GEOMETRIC PRESSURE FOR MULTIMODAL MAPS OF THE INTERVAL

FELIKS PRZYTYCKI† AND JUAN RIVERA-LETELIER‡

ABSTRACT. This paper is an interval dynamics counterpart of three theories founded earlier by the authors, S. Smirnov and others in the setting of the iteration of rational maps on the Riemann sphere: Topological Collet-Eckmann maps, Geometric Pressure, and Nice Inducing Schemes methods leading to results in thermodynamical formalism. We work in a setting of generalized multimodal maps, that is smooth maps f of $\widehat{I}$ being a finite union of compact intervals in $\mathbb{R}$ into $\mathbb{R}$ with non-flat critical points, preserving a compact set $K \subset \widehat{I}$, maximal forward invariant, such that $f|_K$ is topologically transitive and has positive topological entropy. We prove that several notions of nonuniform hyperbolicity of $f|_K$ are equivalent (including uniform hyperbolicity on periodic orbits, TCE and uniform exponential shrinking of diameters of pull-backs of intervals). We prove that several definitions of geometric pressure $P(t)$, that is pressure for the map $f|_K$ and the potential $-t \log |f'|$, give the same value (including pressure on periodic orbits, "tree" pressure, variational pressures and conformal pressure). Finally we prove that in absence of indifferent periodic orbits, the function $P(t)$ is real analytic for t between the "condensation" and "freezing" parameters and for each such t there exists unique equilibrium (and conformal) measure satisfying standard statistical properties.

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1. Introduction. The main results

This paper is devoted to transferring results for iteration of rational functions on the Riemann sphere obtained mainly in [PR-LS1], [PR-LS2], [PR-L1] and [PR-L2], to the interval case. We work with a class of generalized multimodal maps, that is smooth maps f of $\widehat{I}$ being a finite union of compact intervals in $\mathbb{R}$ into $\mathbb{R}$ with non-flat critical points, preserving a compact set $K \subset \widehat{I}$, maximal forward invariant, such that $f|_K$ is topologically transitive and has positive topological entropy.

Many proofs are almost the same, or easier, and we only sketch them. Some proofs however demand an additional care.

The paper concerns three topics, closely related to each other.

1. Extending the results for unimodal and multimodal maps in [NS], [NP] and [R-L] to the generalized multimodal maps considered here, we prove the equivalence of several notions of nonuniform hyperbolicity of $f|_K$, including uniform hyperbolicity on periodic orbits, TCE and uniform exponential shrinking of diameters of pull-backs of intervals. For the complex setting see [PR-LS1] and the references therein.

2. We prove that several definitions of geometric pressure $P(t)$, that is pressure for the map $f|_K$ and the potential $-t\log|f'|$, give the same value (including pressure on periodic orbits, "tree" pressure and variational pressures). For the complex setting, see [PR-LS2] and the references therein.

3. We prove that in absence of indifferent periodic orbits, the function $P(t)$ is real analytic for t (inverse of temperature) between the "condensation" and "freezing" parameters, t_- and t_+, and for each such t there exists unique equilibrium (and conformal) measure satisfying strong statistical properties. All this is contained in Theorem A, the main result of the paper. For the complex setting, see [PR-L2].

The paper about interval multimodal maps [BT1], proving existence and uniqueness of equilibria and the analyticity of pressure, concerns only a small interval of parameters t and assumes additionally a growth of absolute values of the derivatives of the iterates at critical values.

Related are results of [PS, Theorem 8.2]. Also assuming a growth of absolute values of the derivatives of the iterates at critical values and an additional slow recurrence condition, it shows the real analyticity of the geometric pressure function on a neighborhood of [0, 1].

Our main results are stronger in that growth assumptions are not needed and the domain of real analyticity of t is the whole (t_-, t_+), i.e. the maximal possible domain. Precisely in this domain it holds $P(t) > \sup \int \phi\, d\mu$, where supremum is taken over all f-invariant ergodic probability measures on K, and $\phi = -t\log|f'|$ (it is therefore clear that analyticity cannot hold at neither $t = t_-$ nor $t = t_+$, as $P(t)$ is affine to the left of t_- and to the right of t_+).

Let us mention also the paper [IT] where, under the restriction that f has no preperiodic critical points, the existence of equilibria was proved for all $-\infty = t_- < t < t_+$. The authors proved that $P(t)$ is of class C^1 and that their method does not allow them to obtain statistical properties of the equilibria.

Our results are related to papers on thermodynamical formalism for ϕ being Hölder continuous, satisfying the assumption $P(f|_K, \phi) > \sup\phi$ or related ones, see [BT2] and [LR-L].

All these papers, and our [PR-L1], [PR-L2] and this one, use some inducing schemes, that is dynamics of return maps to 'nice' domains.

1.1. **Generalized multimodal maps and related notions.**

Definition 1.1. Let $\widehat{I}$ be the union of a finite family of pairwise disjoint compact intervals in the real line $\mathbb{R}$. We shall assume throughout the paper that intervals are non-degenerate (i.e. non-one point). Consider $f : \widehat{I} \to \mathbb{R}$ a map which is C^2 on each of these intervals. Assume that f has finite number of critical points, i.e. such x that $f'(x) = 0$ (turning and inflection), all of them non-flat (see Section 2 for the formal definition). We call such a map *generalized multimodal map*. (Sometimes we use this name for $f : U \to \mathbb{R}$, replacing a compact set $\widehat{I}$ by an open set U, having a finite number of components.)

Note that we do not assume neither $f(\widehat{I}) \subset \widehat{I}$ nor $f(\partial\widehat{I}) \subset \partial\widehat{I}$.

The set of critical points will be denoted by $\mathrm{Crit}(f)$ turning critical points by $\mathrm{Crit}^T(f)$ and inflection critical points by $\mathrm{Crit}^I(f)$.

Let $K = K(f)$ be the maximal forward invariant set for f, more precisely the set of all points in $\widehat{I}$ whose forward trajectories stay in $\widehat{I}$. We call K a *maximal repeller*.

Next we replace $\widehat{I}$ by a smaller set being a finite union of pairwise disjoint compact non-degenerate intervals, by removing from $\widehat{I}$ open intervals disjoint from K containing all critical points not contained in K. We keep the same notation $\widehat{I} = \bigcup_{j=1,\dots,m} \widehat{I}^j$ in particular m for the number of its components. Thus we always assume that

$$(\widehat{I} \setminus K) \cap \mathrm{Crit}(f) = \emptyset. \tag{1.1}$$

For each $j = 1, ..., m$ denote by $\widehat{I}^j$ the convex hull of $I^j \cap K$ and denote $\widehat{I}_K = \bigcup_{j=1,\dots,m} \widehat{I}^j$. Then we consider the *reduced generalized multimodal map* $f|_{\widehat{I}_K}$. The maximality of K in $\widehat{I}_K$ obviously follows, since $\widehat{I}_K \subset \widehat{I}$.

We shall always assume that K is infinite and that $f|_K$ is topologically transitive (that is for all $U, V \subset K$ open in it, non-empty, there exists $n > 0$

such that $f^n(U) \cap V \neq \emptyset$). Then $f(K) = K$ and K has no isolated points, see Lemma 2.1, hence $\widehat{I}^j$ are also non-degenerate.

Moreover we shall usually assume that K is dynamically sufficiently rich, namely that the topological entropy $h_{\mathrm{top}}(f|_K)$ is positive.

Remark 1.2. All assertions of our theorems concern the action of f on K. So it is natural to organize definitions in the opposite order as follows.

Let $K \subset \mathbb{R}$ be an infinite compact subset of the real line. Let $f : K \to K$ be a continuous topologically transitive mapping.

Assume there exists a covering of K by a finite family of pairwise disjoint closed intervals $\widehat{I}^j$ with end points in K such that f extends to a generalized multimodal map on their union $\widehat{I}_K$ and K is the maximal forward invariant set $K(f)$ in it. By definition f on $\widehat{I}_K$ is a *reduced generalized multimodal map*. Thus we shall consider only reduced generalized multimodal maps.

Notice, that K is either a finite family of compact intervals or a Cantor set, see Lemma 2.1.

Definition 1.3. Consider a C^2 extension of f to a neighbourhood $\mathbf{U}^j$ of each $\widehat{I}^j$. We consider $\mathbf{U}^j$'s small enough that they are pairwise disjoint. Moreover we assume that all critical points in $\mathbf{U}^j$ are in $\widehat{I}^j$. Thus together with (1.1) we assume

$$(\mathbf{U} \setminus K) \cap \mathrm{Crit}(f) = \emptyset. \tag{1.2}$$

Define $\mathbf{U}$, an open neighbourhood of $\widehat{I}_K$, as the union of $\mathbf{U}^j$'s. We consider the quadruple $(f, K, \widehat{I}_K, \mathbf{U})$ and call it a *reduced generalized multimodal quadruple*. In fact it is always sufficient to start with a triple $(f, K, \mathbf{U})$, because this already uniquely defines $\widehat{I}_K$.

As we do not assume $f(\mathbf{U}) \subset \mathbf{U}$, when we iterate f, i.e. consider f^k for positive integers k we consider them on their domains, which can be smaller than $\mathbf{U}$ for $k \geq 2$,

Note that we do not assume K to be maximal forward invariant in $\mathbf{U}$. We assume maximality only in $\widehat{I}_K$.

See Proposition A.2 for an alternative approach, replacing maximality by so-called Darboux property.

Notation 1.4. *1. Let us summarize that the properties of the extension of f beyond K are used only to specify assumptions imposed on this action, in particular a way it is embedded in $\mathbb{R}$. So we will sometimes talk about multimodal pairs (f, K), understanding that this notion involves $\widehat{I}_K$ and $\mathbf{U}$ as above, sometimes about the triples $(f, K, \widehat{I}_K)$, sometimes about the triples $(f, K, \mathbf{U})$.*

2. Denote the space of all reduced generalized multimodal quadruples (pairs, triples) discussed above by $\mathscr{A}$. For f of class C^r we write $\mathscr{A}^r$, so $\mathscr{A} = \mathscr{A}^2$. If $h_{\mathrm{top}}(f|_K) > 0$ is assumed we write $\mathscr{A}_+$ or $\mathscr{A}_+^r$.

