\eta))$ (Comp_x means the component containing x). We denote the set of points "conical" in this sense by $\Lambda_{\mathrm{U}}(f)$.

Of course $\Lambda_{\mathrm{U}}(f)\supset\Lambda_{\mathrm{reg}}(f)$.

Suppose that f has no critical points in J but the set P of periodic parabolic points ($f^k(p)=p,(f^k)'(p)$ is a root of unity) is non-empty. Then $J=\Lambda_{\mathrm{U}}(f)\cup\bigcup_{n\geq 0}f^{-n}(P)$ and $\Lambda_{\mathrm{U}}(f)\cap\bigcup_{n\geq 0}f^{-n}(P)=\emptyset$. This is similar to the geometrically finite Kleinian groups case, see [Maskit, VI.C.3]. This was in fact a motivation for the definition of $\Lambda_{\mathrm{U}}(f)$ in [U2] Unfortunately this is not so for Λ_{cw}:

Proposition A3.1 For each f with no critical points in J, with $P\neq\emptyset$ there exist points which are neither in $\bigcup_{n\geq 0}f^{-n}(P)$ nor in $\Lambda_{\mathrm{cw}}(f)$.

Proof. Consider a finite Markov partition of J, attribute the symbol 0 to all its cells whose closures contain a periodic parabolic point p, and attribute other symbols to other cells. For each $x\in J$ choose a sequence $a_j(x)$ of symbols so that each is attributed to a cell whose closure contains $f^j(x)$. We prove that if

$$\lim_{n\to\infty}\sharp\{0\leq j<n: a_j(x)=0\}/n=1, \tag{a}$$

then $x\notin\Lambda_{\mathrm{cw}}(f,z)$.

Indeed, suppose $y = x_i$ satisfies (i), (iii) for $n = n_i$. Hence for a constant $0 < A < 1$, for n large enough, $\mathrm{dist}(x,y) \le A^n$. We shall prove that in consequence

$$\mathrm{dist}(f^n(x), f^n(y)) \le B^n \mathrm{dist}(x,y), \tag{b}$$

where $B \searrow 1$ as $n \to \infty$. Indeed, join x to y by an interval I. Fix an arbitrary $B : A^{-1} > B > 1$. By (a), due to $|f'| \approx 1$ near P, there exists $n(B)$ such that for every $t \ge n(B)$ it holds $|(f^t)'(x)| \le (\sqrt{B})^t$ and for every $w \in J, v \in \overline{\mathbb{C}}$ with $\mathrm{dist}(w,v) \le (AB)^{n(B)}$ it holds $|f'(v)|/|f'(w)| < \sqrt{B}$. Suppose that (b) fails and $j : n(B) \le j \le n$ is the first integer such that there exists $z \in I$ satisfying $\mathrm{dist}(f^j(x), f^j(z)) > B^j \mathrm{dist}(x,z)$. In particular for every $n(B) \le j' < j$ and $v \in [x,z]$ we have $\mathrm{dist}(f^{j'}(x), f^{j'}(v)] \le B^{j'}A^n \le (AB)^n \le (AB)^{n(B)}$. We have also $\mathrm{dist}(f^{j'}(x), f^{j'}(v)] \le (AB)^{n(B)}$ for $0 \le j' < n(B)$ provided A^n is small, i.e. n is large enough. Hence

$$\mathrm{dist}(f^j(x), f^j(z)) \le \int_{[x,z]} |(f^j)'(v)| dv \le (\sqrt{B})^j (\sqrt{B})^j \mathrm{dist}(x,z) = B^j \mathrm{dist}(x,z)$$

a contradiction. So (b) has been proved

We conclude from (a) and (b) that $|(f^n)'(y)| \le \mathrm{Const} B^n$ with $B \searrow 1$, what contradicts (i). $\square$

(I owe a simplification to M. Urbański.)

We conclude that $\Lambda_{\mathrm{U}}(f) \not\subset \Lambda_{\mathrm{cw}}(f)$.

Clearly the sets $\Lambda_{\mathrm{U}}(f), \Lambda_{\mathrm{c}}(f,z)$ and $\Lambda_{\mathrm{cw}}(f,z)$ are forward invariant.

For our Λ's also the backward invariance hold (unlike for $\Lambda_{\mathcal{U}}$, see for example the paragraph preceding Definition A3.5):

Proposition A3.2. For every rational f and every non-exceptional z (in the sense of Theorem 3.1) $f^{-1}(\Lambda_{\mathrm{c}}) = \Lambda_c$ and $f^{-1}(\Lambda_{\mathrm{cw}}) = \Lambda_{\mathrm{cw}}$.

Proof. Suppose $x = f(x') \in \Lambda_{\mathrm{c}}(f,z)$. Assume $x' \in \mathrm{Crit}(f)$, as for x' non-critical $x' \in \Lambda_{\mathrm{c}}(f,z)$ follows immediately. Let x_i be points approximating x according to Definition 1.2. Then as $z \notin E$ there exists a branch of f^{-n_i} on $B(z, \exp(-\sqrt{n_i}))$ mapping z to x_i, for n_i large enough. So by bounded distortion $\mathrm{dist}(x,x_i) \ge \mathrm{Const}|(f^{n_i})'(x_i)|^{-1}\exp(-\sqrt{n_i})$. Hence, for x_i' the f-preimage of x_i, near x', we obtain

$$|(f^{n_i+1})'(x_i')|^{-1} \le \mathrm{Const}(1+K^{-1})^{-(1/\nu)n_i} \exp((1-1/\nu)\sqrt{n_i})$$

so for x' (i) holds with x_i', n_i+1 for i large enough (and new K). ν is here the multiplicity of f at x'.

The property (ii) for x' follows immediately from $|f'(x_i')|^{-1} \ge \frac{\mathrm{dist}(x',x_i')}{2\nu \mathrm{dist}(x,x_i)}$.

The proof for Λ_{cw} is similar. $\square$

The following rational maps are of interest, see [PR]:

Definition A3.3. f is called *topological Collet-Eckmann* if there exist $M, N, \eta > 0$ such that for every $x \in J$

$$\exists n_j \to \infty, n_j \le Nj, \quad \text{such that each} \quad f^{n_j} \quad \text{has degree at most} \quad M$$

$$\text{on} \ \ \text{Comp}_x f^{-n_j} B(f^{n_j}(x),\eta). \tag{A3.1}$$

(Recall that Comp_x means the component containing x.)

Proposition A3.4. For every rational f, every $x \in J(f)$, for which there exist $M, N, \eta > 0$ such that (A3.1) is satisfied, is conical, more precisely $x \in \Lambda_{\text{c}}(f,z)$ for every $z \notin O(\text{Crit})$. In particular if f is topological Collet-Eckmann then $J(f) = \Lambda_{\text{c}}(f)$.

Proof. It follows from the proof of [PR, Prop.3.1] that there exist $0 < \xi < 1, \eta > 0$ and a sequence of integers $n_j \to \infty$ such that for each $W_j := \text{Comp}_x f^{-n_j}(B(f^{n_j}(x),\eta))$ we have together with (A3.1) also

$$\text{diam} W_j \leq \xi^{n_j}. \tag{A3.2}$$

Due to (A3.1) we can also assume, taking η small enough, that all $f^k(W_j), k = 0,1,...,n_j$ have small diameters, see [M].

(Later, in [P6, Lemma 3.6], it has been proved that (A3.2) holds for all n.)

Fix an arbitrary $n = n_j$. We claim that there exists a disc $D \subset B(f^n(x),\eta)$ of diameter at least $A\eta$ for a constant A depending only on M such that there exists $y \in W \cap f^{-n}(D)$ satisfying