1.2. **Periodic orbits and basins of attraction. Bounded distortion property.**

Definition 1.5. As usually we call a point $p \in \mathbf{U}$ *periodic* if there exists m such that $f^m(p) = p$. We denote by $O(p)$ its periodic orbit.

Define the *basin of attraction* of the periodic orbit $O(p) \subset \mathbf{U}$ by

$$B(O(p)) := \text{interior}\{x \in \mathbf{U} : f^n(x) \to O(p), \text{ as } n \to \infty\}.$$

The orbit $O(p)$ is called *attracting* if $O(p) \subset B(O(p))$. Notice that this happens if $|(f^m)'(p)| < 1$ and it can happen also if $|(f^m)'(p)| = 1$. The orbit is called *repelling* if $|(f^m)'(p)| \geq 1$ and for $g := f|_W^{-1}$ where W is a small neighbourhood of $O(p)$ in $\mathbb{R}$ we have $g(W) \subset W$ and $g^n(W) \to O(p)$ as $n \to \infty$. If $|(f^m)'(p)| > 1$ we call the orbit *hyperbolic repelling.*

In the remaining cases we call $O(p)$ *indifferent.*

We say also that a periodic point p is *attracting, (hyperbolic) repelling* or *indifferent* if $O(p)$ is attracting, (hyperbolic) repelling or indifferent respectively.

The union of the set of attracting periodic orbits will be denoted by $A(f)$ and the union of the set of indifferent periodic orbits by $\text{Indiff}(f)$.

If $O(p)$ is indifferent, i.e neither attracting nor repelling, then denoting by m a period of p such that for $F := f^m$, either p attracts under iteration of F from the left side or from the right side or from none of the sides. Similarly to the definition of repelling we can define repelling from the left or from the right side.

Notice that indifferent p being neither attracting nor repelling from the left (or right) side, must be an accumulation point of periodic points of period m from that side.

Remark 1.6. For $(f, K) \in \mathscr{A}$ it follows by the maximality of K that there are no periodic orbits in $\widehat{I} \setminus K$. By the topological transitivity of f on K, there are no attracting periodic orbits in $\widehat{I}_K$. By the same reasons and smoothness of f the number of indifferent and non-hyperbolic repelling periodic orbits in K is finite. This relies on [dMvS, Ch. IV, Theorem B]. For an explanation see remarks after Corollary A.3 in Appendix A.

Moreover by changing f on $\mathbf{U}$ if necessary one can assume that the only periodic orbits in $\mathbf{U} \setminus K$ are hyperbolic repelling. See Appendix A, Lemma A.4, for more details.

Definition 1.7. We shall use the notion of the immediate basin $B_0(O(p))$ defined as the union of the components of $B(O(p))$ whose closures intersect $O(p)$. Notice that if $O(p)$ is attracting then $O(p) \subset B_0(O(p))$. The component of $B_0(O(p))$ whose closure contains p will be denoted by $B_0(p)$. If $O(p)$ is indifferent attracting from one side then p is a boundary point of $B_0(p)$ and *vice versa.* Finally notice that for each component T of $B(O(p))$ there exists $n \geq 0$ such that $f^n(T) \subset B_0(f) \cup O(p)$, (see an argument in Proof of Proposition A.2 in Appendix A. We need to add $O(p)$ above in case $O(p)$ is indifferent and some $f^j(T)$ contains a turning critical point whose forward trajectory hits p. (Compare also Example 1.8 below, where Julia set need not be backward invariant.)

Example 1.8. If for a generalized multimodal map f its domain $\widehat{I}$ is just one interval I and $f(I) \subset I$, then we have a classical case of an interval multimodal map. However the set of non escaping points $K(f)$ is the whole I in this case, usually too big (not satisfying topological transitivity, and not even mapped by f onto itself). So one considers smaller f-invariant sets, see below.

Write $B(f) = \bigcup_p B(O(p))$ and $B_0(f) = \bigcup_p B_0(O(p))$.

Write $I^+ := \bigcap_{n=0}^{\infty} f^n(I)$. Define Julia set by

$$J(f) := (I \setminus B(f)),$$

compare [dMvS, Ch.4], and its *core Julia set* by

$$J^+(f) := (I \setminus B(f)) \cap I^+.$$

Notice that the sets $J(f)$ and $J^+(f)$ are compact and forward invariant. They need not be backward invariant, even $J^+(f)$ for $f|_{I^+}$, namely a critical preimage of a indifferent point $p \in \mathrm{Indiff}$ can be in $B(O(p)))$, whereas $p \notin B(O(p))$.

The definition of $J(f)$ is compatible with the definition as a complement of the domain of normality of all the forward iterates of f as in the complex case, provided there are no wandering intervals.

Notice however that without assuming that f is topologically transitive on $J^+(f)$ the comparison to Julia set is not justified. For example for f mapping $I = [0,1]$ into itself defined by $f(x) = f_\lambda(x) = \lambda x(1-x)$ where $3 < \lambda < 4$ (to exclude an attracting or indifferent fixed point or an escape from $[0,1]$), we have $I^+ = I$ where f is not topologically transitive. However if we restrict f to $[\epsilon, 1]$ then f is topologically transitive on $K = I^+ \setminus B(f)$ for $I^+ = [c_2, c_1]$, where $c_2 = f^2(c), c_1 = f(c)]$, where $c := 1/2$ is the critical point, provided f is not renormalizable. Our $\widehat{I}$ is I^+ in this example.

Notice that since $f(c_2)$ belongs to the interior of I^+ the set K is not maximal invariant in U being a neighbourhood of I^+ in $\mathbb{R}$ whatever small U we take.

Example 1.9. Multimodal maps f considered in the previous example, restricted to $J^+(f)$ still need not be topologically transitive. Then, instead, examples of generalized multimodal pairs in $\mathscr{A}_+$ are provided by (f, K), where K is an arbitrary maximal topologically transitive set in Ω_n's in the spectral decomposition of the set of nonwandering points for f, as in [dMvS, Theorem III.4.2, item 4.], so-called *basic set*, for which $h_{\mathrm{top}}(f|_K) > 0$). It is easy to verify that basic sets are weakly isolated.

In a non-smooth multimodal case there can exist a wandering interval namely an open interval T such that all intervals $T, f(T), f^2(T), ...$ are pairwise disjoint and not in $B(f)$. In the C^2 multimodal case wandering intervals cannot exist, see [dMvS, Ch.IV, theorem A]. We shall use this fact many times in this paper.

Definition 1.10. Following [dMvS] we say that for $\epsilon > 0$ and an interval $I \subset \mathbb{R}$, an interval $I' \supset I$ is an ϵ-scaled neighbourhood of I if $I' \setminus I$ has two components, call them left and right, L and R, such that $|L|/|I|, |R|/|I| = \epsilon$.

We say that a generalized multimodal map $f : U \to \mathbb{R}$ and its compact invariant set X (in particular a generalized multimodal triple $(f, K, \mathbf{U}) \in \mathscr{A}$) satisfies *bounded distortion*, abbr. BD, condition if there exists $\delta > 0$ such that for every $\epsilon > 0$ there exists $C = C(\epsilon) > 0$ such that the following holds: For every pair of intervals $I_1, I_2 \subset U$ such that I_1 intersects X and $|I_2| \le \delta$ and for every $n > 0$, if f^n maps diffeomorphically an interval I_1' containing I_1 onto an interval I_2' being an ϵ-scaled neighbbourhood of I_2, then for every $x, y \in I_2$ we have for $g = (f^n|_{I_1})^{-1}$

$$|g'(x)/g'(y)| \le C(\epsilon).$$

(Equivalently, for every $x, y \in I_1$, $\quad |(f^n)'(x)/(f^n)'(y)| \le C(\epsilon)$.)

Notice that BD easily implies that for every $\epsilon > 0$ there is $\epsilon' > 0$ such that if in the above notation I_2' is an ϵ-scaled neighbourhood of I_2 then I_1' contains an ϵ'-scaled neighbourhood of I_1.

For related definitions of distortion to be used in the paper and further discussion see Section 2: Definition 2.13 and Remark 2.14.

We denote the space of $(f, K) \in \mathscr{A}$ or $(f, K) \in \mathscr{A}_+$ satisfying BD, by $\mathscr{A}^{\mathrm{BD}}$ or $\mathscr{A}_+^{\mathrm{BD}}$, respectively.

Remark 1.11. Notice that C^2 and BD imply that all repelling periodic orbits in K are hyperbolic repelling, so the only periodic orbits in K are indifferent one-side repelling (there is at most finite number of them) and hyperbolic repelling. The finiteness of the set of indifferent one-side repelling orbits in K was noted already in Remark 1.6 without assuming BD.

The finiteness of this set in $\mathbf{U}$ (where we treat an interval of periodic points as one point) follows from the standard fact that by BD for every attracting or indifferent $O(p)$, the immediate basin $B_0(O(p))$ must contain a critical point or a point belonging to $\partial \mathbf{U}$ and that there is only a finite number of such points (in fact critical points cannot be in $B_0(O(p))$ since we have assumed $\mathrm{Crit}(f) \subset K$). See Appendix, Proposition A.5 for more details.

In particular, by shrinking $\mathbf{U}$, one can assume that there are no attracting or indifferent periodic orbits in $\mathbf{U} \setminus K$.

1.3. **Statement of Theorem A: Analytic dependence of geometric pressure on temperature, equilibria.**

Analogously to [PR-L2] let $\mathscr{M}(f,K)$ be the space of all probability measures supported on K that are invariant by f. For each $\mu \in \mathscr{M}(f,K)$, denote by $h_\mu(f)$ the *measure theoretic entropy of* μ, and by $\chi_\mu(f) := \int \log|f'| d\mu$ the *Lyapunov exponent of* μ.