$$|(f^n)'(y)| \geq A\xi^{-n} \tag{A3.3}$$

and

$$\text{dist}(x,y) \leq A^{-1}|(f^n)'(y)|^{-1}. \tag{A3.4}$$

Indeed, there exists an integer $0 \leq k \leq M$ such that $B(f^n(x),\eta\frac{k+1}{M+1}) \setminus B(f^n(x),\eta\frac{k}{M+1})$ does not contain any critical value for $f^n|_W$. We choose D an arbitrary disc in the annulus $B(f^n(x),\eta\frac{3k+2}{3(M+1)}) \setminus B(f^n(x),\eta\frac{3k+1}{3(M+1)})$. Denote $W' = \text{Comp}_x f^{-n}(B(f^n(x),\eta(3k+2)/3(M+1)))$ and fix $y \in W' \cap f^{-n}(D)$. Let $0 = m_0 < m_1 < m_2 < ... < m_{M'} = n$ be all consecutive integers such that $f^{m_t}(W')$ contains an f-critical point, except maybe $t = 0$ and $t = M'$. (Observe that $M' \leq M(\text{deg} f) + 1$.) Then for every $m_t, t < M'$ we have

$$\frac{\text{diam} f^{m_t+1}(W')}{\text{diam} f^{m_{t+1}}(W')} \approx |f^{m_{t+1}-m_t-1})'(f^{m_t+1}(y))|^{-1} \tag{A3.5}$$

and

$$\frac{\text{diam} f^{m_t}(W')}{\text{diam} f^{m_t+1}(W')} \approx |f'(f^{m_t}(y))|^{-1}. \tag{A3.6}$$

Here $\approx$ means the ratios of the left and right sides, and *vice versa* are bounded by a constant depending only on M. The former $\approx$ follows from bounded distortion, the latter holds because the distance of $f^{m_t+1}(y)$ from $f(\text{Crit}(f) \cap f^{m_t}(W'))$ is at least $C_t \text{diam} f^{m_t+1}(W')$ for a constant $C_t > 0$.

Combining (A3.5) and (A3.6) over all t we obtain for a constant $A > 0$

$$A \ \text{diam} W' \leq |(f^n)'(y)|^{-1}\eta \leq A^{-1}\text{diam} W'. \tag{A3.7}$$

The right hand side inequality together with (A3.2) give (A3.3). The left hand side inequality in (A3.7) gives immediately (A3.4).

(The above proof is only sketched, a precise proof needs induction over decreasing t's to control distortion and constants C_t. For details see [P5, Proof of L.1.4, (1.4)]; though only the left hand side inequality (A3.7) is proved there, the technique is the same.)

It is known that there exists k depending only on z and on the diameter of D (i.e. on η and M) such that $f^k(D) \ni z$. So for every n_j we could choose y with $f^{n_j+k}(y) = z$. Moreover $|(f^k)'(f^{n_j}(y))| > \text{Const} > 0$ (because $z \notin O(\text{Crit})$). This and (A3.3) give (i) for $n_j + k$. The upper bound for $|f^k)'|$ and (A3.4) give (ii). $\square$

Notice that this Proposition, (2.2) and [PR, L.2.2] implies for f Collet-Eckmann that $HD_{\text{hyp}}(J) = HD(J)$.

(Thus we have obtained a new proof of a part of [P5, Th.A]. Recall that f is called Collet-Eckmann if for every $c \in \text{Crit} \cap J$, whose forward trajectory does not meet other critical points, $|(f^n)'(f(c))|$ grows exponentially fast as $n \to \infty$. [PR, L.2.2] says that Collet-Eckmann implies topological Collet-Eckmann.)

Notice that, by Proposition A3.4, if f is topological Collet-Eckmann with $\text{Crit} \cap J \neq \emptyset$, then $\Lambda_{\text{c}}(f) \not\subset \Lambda_{\text{U}}(f)$.

This is so because, by the definition, $c \notin \Lambda_{\text{U}}$.

Recall [CJY] that f is called semi-hyperbolic if it has no recurrent critical points and no parabolic periodic points. This class of maps is contained in the class of topological Collet-Eckmann maps. It follows from [U1] that if f is semi-hyperbolic, then all points in J except $\bigcup_{n\geq 0} f^{-n}(\text{Crit})$ belong to $\Lambda_{\text{U}}(f)$. So $\Lambda_{\text{U}}(f)$ is not backward invariant.