If μ is supported on a periodic orbit $O(p)$ we use the notation $\chi(p)$.

Given a real number t we define the *pressure of* $f|_K$ *for the potential* $-t\log|f'|$ by,

$$P(t) := \sup\left\{h_\mu(f) - t\chi_\mu(f) : \mu \in \mathscr{M}(f,K)\right\}. \tag{1.3}$$

For each $t \in \mathbb{R}$ we have $P(t) < +\infty$ since $\chi_\mu(f) \geq 0$ for each $\mu \in \mathscr{M}(f,K)$, see [P-Lyap] (or [R-L, Appendix A] for a simplified proof). Sometimes we call $P(t)$ *variational pressure* and denote it by $P_{\mathrm{var}}(t)$.

μ is called an *equilibrium state of* f *for the potential* $-t\log|f'|$, if the supremum in (1.3) is attained for this measure.

As in [PR-L2] define the numbers,

$$\chi_{\inf}(f) := \inf\left\{\chi_\mu(f) : \mu \in \mathscr{M}(f,K)\right\},$$

$$\chi_{\sup}(f) := \sup\left\{\chi_\mu(f) : \mu \in \mathscr{M}(f,K)\right\},$$

In Section 8 we use another definition, see Section 4, compare [PR-L2] in the complex case. See also [R-L]).

Define the *phase transition* points

$$t_- := \inf\{t \in \mathbb{R} : P(t) + t\chi_{\sup}(f) > 0\},$$

$$t_+ := \sup\{t \in \mathbb{R} : P(t) + t\chi_{\inf}(f) > 0\},$$

the *condensation point* and the *freezing point* of f, respectively. As in the complex case the condensation (resp. freezing) point can take the value $-\infty$ (resp. $+\infty$).

Similarly to [PR-L2] we have the following properties:

- $t_- < 0 < t_+$;
- for all $t \in \mathbb{R} \setminus (t_-, t_+)$ we have $P(t) = \max\{-t\chi_{\sup}(f), -t\chi_{\inf}(f)\}$;
- for all $t \in (t_-, t_+)$ we have $P(t) > \max\{-t\chi_{\inf}(f), -t\chi_{\sup}(f)\}$.

Definition 1.12. We call a finite Borel measure on K for $(f,K) \in \mathscr{A}$ (or more generally for an f-invariant set $K \subset U$ for a generalized multimodal map $f : U \to \mathbb{R}$) and a function $\phi : K \to \mathbb{R}$ a ϕ-conformal measure if it is forward quasi-invariant, i.e. $\mu \circ f \prec \mu$, compare [PU, Section 5.2], and for every Borel set $A \subset K$ on which f is injective

$$\mu(f(A)) = \int_A \phi \, d\mu. \tag{1.4}$$

Definition 1.13. We say that for intervals W, B in $\mathbb{R}$, the map $f^n : W \to B$ is a *K-diffeomorphism* if it is a diffeomorphism, in particular it is well defined i.e. for each $j = 0, ..., n-1$ $\quad f^j(W) \subset \mathbf{U}$ and $f^j(W) \cap \mathrm{NO}(f,K) = \emptyset$, and moreover $f(W \cap K) = f(W) \cap K$, compare Definition A.1 and Lemma 2.2.

Definition 1.14. Denote by $K_{\rm con}(f)$ the "conical limit part of K" for f and K, defined as the set of all those points $x \in K$ for which there exists $\rho(x) > 0$ and an arbitrarily large positive integer n, such that f^n on $W \ni x$, the component of the f^{-n}-preimage of the interval $B := B(f^n(x), \rho(x))$ containing x, is a K-diffeomorphism onto B.

The main theorem of the paper corresponding to [PR-L1, Main Theorem] is

Theorem A. *Let $(f,K) \in \mathscr{A}_+^3$ (in particular f is topologically transitive on K and has positive entropy) and let all f-periodic orbits in K be hyperbolic repelling. Then*[1] *the following hold.*

- **Analyticity of the pressure function:** *The pressure function $P(t)$ is real analytic on (t_-, t_+), and linear with slope $-\chi_{\sup}(f)$ (resp. $-\chi_{\inf}(f)$) on $(-\infty, t_-]$ (resp. $[t_+, +\infty)$).*
- **Conformal measure:** *For each $t \in (t_-, t_+)$, the least value p for which there exists an $(\exp p)|f'|^t$-conformal probability measure μ_t on K is $p = P(t)$. μ_t is unique among all $(\exp p)|f'|^t$-conformal probability measures positive on sets open in K and zero on all weakly S'-exceptional subsets of K, or S'-exceptional if there are no singular connections (see Definition 1.17). Moreover μ_t is non-atomic, ergodic, and it is supported on $K_{\rm con}(f)$.*
- **Equilibrium states:** *For each $t \in (t_-, t_+)$, for the potential $\phi = -t \log |f'|$, there is a unique equilibrium measure of f. It is ergodic and absolutely continuous with respect to μ_t with the density bounded from below by a positive constant almost everywhere. If furthermore f is topologically exact on K, then this measure is mixing, moreover exponentially mixing and it satisfies the Central Limit Theorem for Lipschitz gauge functions.*

The assertion about the analyticity of $P(t)$ can be false without the topological transitivity assumption, see [Dobbs].

We expect that the assumption $(f,K) \in \mathscr{A}_+^{\rm BD}$ (more precisely $\mathscr{A}_+^{\rm HBD}$, see Definition 2.13), is sufficient. However in the proof we use inducing, i.e. a return map to a *nice set* belonging to a *nice couple*, see Section 6. For the existence of nice couples we refer to [CaiLi], where C^3 is assumed.

The smoothness C^3 is sufficient for BD if we allow to modify f outside $\widehat{I}_K$, see Remark 2.14 and Lemma A.4.

Let us now comment on the properties of topological transitivity and positive entropy assumed in Theorem A.

[1]In this preliminary version of the paper we assume that there exist nice couples having components of arbitrarily small diameters, see Definition 6.5. This assumption seems substantial in the complex case, [PR-L2]. In our real setting it automatically holds under the condition that the only points where $f|_K$ is not open, so are to be contained in nice couples, are critical points, see Definition 1.17. This relies on [CaiLi], see Lemma 6.6, where this condition also must be assumed. We expect that a careful analysis of [CaiLi] allows to remove this condition.

Definition 1.15. Let $h : X \to X$ be a continuous mapping of a compact metric space X.

We call h *weakly topologically exact* (or just *weakly exact*) if there exists $N > 0$ such that for every non-empty open $V \subset X$, there exists $n(V) \geq 0$ such that

$$\bigcup_{j=0}^{N} h^{n(V)+j}(V) = X. \tag{1.5}$$

This property clearly implies that h maps X onto X, compare Lemma 2.1.

Notice that the equality (1.5) implies automatically the similar one with $n(V)$ replaced by any $n \geq n(V)$. To see this apply $h^{n-n(V)}$ to both sides of the equation and use $h(X) = X$.

Finally according to the standard definition $h : X \to X$ is called *topologically exact* if for every non-empty open $V \subset X$ there is $m > 0$ such that $h^m(V) = K$.

In the sequel we shall usually use these definitions for $(f, K) \in \mathscr{A}$ setting $X = K$, $h = f|_K$. For some immediate technical consequences of the property of weak exactness see Remark 2.6 below.

Remark 1.16. Clearly topological exactness implies weak topologically exactness which in turn implies topological transitivity. In Appendix A, Lemma A.7, we provide a proof of the converse fact for $f|_K$, saying that topological transitivity and positive topological entropy of $f|_K$ imply weak topological exactness. <u>This allows in our theorems to assume only *topological transitivity* and to use in proofs the formally stronger *weak exactness*</u>.

This fact seems to be folklore. Most of the proof in Appendix A was told to us by Michał Misiurewicz. We are also grateful to Peter Raith for explaining us how this fact follows from [Hof]. See also [Buzzi, Appendix B].

In fact for $(f, K) \in \mathscr{A}$, for $f|_K : K \to K$ the properties: weak topological exactness and positive topological entropy are equivalent, see Proposition A.8.

Now we provide the notion of *exceptional* and related ones, used in the formulation of Theorem A.

Definition 1.17. 1. End points. Let $(f, K, \widehat{I}_K, \mathbf{U}) \in \mathscr{A}$. We say that $x \in K$ is an *end point* if $x \in \partial \widehat{I}_K$. We shall use also the notion of the *singular set* of f in K, defined by $S(f, K) := \mathrm{Crit}(f) \cup \partial(\widehat{I}_K)$.

We shall use also the notion of the *restricted singular set* $S'(f, K)$ being the union of $\mathrm{Crit}(f)$ and the set $\mathrm{NO}(f, K)$ of points where $f|_K$ is not an open map, i.e. points $x \in K$ such that there is an arbitrarily small neighbourhood V of x in K whose f-image is not open in K.

We have $\mathrm{NO}(f, K) \subset \mathrm{Crit}^T(f) \cup \partial(\widehat{I}_K)$, see Lemma 2.2.

2. Exceptional sets. (Compare [MS] and [GPRR].) We say that a nonempty forward invariant set $E \subset K$ is S-*exceptional* for f, if it is not

dense in K and

$$(f|_K)^{-1}(E) \setminus S(f,K) \subset E.$$

Analogously, replacing $S(f,K)$ by $S'(f,K)$, we define S'-*exceptional* subsets of K.

Another useful variant of this definition is *weakly S-exceptional* or *weakly S'-exceptional*, where we do not assume E is forward invariant. For example each unimodal map f of interval, i.e. with just one turning critical point c, has the one-point set $\{f(c)\}$ being a weakly S-exceptional set.

It is not hard to see that if weak topological exactness of f on K is assumed, then each non-dense S-exceptional set is finite, moreover with number of elements bounded by a constant, see Proposition 2.7. Therefore the union of S-exceptional sets is S-exceptional and there exists a maximal S-exceptional set $E_{\max}$ which is finite. It can be empty. If it is non-empty we say that (f,K) is *S-exceptional*. Analogous terminology is used and facts hold for weakly exceptional sets and for S' in place of S.

More generally the above facts hold for an arbitrary finite $\Sigma \subset K$ in place of S or S', where $E \subset K$ is called then Σ-exceptional or weakly Σ-exceptional, depending as we assume it is forward invariant or not, if $(f|_K)^{-1}(E) \setminus \Sigma \subset E$, see Proposition 2.7.

3. To simplify notation we shall sometimes assume that no critical point is in the forward orbit of a critical point. This is a convention similar to the complex case. Moreover we shall sometimes assume that no point belonging to $\mathrm{NO}(f,K)\cup\mathrm{Crit}(f)$ is in the forward orbit of a point in $\mathrm{NO}(f,K)\cup\mathrm{Crit}(f)$, calling it *no singular connection* condition.

These assumptions are justified since no critical point, neither a point belonging to $NO(f,K)$, can be periodic, see Lemma 2.2, hence each trajectory in K can intersect $S'(f,K)$ in at most $\#S'(f,K)$ number of times, hence with difference of the moments between the first and last intersection bounded by a constant. In consequence several proofs hold in fact without these assumptions.

1.4. **Characterizations of geometric pressure.**

All the definitions of pressure introduced in the rational functions case, see [PR-LS2], make sense for $f|_K$ for $(f,K,\mathbf{U}) \in \mathscr{A}_+^{\mathrm{BD}}$. In particular

Definition 1.18 (Hyperbolic pressure)**.**

$$P_{\mathrm{hyp}}(t) := \sup_X P(f|_X, -t\log|f'|),$$

supremum taken over all compact f-invariant (that is $f(X) \subset X$) isolated hyperbolic subsets of K.

Isolated (or *forward locally maximal*), means that there is a neighbourhood U of X in $\mathbf{U}$ such that $f^n(x) \in \mathbf{U}$ for all $n \geq 0$ implies $x \in X$.

Hyperbolic or *expanding* means that there is a constant $\lambda_X > 1$ such that for all n large enough and all $x \in X$ we have $|(f^n)'(x)| \geq \lambda_X^n$. We call such isolated expanding sets *expanding repellers* following Ruelle.

We shall prove that the space of such sets X is non-empty.

Notice that by our definitions X is a maximal invariant set in U its neighbourhood in $\mathbb{R}$, whereas the whole K need not be maximal invariant in any its neighbourhood in $\mathbb{R}$, see Definition 1.3 and Example 1.8.

$P(f|_X, -t\log|f'|)$ (we shall use also notation $P(X,t)$) denotes the standard topological pressure for the continuous mapping $f|_X$ and the continuous real-valued potential function $-t\log|f'|$ on X , *via*, say, (n,ϵ)-separated sets, see for example [Walters] or [PU].

From this definition it immediately follows (compare [PU, Corollary 12.5.12] in the complex case) the following

Proposition 1.19. (Generalized Bowen's formula) *The first zero of $t \mapsto P_{\mathrm{hyp}}(K,t)$ is equal to the hyperbolic dimension of K, that is*

$$\mathrm{HD}_{\mathrm{hyp}}(K) := \sup_{X\subset K} \mathrm{HD}(X),$$

supremum taken over all compact forward f-invariant isolated hyperbolic subsets of K.

Sometimes we shall assume the following property

Definition 1.20. We call an f-invariant compact set $K \subset \mathbb{R}$ *weakly isolated* if there exists U an open neighbourhood of K in the domain of f such that for every f-periodic orbit $O(p) \subset U$, if it is in U, then it is in K. We abbreviate this property by (wi).

In the case of a reduced generalized multimodal quadruple $(f,K,\widehat{I}_K,\mathbf{U})$ it is sufficient to consider in this property $U = \mathbf{U}$ a neighbourhood of $\widehat{I}_K$. Indeed, by the maximality property if $O(p)$ is not contained in K it is not contained in $\widehat{I}_K$. For an example of a topologically exact multimodal pair which does is not satisfy (wi) see Example2.12.

Definition 1.21 (Tree pressure)**.** For every $z \in K$ and $t \in \mathbb{R}$ define

$$P_{\mathrm{tree}}(z,t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{f^n(x)=z, x\in K} |(f^n)'(x)|^{-t}.$$

Under suitable conditions for z (safe and expanding, as defined below) limsup can be replaced by liminf, i.e. limit exists in in this definition, see Proof of Theorem B, more precisely Lemma 4.4 and the Remark following it.

To discuss the (in)dependence of tree pressure on z we need the following notions.

Definition 1.22 (safe)**.** See [PU, Definition 12.5.7]] We call $z \in K$ *safe* if $z \notin \bigcup_{j=1}^\infty (f^j(S(f,K)))$ and for every $\delta > 0$ and all n large enough $B(z, \exp(-\delta n)) \cap \bigcup_{j=1}^n (f^j(S(f,K))) = \emptyset$.

Notice that by this definition all points except at most a set of Hausdorff dimension 0, are safe.

Definition 1.23 (expanding or hyperbolic)**.** See [PU, Definition 12.5.9]]. We call $z \in K$ *expanding* or *hyperbolic* if there exist $\Delta > 0$ and $\lambda = \lambda_z > 1$ such that for all n large enough f^n maps 1-to-1 the interval $\mathrm{Comp}_z(f^{-n}(B(f^n(z),\Delta))$ to $B(f^n(z),\Delta)$ and $|(f^n)'(z)| \geq \mathrm{Const}\,\lambda^n$. Here and further on Comp_z means the component containing z.

Sometimes we shall use also the following technical condition

Definition 1.24 (safe forward)**.** A point $z \in K$ is called *safe forward* if there exists $\Delta > 0$ such that $\mathrm{dist}(f^j(z), \partial \widehat{I}_K) \geq \Delta$ for all $j = 0, 1, ...$

Proposition 1.25. *For every $(f,K) \in \mathscr{A}_+^{\mathrm{BD}}$, there exists $z \in K$ which is safe and expanding. The pressure $P_{\mathrm{tree}}(z,t)$ does not depend on z for all $z \in K$ safe and expanding.*

For such z we shall just use the notation $P_{\mathrm{tree}}(t)$ and use the name *tree pressure.*

For f rational function it is enough to assume z is safe, i.e. $P_{\mathrm{tree}}(z,t)$ does not depend on z, except z in a thin set (of Hausdorff dimension 0). In the interval case we do not know how to get rid of the assumption z is expanding.

One defines periodic orbits pressure hyperbolic variational pressure and conformal pressure for $(f,K) \in \mathscr{A}_+^{\mathrm{BD}}$, analogously to [PR-LS2], by

Definition 1.26 (Periodic orbits pressure)**.** Let Per_n be the set of all f-periodic points in $\mathbf{U}$ of period n (not necessarily minimal period). Define

$$P_{\mathrm{Per}}(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{z \in \mathrm{Per}_n(f)\cap K} |(f^n)'(z)|^{-t}.$$

Definition 1.27 (Hyperbolic variational pressure)**.**

$$P_{\mathrm{varhyp}}(t) := \sup\left\{h_\mu(f) - t\chi_\mu(f) : \mu \in \mathscr{M}^+(f,K), \mathrm{ergodic}, \right\}, \tag{1.6}$$

where $\mathscr{M}^+(f,K) := \{\mu \in \mathscr{M}(f,K) : \chi_\mu(f) > 0\}$.

Notice that compared to variational pressure in (1.3) we restrict here to hyperbolic measures, i.e. measures with positive Lyapunov exponent. The space of hyperbolic measures is non-empty, since $h_{\mathrm{top}}(f) > 0$. Indeed, then there exists μ, an f-invariant measure on K of entropy $h_\mu(f)$ arbitrarily close to $h_{\mathrm{top}}(f)$ by Variational Principle, hence positive. Hence, by Ruelle's inequality, $\chi_\mu(f) \geq h_\mu(f) > 0$.

Theorem B. *For every $(f,K) \in \mathscr{A}_+^{\mathrm{BD}}$, weakly isolated, all pressures defined above coincide for all $t \in \mathbb{R}$. Namely*

$$P_{\mathrm{Per}}(K,t) = P_{\mathrm{tree}}(K,t) = P_{\mathrm{hyp}}(K,t) = P_{\mathrm{varhyp}}(K,t) = P_{\mathrm{var}}(K,t).$$

For $t < t_+$ the assumption (wi) can be skipped.

Denote any of these pressures by $P(K,t)$ and call *geometric pressure*.

The first equality holds for complex rational maps, under an additional assumption H, see [PR-LS2], and we do not know whether this assumption can be omitted there. The proofs of the equalities $P_{\mathrm{Per}}(K,t) = P_{\mathrm{varhyp}}(K,t) = P_{\mathrm{var}}(K,t)$ in the interval case must be modified since we cannot use the tool of short chains of discs joining two points, omitting critical values and their images, see e.g. [PR-LS1, Geometric Lemma]. This is also the reason we need to assume z is expanding when proving that $P_{\mathrm{tree}}(K,t)$ is independent of z. The proofs here use the notion of Topological Collet-Eckmann maps, TCE, and equivalent notions, see [P-Holder] and [PR-LS2] in the complex rational case and [NP] in the interval case, see Section 3.