M. Lyubich and M. Minsky's conical limit set defined in [LM], which we denote by Λ_{LM}, has a complicated definition and we shall not rewrite it here. Instead, let us introduce the following notion:

Definition A3.5. We call $x \in J$ *strong LM-conical* if there exist $\eta, M > 0$ and sequences of integers $n_j \to \infty, k_i \to \infty$ and $i(j) \to \infty$ such that for every n_j and $i = 1, ..., i(j)$ the map $f^{n_j - k_i}$ has degree bounded by M on $\text{Comp}_x f^{-(n_j-k_i)}(B(f^{n_j-k_i}(x), \eta))$. Denote the set of all strong LM-conical points by $\Lambda_{\text{sLM}}(f)$

One can show that $\Lambda_{\text{sLM}}(f) \subset \Lambda_{\text{LM}}(f)$. In [LM] x together with a backward trajectory $\hat{x}$ with "parabolic global unstable leaf" is considered. Note that such $\hat{x}$ always exists. Just take $\hat{x}$ coverging to a repelling periodic orbit.

By [LM, Prop. 8.8 and L.8.4] if f is semi-hyperbolic, then $\Lambda_{\text{LM}}(f) = J$. It is easy to see that if x satisfies (A3.1) then x is strong LM-conical. Thus we obtain $\Lambda_{\text{LM}}(f) = J$ for every topological Collet-Eckmann map f.

By the way we obtain also $\Lambda_{\text{LM}}(f) \supset \Lambda_{\text{sLM}}(f) \supset \Lambda_{\text{reg}}(f)$.

Notice finally that similarly to $\Lambda_{\text{cw}}(f)$ one has

$$\text{HD}(\Lambda_{\text{U}}(f)) = \text{HD}(\Lambda_{\text{LM}}(f)) = \text{HD}_{\text{hyp}}(J) \tag{A3.8}$$

$\mathrm{HD}(\Lambda_{\mathrm{U}}(f)) = \mathrm{HD}_{\mathrm{hyp}}(J)$ follows from $\mathrm{HD}(\Lambda_{\mathrm{U}}(f)) \leq \alpha$ for every α-conformal measure, see [U2] and [DMNU] (and from (2.2): $\alpha(f) \leq \mathrm{HD}_{\mathrm{hyp}}(J)$, $\Lambda_{\mathrm{U}}(f) \supset \Lambda_{\mathrm{reg}}(f)$ and $\mathrm{HD}_{\mathrm{hyp}}(J) \geq \mathrm{HD}(\Lambda_{\mathrm{reg}}(f))$).

This follows however, as well as the second equality in (A3.8), from Proposition A3.7 below.

Definition A3.6. We say $x \in \Lambda_{\mathrm{LM1}}(f)$ if there exist $\eta, M > 0$ and a sequence of integers $n_j \to \infty$ such that each f^{n_j} has degree bounded by M on $\mathrm{Comp}_x f^{-n_j}(B(f^{n_j}(x), \eta))$.

Of course $\Lambda_{\mathrm{LM1}}(f) \supset \Lambda_{\mathrm{U}}(f)$ and by [LM, Prop.8.7] $\Lambda_{\mathrm{LM1}}(f) \supset \Lambda_{\mathrm{LM}}(f)$.

Proposition A3.7. $\mathrm{HD}(\Lambda_{\mathrm{LM1}}(f)) \leq \alpha(f)$. Hence $\mathrm{HD}(\Lambda_{\mathrm{LM1}}(f)) = \mathrm{HD}_{\mathrm{hyp}}(J)$.

Proof. The inequality follows as in [P5, Proof of Th. A] from $\mu(B(x, r_j)) \geq \mathrm{Const} r_j^\alpha$ for $x \in \Lambda_{\mathrm{LM1}}(f)$ for μ any α-conformal measure and $r_j = r_j(x) \to 0$ as $j \to \infty$. Const depends on M only. The latter inequality uses bounded distortion for finite criticality as [P5, L.1.4].

Question Is every conformal measure μ on $\Lambda_{\mathrm{c}}(f)$ ergodic? Is μ unique, provided it has no atoms at critical points?

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