The definition of conformal pressure is also the same as in the complex case:

Definition 1.28 (conformal pressure)**.**

$$P_{\mathrm{conf}}(K,t) := \log \lambda(t),$$

where

$$\lambda(t) = \inf\{\lambda > 0 : \exists \mu \text{ on } K \text{ which is } \lambda|f'|^t - \text{conformal}\}. \tag{1.7}$$

The proof of Theorem A yields the extension of Theorem B to the conformal pressure for $t_- < t < t_+$, provided we impose stronger assumptions than in Theorem B, namely the assumptions from Theorem A. We obtain

Corollary 1.29. *For every $(f,K) \in \mathscr{A}_+^3$ whose all periodic orbits in K are hyperbolic repelling for every $t_- < t < t_+$*

$$P_{\mathrm{conf}}(K,t) = P(K,t).$$

Our proof will use inducing and will accompany Proof of Theorem A, see Subsection 7.2.

Remark 1.30. In Appendix C we provide Patterson-Sullivan's constructions. However they result only with *conformal** measures, the property weaker than conformal, see (C.2). Their use will yield a different conformal pressure $P^*_{\mathrm{conf}}(t)$. In Appendix C we shall discuss its relation to $P(K,t)$ for all real t.

1.5. **Topological Collet-Eckmann maps.**

The following is an extension to generalized multimodal maps of the results for multimodal maps in [NP] and [R-L], see also [NS] for the case of unimodal maps and [PR-LS1] for the case of complex rational maps.

Theorem C. *For every $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$, weakly isolated the following properties are equivalent.*

- *TCE (Topological Collet-Eckmann) and no periodic indifferent points in K,*
- *ExpShrink (exponential shrinking of components),*
- *CE2*(z) (backward Collet-Eckmann condition at $z \in K$ for preimages close to K),*
- *UHP. (Uniform Hyperbolicity on periodic orbits in K.)*
- *UHPR. (Uniform Hyperbolicity on repelling periodic orbits in K.)*
- *Lyapunov (Lyapunov exponents of invariant measures are bounded away from 0),*
- *Negative Pressure ($P(t) < 0$ for t large enough)*

Here CE2*(z) means the uniform exponential decay of $|(f^n)'(z_n)|^{-1}$ for (z_n) being all backward trajectories of z contained in pull-backs of a neighbourhood of z intersecting K, rather than z_n only in K, that we denote CE(z). All the definitions will be recalled or specified in Section 3. See Definition 2.8 for the "pull-back".

The only place we use (wi) is that UHP on periodic orbits in K is the same as on periodic orbits in a sufficiently small neighbourhood of K, since both sets of periodic orbits are the same by the definition of (wi).

A novelty compared to the complex rational case is the proof of the implication CE2*(z_0) $\Rightarrow$ ExpShrink, done by the second author for multimodal maps in [R-L]. Here we present a slight generalization, following the same strategy, with a part of the proof being different.

2. Preliminaries

2.1. Basic properties of generalized multimodal pairs.

We shall start with Lemma 2.1 (its first paragraph), which explains in particular why in Definition 1.1 the intervals $\widehat{I^j}$ are non-degenerate.

Lemma 2.1. *If $f : X \to X$ is a continuous map for a compact metric space X and it is topologically transitive, then f maps X onto X and if X is infinite, it has no isolated points and is uncountable.*

If we assume additionally that $X \subset \mathbb{R}$ and moreover that for $\mathbf{U}$ being an open neighbourhood of X, $(f, X, \mathbf{U}) \in \mathscr{A}$, then X is either the union of a finite collection of compact intervals or a Cantor set.

Proof. Recall one of the equivalent definitions of topological transitivity which says that for every non-empty $U, V \subset X$ open in X, there exists $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$. If $x \in X$ were isolated, then for $U = V = \{x\}$ there would exist $n(x) \geq 1$ such that $f^{n(x)}(x) = x$. Since X is infinite there is $y \in X \setminus O(x)$. Let V be a neighbourhood of y in X disjoint from $O(x)$. Then $f^n(\{x\}) \cap V = \emptyset$ for all $n \geq 1$ what contradicts the topological transitivity since $\{x\}$ is open. f is onto X by the topological transitivity since otherwise for the non-empty open sets $U = X$, $V = X \setminus f(X)$ for all $n \geq 1$ $f^n(U) \cap V = \emptyset$. K is uncountable by Baire property.

To prove the last assertion suppose that X contains a non-degenerate closed interval T. Extend $f|_X$ to a C^2 multimodal map $g : I \to I$ of a closed interval I containing $\mathbf{U}$ with all critical points non-flat. Then by the absence of wandering intervals, see [dMvS, Ch.IV, Theorem A], there exist $n \geq 0, m \geq 1$ such that for $L := f^n(T)$ the interval $f^m(L)$ intersects L (the other possibility, that $f^n(T) \to O(p)$ for a periodic orbit in X, is excluded by the topological transitivity of f on X). Let M be the maximal closed interval in X containing L. Then $f^m(M) \subset M$. By the topological transitivity of f on X, for M° denoting the interior of M in X, the union $\bigcup_{j=0}^{\infty} f^j(M^\circ)$ is a dense subset of X. Hence the family of intervals $M, f(M), ..., f^{m-1}(M)$ covers a dense subset of X. But $\bigcup_{j=0}^{m-1} f^j(M)$ is closed, as the finite union of closed sets, hence equal to X. We conclude that X is the union of a finite collection of compact intervals.

If X does not contain any closed interval, then since all its points are accumulation points, X is a (topological) Cantor set. $\square$

Let us provide now some explanation concerning the set $S'(f, K)$, in particular $NO(f, K)$, see Definition 1.17, item 1.

Lemma 2.2. *Consider $(f, K) \in \mathscr{A}$. Then $\mathrm{NO}(f, K) \subset \partial(\widehat{I}_K) \cup \mathrm{Crit}(f)$ is finite. None of points in $S'(f, K) = \mathrm{Crit}(f) \cup NO(f, K)$ is periodic. The set $\mathrm{Crit}(f)$ cuts $\widehat{I}_K = \widehat{I}^1 \cup ... \cup \widehat{I}^{m(K)}$ into smaller intervals $I'^1, ..., I'^{m'(K)}$ such that for each j the restriction of f to the interior of I'^j is a diffeomorphism onto its image and $f(I'^j \cap K) = f(I'^j) \cap K$, i.e. a K-diffeomorphism according to Definition 1.13.*

Proof. The equalities $f(I'^j \cap K) = f(I'^j) \cap K$ hold by the maximality of K. Indeed, if the strict inclusion $\subset$ held for some j we could add to $I'^j \cap K$ the missing preimage of a point in $f(I'^j) \cap K$. (Equivalently this equality holds by Darboux property, see Proposition A.2). So the only points where $f|_K$ is not open can be some end points of I'^j. Hence $\mathrm{NO}(f, K) \subset \partial(\widehat{I}_K) \cup \mathrm{Crit}(f)$ (in fact one can write $\mathrm{Crit}^T(f)$ here) and the set is finite.

No critical point is periodic. Otherwise, if $c \in \mathrm{Crit}(f)$ were periodic, then its periodic orbit $O(c)$ would be attracting in K, what would contradict the topological transitivity of $f|_K$, see Remark 1.6. (We recall an argument: a small neighbourhood od $O(c)$ could not leave it under f^n, therefore could not intersect nonempty open set in K disjoint from $O(c)$.)

Suppose now that $x \in \mathrm{NO}(f, K) \setminus \mathrm{Crit}(f)$ is periodic. Then $x \in \partial(\widehat{I}_K)$ is a limit of K only from one side; denote by L a short interval adjacent to x from this side. Then $f(x)$ is the limit of K from both sides since otherwise $x \notin \mathrm{NO}(f, K)$ since f is a homeomorphism from $L \cap K$ to $f(L) \cap K$. We have already proved that the periodic orbit $O(x)$ is disjoint from $\mathrm{Crit}(f)$.

Hence $f^m(x) = x$ is a limit of K from both sides, as f^{m-1} image of $f(x)$. We arrive at a contradiction. $\square$

The following property, stronger than topological transitivity and weaker than weak exactness, is useful:

Definition 2.3. For $f : X \to X$ a continuous map for a compact metric space X and $x \in X$ denote $A_\infty(x) := \bigcup_{j\geq 0} f^{-j}(x)$. Then f is said to satisfy *density of preimages property*, abbr. (dp), if for every $x \in X$ the set $A_\infty(x)$ is dense in X.

It is clear that weak exactness implies (dp). In fact for generalized multimodal pairs, (dp) corresponds to topological transitivity. Namely

Proposition 2.4. *For every $(f, K) \in \mathscr{A}$ the map $f|_K$ satisfies density of preimages property.*

The proof is not immediate. We provide it in Appendix A, in Proof of A.7, Step 1.

In the general continuous mappings setting, it is easy to provide examples of topologically transitive maps which do not satisfy (dp).

Similarly to Julia sets in the complex case, the following holds in the interval case:

Proposition 2.5. *For every $(f, K) \in \mathscr{A}_+$ the set of repelling periodic orbits for $f|_K$ is dense in K.*

Below are sketches of two proofs. Unfortunately the proofs are not immediate and refer to material discussed further on in this paper. On the other hand they are rather standard.

Proof 1. Apply Theorem 4.1 or facts in Section 4 following it. To be concrete, consider an arbitrary safe, safe forward, expanding point z_0. In Proof of Lemma 4.4, we find an invariant isolated hyperbolic set $X \subset K$ whose all trajectories (in particular a periodic one) pass arbitrarily close to z_0.

Due to $h_{\text{top}}(f|_K) > 0$ we find infinitely many such periodic orbits and all except at most a finite set of them are hyperbolic repelling, see Remark 1.6. Finally notice that safe, safe forward, and expanding points form in K a backward invariant set (i.e. its preimage is contained in it). Therefore having just one such point z_0 whose existence is asserted in Lemma 4.4, we have the set $A_\infty(z_0)$ of safe, safe forward, expanding points. Clearly this set is dense in K, see Definition 2.3 and the discussion following it.

Proof 2. It is easy to prove the density of periodic orbits for piecewise continuous piecewise affine topologically transitive maps of interval with fixed slope $\beta > 1$. It follows for $f|K$ due to the semiconjugacy as in Proof of Lemma A.7, Step 2.

Finally let us discuss some easy technical facts related to weak exactness property or to (dp).

Remark 2.6. For $f : X \to X$ a continuous map for a compact metric space X, for $x \in X$ and for each positive integer k, denote similarly to $A_\infty(x)$,

$$A_k(x) := \bigcup_{0 \leq j \leq k} f^{-j}(x)$$

Assume that f is weakly exact. Then it is easy to see, that there exists $N \geq 0$ depending only on f such that for every $\epsilon > 0$ there exists $n(\epsilon) \geq 0$ such that for every $n \geq n(\epsilon)$ and every ball $B(x_0, \epsilon)$ for $x_0 \in X$, we have $\bigcup_{j=0}^{N} f^{n+j}(B(x_0, \epsilon)) = X$.

In particular $A_{n+N}(x)$ is ϵ-dense in X (that is, for each $y \in X$ there exists $y' \in A_{n+N}(x)$ such that $\mathrm{dist}(y, y') < \epsilon$).

2.2. **On exceptional sets.**

We shall explain here why exceptional sets must be finite, see Definition 1.17, and provide some estimates.

Proposition 2.7. *For every $(f, K) \in \mathscr{A}_+$ (or equivalently $(f, K) \in \mathscr{A}$ such that $f|_K$ is weakly topologically exact), for every finite $\Sigma \subset K$ each weakly Σ-exceptional (in particular weakly S-exceptional) set is finite and its cardinality is bounded by a constant depending only on $\#\Sigma$ and (f, K).*

Proof. Part 1. Let an interval T centered in K be disjoint from a weakly Σ-exceptional set E. It exists since by definition E is not dense in K. Denote $epsilon = |T|/2$.

Let $n(\epsilon)$ be an integer found for ϵ as in Remark 2.6. Then for every $z \in E$ and $k = n(\epsilon) + N$ for N as in Remark 2.6, the set $A_k(z)$ intersects T. Suppose E is infinite. Hence, as all $\#A_k(z)$ are uniformly bounded, given k, with respect to z, there is an infinite sequence of points $z_t \in E$ such that all $A_k(z_t)$ are pairwise disjoint. Indeed, it is sufficient to define inductively an infinite sequence of points $z_t \in E$ such that for every $i < j$ and every $s : 0 \leq s \leq k \quad f^s(z_i) \neq z_j$. If there exists $x \in A_k(z_i) \cap A_k(z_j)$ then there exist s_1, s_2 between 0 and k such that $f^{s_1}(x) = z_i$ and $f^{s_2}(x) = z_j$. Then $f^{s_2 - s_1}(z_i) = z_j$ or $f^{s_1 - s_2}(z_j) = z_i$ depending as $s_2 > s_1$ or $s_1 > s_2$, a contradiction.

Since Σ is finite, we obtain at least one $A_n(z_k)$ disjoint from Σ. Then $A_n(z_k) \subset E$ by the definition of an exceptional set, hence E intersects T, a contradiction.

Part 2. Now we find a common upper bound for $\#E$ for all weakly Σ-exceptional E, depending on $\#\Sigma$ and (f, K). For this we repeat the consideration in Part 1. with more care.

First, consider $O(p)$ a repelling periodic orbit in $K \cap \mathrm{interior}\, \widehat{I}_K$. It exists because there are infinitely many such orbits in K by $h_{\mathrm{top}}(f|_K) > 0$, see e.g. Theorem 4.1. Let $T = T(p)$ be now an open interval centered at p with $|T| = 2\epsilon$ small enough that the backward branches g_n inverse to f^n, along $O(p)$, exist on T, $g_n(T) \subset \widehat{I}_K$ and $g_n(T) \to O(p)$ as $n \to \infty$, and else $(g_n(T) \setminus O(p)) \cap \Sigma = \emptyset$.

Define $k = n(\epsilon) + N$ as in Part 1 of the Proof. Denote $\sup_{z\in K} \#A_k(z)$ by $C = C(\epsilon)$. Find consecutive points $z_t \in E$ by induction as follows. Take an arbitrary $z_1 \in E$. Next find $z_2 \in E$ so that $A_k(z_1) \cap A_k(z_2) = \emptyset$. This is possible if $\#E > C + k$. The summand C is put to avoid $z_2 \in A_k(z_1)$, the summand k is put to avoid $z_2 = f^s(z_1)$ for $0 < s \leq k$. To be able to repeat this until finding z_m for an arbitrary integer $m > 0$, we need $\#E > (m-1)(C + k)$.

Assume that E is disjoint from $T(p)$. If $m > \#\Sigma$, then at least one $A_k(z_t)$ for $t = 1, ..., m$ is disjoint from Σ and intersects $T(p)$. Hence E intersects $T(p)$, a contradiction. Therefore

$$\#E \leq C(f, K, \epsilon) := (\#\Sigma)(C(\epsilon) + k). \tag{2.1}$$

Consider finally E being an arbitrary non-dense weakly Σ-exceptional set (previous E was assumed to be disjoint from $T(p)$). Notice that if $\#E > \#O(p)$ then $E' := E \setminus O(p)$ is also a weakly Σ-exceptional (notice that removal of a forward invariant set from a weakly Σ-exceptional set leaves the rest weakly Σ-exceptional or empty).

We conclude with

$$\#E \leq \#O(p) + (\#\Sigma)(C(\epsilon) + k), \tag{2.2}$$

see (2.1). Indeed, if this does not hold, then $\#E' > C(f, K, \epsilon)$, hence there exists $w \in E' \cap T(p)$. Then the trajectory of w under g_n omits Σ hence it is in $E \setminus O(p)$ (notice that it is in $\widehat{I}_K$ hence in K by maximality of K) and therefore E is infinite which contradicts finiteness proved already in Part 1. of the Proof. $\square$

2.3. **Backward stability.**

The following notion is useful

Definition 2.8. Let $(f, K, \mathbf{U}) \in \mathscr{A}$. For any interval $T \subset \mathbf{U}$ intersecting K, or a finite union of intervals each intersecting K, and for any positive integer n, we call an interval T' its *pull-back* for f^n, of order n, or just a *pull-back* if it is a component of $f^{-n}(T)$ intersecting K

Notice that, unlike in the complex case, f^n need not map T' onto T. This can happen either if T' contains a turning point or if an end point of T' coincides with a boundary point of $\mathbf{U}$. We shall prevent the latter possibility by considering T small enough (or having all components small enough), see Lemma 2.10.

The absence of wandering intervals, [dMvS, Ch. IV, Th. A], see also comments in Example1.8 and Proof of Proposition A.2, and Lemma 2.1, is reflected in the following

Lemma 2.9. *Let $f : U \to \mathbb{R}$ for open set $U \subset \mathbb{R}$ be a C^2 map with non-flat critical points and $X \subset U$ be a compact f-invariant set. Let $T \subset U$ be an open interval such that one end x of T belongs to X and $f^k(T) \cap X = \emptyset$ for all $k \geq 0$ (we consider f^k on its domain which can be smaller than T).*

Then either x is eventually periodic, that is there exists $n, m > 0$ such that $f^n(x) = f^{n+m}(x)$, or attracted to a periodic attracting or indifferent periodic orbit in X.

If $(f, X, U) \in \mathscr{A}$, then in the former case the periodic orbit $O(f^n(x))$ is disjoint from $\mathrm{Crit}(f)$ *and contains a point belonging to $\partial(\widehat{I}_X)$.*

Proof. We shall only use f restricted to a small neighbourhood U_X of X in U. We consider $U_X = \{x : \mathrm{dist}(x, X) < \delta\}$ for a positive δ. So U_X has finite number of components. We assume that δ is small enough that $U_X \setminus X$ does not contain critical points.

Let us define by induction $T_0 = T$ and T_n the component of $f(T_{n-1}) \cap U_X$ containing $f^n(x)$ in its closure. We consider two cases:

Case 1. There exists n_0 such that for all $n \geq n_0$, $f(T_{n-1}) \subset U_X$, that is $T_n = f(T_{n-1})$, in particular $f(T_{n-1}) \cap U_X$ is connected so there is no need to specify a component.

Suppose that all T_n are pairwise disjoint for $n \geq n_0$. As in the previous proofs extend $f|_{U_X}$ to a C^2 multimodal map $g : I \to I$ of a closed interval containing U_X with all critical points non-flat. Then T_{n_0} is a wandering interval for g since on its forward orbit f and g coincide (unless it is attracted to a periodic orbit) which is not possible by [dMvS, Ch. IV, Th. A].

So let $n' > n \geq n_0$ be such that $T_n \cap T_{n'} \neq \emptyset$. Then the end points $f^n(x)$ and $f^{n'}(x)$ of T_n and $T_{n'}$ respectively, belonging to X, do not belong to the open interval $T^1 := T_n \cup T_{n'}$. If these end points coincide, then $f^n(x) = f^{n'}(x)$ i.e. x is eventually periodic. If they do not coincide consider $T^2 := f^{n'-n}(T^1)$. The intervals T^1 and T^2 have the common end $x^1 := f^{n'}(x)$. Since $f^{n'-n}$ changes orientation on $f^n(T)$ (i.e has negative derivative) and there are no critical points in $T^1 \subset U_X \setminus X$, it changes orientation on T^1. Hence $T^2 = T^1$ and x^1 is periodic of period $2(n' - n)$.

Case 2. There is a sequence $n_j \to \infty$ such that $f(T_{n_j - 1})$ are not contained in U_X. For each such n_j one end of T_{n_j} is $f^{n_j}(x) \in X$ and the other end is in ∂U_X. Remember that T_{n_j} is disjoint from X. Now notice that there is only a finite number of such intervals. So again there are two different n_j and $n_{j'}$ such that $f^{n_j}(x) = f^{n_{j'}}(x)$, hence x is eventually periodic.

Suppose now that $(f, X, U) \in \mathscr{A}$. Suppose that x is preperiodic and put $p = f^n(x)$ periodic. Then $p \notin \mathrm{Crit}(f)$ since otherwise p would be a (super)attracting periodic point so it could not be in K.

If $O(p) \cap \partial(\widehat{I}_X) = \emptyset$ then the forward orbit of $T' := f^n(T)$ would stay in $\widehat{I}_X$. It cannot leave this set because it would capture a point belonging to $\partial(\widehat{I}_X)$ hence belonging to X. This is however not possible since by the maximality of X in $\widehat{I}_X$ the forward orbit of T' would be in X.

□

Now we are in the position to prove a general lemma about shrinking of pull-backs.

Lemma 2.10. *For every $(f, K, \mathbf{U}) \in \mathscr{A}_+$ and for every $\epsilon > 0$ there exists $\delta > 0$ such that if T is an arbitrary open interval in $\mathbb{R}$ intersecting K, disjoint from* Indiff(f)*, and satisfying $|T| \leq \delta$, then for every $n \geq 0$ and every component T' of $f^{-n}(T)$ intersecting K (i.e pull-back, see Definition 2.8) we have $|T'| \leq \epsilon$. Moreover the lengths of all components of $f^{-n}(T)$ intersecting K converge to 0 uniformly as $n \to \infty$.*

In particular, for δ small enough the closures in $\mathbb{R}$ of all T' are contained in U.

Proof. Since $(f, K, \mathbf{U}) \in \mathscr{A}_+$ then $f|_K$ is weakly topologically exact, Lemma A.7. (We shall not use $h_{\mathrm{top}}(f|_K) > 0$ anymore in Proof of Lemma 2.10.)

Suppose there is a sequence of intervals T_n intersecting K, disjoint from Indiff(f), with $|T_n| \to 0$, a sequence of integers $j_n \to \infty$ and a sequence of intervals T_n' being some components of $f^{-j_n}(T_n)$ respectively, intersecting K, with $|T_n'|$ bounded away from 0. Then, passing to a subsequence if necessary, we find a non-trivial open interval $L \subset I$ such that $L = \lim_{n\to\infty} T_n'$ (in the sense of convergence of the end points).

Suppose that L intersects K. Then there exists an open interval $L' \subset L$ with closure contained in L such that L' intersects K. Then $T_n' \supset L'$ for n large enough, hence $L_n := f^{j_n}(L') \subset T_n$. By weak topological exactness, see Definition 1.15, there exists $N > 0$ such that for all n large enough $\bigcup_{j=0}^{N} f^j(L_n) \supset K$ which is not possible, since the lengths of $L_n \subset T_n$ tend to 0 and K is infinite.

Suppose now that L does not intersect K. Since all T_n' intersect K we can choose $a_n \in T_n' \cap K$. Let a be the limit of a convergent subsequence a_{n_t} (to simplify notation we shall omit the subscript t). Then $a \in K$ by the compactness of K, hence it is one of the end points of L.

All iterates f^k are well defined on the whole L since for every n the map f^{j_n} is well defined on T_n' hence all $f^k, k \leq j_n$ are well defined on T_n', $j_n \to \infty$ and $T_n' \to L$. If we replace L by L' having the same end a but shorter at the other end, then as in the previous case $T_n' \supset L'$ for n large enough (all T_n' for n large enough contain a since the points a_n lie on the other side of a than L; otherwise L would intersect K). Hence $f^{j_n}(L') \subset T_n$.

(Though the f^{j_n} are defined on L we need to shorten it to L' to have the latter inclusion. So we shall use L' instead of L in the sequel.)

If there is $k>0$ such that $f^k(L')$ intersects K then this contradicts weak topological exactness as in the previous case.

If no $f^k(L')$ intersects K we can apply Lemma 2.9 and conclude that a is eventually periodic. Moreover by Lemma 2.9 we can find periodic $a' = f^s(a)$ such that $a' \in \partial(\widehat{I}_K) \setminus \mathrm{Crit}(f)$.

We conclude that all $|f^k(L')|$ for k large enough are bounded away from 0 by a constant $D(a')$. This is obvious if a' (more precisely its orbit $O(a')$)) is repelling. We just define $D(a') := \mathrm{dist}(O(a'), \partial W)$ for W as in Definition 1.5. Finally notice that a' cannot be indifferent. Indeed, if a' were indifferent, attracting from one side, this side would be the side on which $f^s(L')$ lied, disjoint from K. Hence all $f^s(a_n)$ would be on the other side. Hence $f^s(T_n')$ and therefore T_n would contain indifferent periodic points (belonging to $O(a')$) for n large enough, what would contradict the assumptions.

We again arrive at contradiction with $|T_n| \to 0$.

The proof that if $|T|$ is small enough then for any sequence T_n' being components of $f^{-j_n}(T)$ respectively, for varying T, intersecting K (i.e. pull-backs, see Definition 2.8) we have uniformly $|T_n'| \to 0$, is virtually the same.

Let us be more precise. Since the set $\partial(\widehat{I}_K)$ is finite, $D = \min_{a'} D(a')$ is positive, where a' are repelling periodic points belonging to $\partial(\widehat{I}_K)$. Let $\widehat{D} > 0$ be an arbitrary number less than D and such that for no T with $|T| \le \widehat{D}$ the inclusion $\bigcup_{j=0}^{N} f^j(T) \supset K$ is possible.

Fix an arbitrary integer $k>0$. Suppose there is a sequence of intervals $T(n)$ intersecting K with $|T(n)| \le \widehat{D}$, disjoint from $\mathrm{Indiff}(f)$ and for each $T(n)$ there exist its pull-back T_n' of order j_n with $j_n \to \infty$ as $n \to \infty$, such that for all n, $|T_n'| \ge 1/k$. Then we find a limit L and $L' \subset L$ and arrive at contradiction as before. Therefore there exists $j(k)$ such that for all T with $|T| \le \widehat{D}$ every pull-back of T of order at least $j(k)$ is shorter than $1/k$. This applied to every k gives for $k \to \infty$ the asserted uniform convergence of $|T_n'|$ to 0.

□

Remark 2.11. The first conclusion of Lemma 2.10, that $|T'| < \varepsilon$, is called *backward Lyapunov stability*. To obtain this conclusion it is sufficient to assume $(f,K) \in \mathscr{A}$. In particular weak exactness assumption is not needed. The proof of backward Lyapunov stability can be then proved as follows.

In the case where no $f^k(L')$ intersects K, only Lemma 2.9 has been used in Proof of Lemma 2.10, where weak exactness was not assumed.

In the remaining case we can use [dMvS, Contraction Principle 5.1]. Namely, extend f to a C^2 multimodal map $g : I \to I$, where $I \supset \mathbf{U}$, as in Proof of Lemma 2.9 (keep for g the notation f). Then, for L' as in Proof of Lemma 2.10, due to $\liminf_{n\to\infty} |f^n(L')| \le \liminf_{n\to\infty} |T_n| = 0$, by Contraction Principle, either L' is wandering or attracted to a periodic orbit O.

The former, wandering case, is not possible by [dMvS, Ch.IV, Th. A]. In the latter case $O \subset K$, since all $f^n(L')$ intersect K for n large enough by forward invariance of K. This however contradicts topological transitivity of $f|_K$, see Proof of Lemma 2.1.

Backward Lyapunov stability can fail in the complex case, e.g. for f on its Julia set, even for f being a quadratic polynomials without indifferent periodic orbits. See [Levin, Remark 2].

2.4. **Weak isolation.**
Below is the example of a generalized multimodal pair not satisfying (wi), promised in Introduction

Example 2.12. First consider $f : [-1,1] \to [-1,1]$ defined by $f(x) := 4x^3 - 3x$. This is degree 3 Chebyshev polynomial with fixed points -1 and 1 and critical points $-1/2, 1/2$ mapped to 1 and -1 respectively. For $x : -1 \leq x \leq 1, x \neq \mp 1/2$ let $i(x) = 0$, 1 or 2, depending as $x < -1/2, -1/2 < x < 1/2$ or $x > 1/2$. Define the itinerary of each $x \in (-1,1)$ whose trajectory does not hit $\mp 1/2$ by the infinite sequence $\underline{i}(x) = (i(x), i(f(x)), ..., i(f^n(x)), ...)$ and of each x such that $f^n(x) = -1/2$ or $1/2$ by the finite sequence $(i(x), i(f(x)), ..., i(f^{n-1}(x)))$, empty for $x = \mp 1/2$. Define the itinerary as empty set also for $x = \mp 1$. Later on we shall usually omit commas in the notation of these sequences. Let T be the open interval of points in $[-1,1]$ for which the itinerary starts with the block 11. Then the forward trajectory of ∂T is disjoint from the closure of T (compare Definition 6.5). We define

$$K := [0,1] \setminus \bigcup_{n=0}^{\infty} f^{-n}(T).$$

Notice that K can be defined as the set of all points $[-1,1]$ for which the itinerary does not contain the forbidden block 11.

Finally define $\widehat{I}_K := [-1,1] \setminus T$, and consider $(f, K, \widehat{I}_K, \mathbf{U}) \in \mathscr{A}$, where $\mathbf{U}$ is an arbitrary two components neighbourhood of $[-1,1] \setminus T$ and f is smoothly extended to $\mathbf{U} \setminus [-1,1]$ keeping the previous f on $[-1,1]$, in particular on $[-1,1] \cap \mathbf{U}$.

The map $f|_K$ is topologically transitive and even topologically exact. Here is the standard proof: $f|_K$ is the factor of the topological Markov chain (σ, Σ_A), with the 3×3 transition 0-1 matrix $A = (a_{ij})$ having all entries equal to 1, except $a_{11} = 0$. Σ_A is by definition the space of all infinite sequences $a_0 a_1 ...$ where $a_i = 0, 1$ or 2 with no two consecutive 1's and σ is the shift to the left, $\sigma((a_j))_k = a_{k+1}$. The projection, i.e. the 'coding' π, is defined in the standard[2] way, as follows. Consider an arbitrary sequence $a_0a_1...$ of integers 0,1, or 2, not containing the forbidden block 11 and for each integer n consider the associated cylinder $C_n(a_0...a_n)$, i.e. the set of all sequences in Σ_A starting from the this block. Define $\pi'(C_n(a_0...a_n))$ as the set of all points in $[-1,1]$ whose itinerary starts from the block $a_0...a_n$. Then $\pi((a_0a_1...))$ is (uniquely) defined as $\bigcap_{n\to\infty}\overline{\pi'(C_n(a_0...a_n))}$. We obtain by construction $\pi\circ\sigma = f\circ\pi$. The continuity of π also follows easily from the construction.

Consider an arbitrary $C_n = C_n(a_0...a_n)$. Then $\sigma^{n+2}(C_n) = \Sigma_A$, i.e. the whole space. This is so because every sequence $(b_j)\in\Sigma_A$ is the σ^{n+2}-image of $(a_0...a_na_{n+1}b_0b_1...)$ belonging to C_n for a_{n+1} being 0 or 2, since the latter sequence does not contain forbidden 11. (Notice that in the case $a_n\neq 1$ we do not need a_{n+1} hence σ^{n+1} would be sufficient.) The topological exactness of σ on Σ_A immediately implies the topological exactness of the factor $f|_K$.

It is easy to calculate that $h_{\mathrm{top}}(\sigma) = \log(1+\sqrt{3})$, which is positive. Since π is a coding *via* Markov partition, $h_{\mathrm{top}}(\sigma) = h_{\mathrm{top}}(f|_K)$, compare e.g. [PU, Theorem 4.5.8]; in fact this equality easily follows from the fact that our coding π is at most 2-to-1 (use the definition of the topological entropy *via* (n,ϵ)-nets, for any such net in K consider its π-preimage in Σ_A). In particular $h_{\mathrm{top}}(f|_K)$ is positive. Thus $(f,K)\in\mathscr{A}_+$. (In fact 'positive entropy' follows already from the topological exactness, see Proposition A.8.)

For each $n\geq 2$ there is a periodic point in $[-1,1]\setminus K$ of period n with the periodic orbit having the itinerary being the concatenation of the 1100...0 (block of length n) with itself infinitely many times, hence intersecting T, therefore not in $\widehat{I}_K$. It is arbitrarily close to K for n large, since the interval encoded by the block 1100...0 of length n is adjacent to the interval encoded by the block 1200...0 of length n. Therefore the weak isolation condition (wi) is not satisfied.

2.5. **Bounded Distortion and related notions.**

In Section 6 we shall use 'bounded distortion' properties formally stronger than BD defined in Definition 1.10.

Definition 2.13. We say that $(f,K,\mathbf{U})$ satisfies *Hölder bounded distortion* condition, abbr. HBD, if there exist constants $\alpha: 0<\alpha\leq 1$ and $\delta>0$ such that for every $\tau>0$ there exists a constant $C(\tau)$ such that the following holds:

For every pair of intervals $I_1,I_2\subset\mathbf{U}$ such that I_1 intersects K and $|I_2|\leq\delta$ and $f^n(I_1)=I_2$ for a positive integer n, for every interval $T\subset I_2$ such that I_2 is a τ-scaled neighbourhood of T, for $g=(f^n|_{I_1})^{-1}$ and for all $x,y\in T$

$$|g'(x)/g'(y)|\leq C(\tau)(|x-y|/|T|)^{\alpha}.$$

[2]In our Chebyshev case we can just write $\pi(a_0a_1...) = -\cos(\pi\sum_{n=0}^{\infty}a_n/3^n)$, and use the identity $f(\cos(\theta)) = \cos(3\theta)$ to get $f\circ\pi = \pi\circ\sigma$.

For $\alpha = 1$ we call this condition: *Lipschitz bounded distortion* condition and abbreviate to LBD.

Remark 2.14. The conditions BD and even LBD are true if f is C^3 and its Schwarz derivative is negative, (without assuming I_1 intersects K, as K is in this situation irrelevant), see [dMvS]. More precisely

$$|g'(x)/g'(y)| < (\frac{1+2\tau}{\tau^2} + 1)\frac{|g(x) - g(y)|}{|g(T)|},$$

see e.g. [BT1, Section 2.2] or [dMvS, Ch.IV, Theorem 1.2]. In consequence

$$|g'(x)/g'(y)| < C(\tau)\frac{|x-y|}{|T|}.$$

Seemingly, see [vSV], BD and LBD hold for all $(f,K) \in \mathscr{A}$ (i.e. C^2) provided the set of all periodic orbits in $\mathbf{U}$ except hyperbolic repelling orbits, is in a positive distance from K, i.e. there are no such orbits if we shrink $\mathbf{U}$.

They are true for every multimodal map of interval $f : I \to I$ if f is C^3 and all periodic orbits are hyperbolic repelling, by an argument in [BRSS, Chapter 3] using [vSV], decomposing f^n into a negative Schwarzian block till the last occurrence of $f^j(I_1)$ close to $\mathrm{Crit}(f)$ and an expanding one, within a distance from $\mathrm{Crit}(f)$.

Hence we can assume that BD and even LBD are true for $(f,K) \in \mathscr{A}^3$ if all periodic orbits in K are hyperbolic repelling, by an appropriate modification (if necessary) of f outside K (in fact only outside $\widehat{I}_K$) so that also outside K all periodic orbits are hyperbolic repelling. See Appendix A, Lemma A.4 for details, and [BRSS, Chapter 3] cited above.

In this paper if we need to use BD (similarly LBD or HBD) for $(f,K) \in \mathscr{A}$ but do not need smoothness higher than C^2, we just assume it. If a higher smoothness, say C^3, is needed by other reasons (e.g. existence of nice couples in Proof of Theorem A), we assume $(f,K) \in \mathscr{A}^3$ and then we have BD (or LBD, HBD), provided f is appropriately modified outside $\widehat{I}_K$. Then we *a priori* consider this modified f sometimes without mentioning it.

3. Topological Collet-Eckmann maps

In this Section we shall prove Theorem C. Let us recall some definitions, see e.g. [PR-LS1], [NP] and [R-L], adapted to C^2 generalized multimodal quadruples, namely for $(f, K, \widehat{I}_K, \mathbf{U}) \in \mathscr{A}$, see Introduction.

- TCE. *Topological Collet-Eckmann condition.* There exist $M \geq 0, P \geq 1$ and $r > 0$ such that for every $x \in K$ there exists a strictly increasing sequence of positive integers n_j, for $j = 1, 2, ...$ such that $n_j \leq P \cdot j$ and for each j

$$\#\{i : 0 \leq i < n_j, \mathrm{Comp}_{f^i(x)} f^{-(n_j - i)} B(f^{n_j}(x), r) \cap \mathrm{Crit}(f) \neq \emptyset\} \leq M,$$

• ExpShrink. *Exponential shrinking of components.* There exist $\lambda_{\text{Exp}} > 1$ and $r > 0$ such that for every $x \in K$, every $n > 0$ and every connected component W of $f^{-n}(B(x,r))$ intersecting K (pull-back, see Definition 2.8), we have

$$|W| \leq \lambda_{\text{Exp}}^{-n}.$$

• Lyapunov. *Lyapunov exponents of invariant measures are bounded away from zero.* There is a constant $\lambda_{\text{Lyap}} > 1$ such that the Lyapunov exponent of any invariant probability measure μ supported on K satisfies $\Lambda(\mu) \geq \log \lambda_{\text{Lyap}}$.

• Negative Pressure. *Pressure for large t is negative.* For large values of t the pressure function $P(t)$ is negative.

• UHP. *Uniform Hyperbolicity on periodic orbits.* There exists $\lambda_{\text{Per}} > 1$ such that every periodic point $p \in K$ of period $k \geq 1$ satisfies,

$$|(f^k)'(p)| \geq \lambda_{\text{Per}}^k.$$

• UHPR. *Uniform Hyperbolicity on hyperbolic repelling periodic orbits.* The same as UHP but only for hyperbolic repelling periodic points in K.

• CE2(z_0). *Backward* or *the Second Collet-Eckmann condition at $z_0 \in K$.* There exist $\lambda_{\text{CE2}} = \lambda_{\text{CE2}}(z_0) > 1$ and $C > 0$ such that for every $n \geq 1$ and every $w \in f^{-n}(z_0) \cap K$,

$$|(f^n)'(w)| \geq C\lambda_{CE2}^n.$$

• CE2*(z_0). *Backward* or *the second Collet-Eckmann condition at $z_0 \in K$ for preimages close to K.* The same as CE2(z_0), but for all $w \in f^{-n}(z_0)$ such that for a constant $R > 0$ (not depending on z_0, nor on n) all pull-backs of $B(z_0, R)$ for f^j containing respective $f^{n-j}(w)$, for $j = 0, ..., n$ intersect K. (This condition is stronger than $w \in K$ in CE2(z_0).)

The appropriate λ will be denoted by λ_{CE2^*}.

To prove Theorem C we shall use the following important

Lemma 3.1. *Let $(f, K) \in \mathscr{A}$. Then*

$$\sum_{j=0}^{n}{}' - \log |T^j(x) - c| \leq Qn \tag{3.1}$$

for a constant $Q > 0$ an arbitrary $c \in \text{Crit}(f)$, every $x \in K$ and every integer $n > 0$. Σ' means we omit in the sum the index j of smallest distance $|T^j(x) - c|$.

This Lemma first appeared in [DPU] in the complex setting, named Rule II. In subsequent [NP] where the lemma appeared in the real setting, ... the absence of strictly attracting periodic orbits was assumed. Here the absence of such orbits in a small neighbourhood of K also takes place by the assumption of its maximality of K if we assume Bounded Distortion, see Remark 1.11. In fact the absence of such orbits with large periods is sufficient, which follows from [dMvS, Ch. IV, Theorem B].

Lemma 3.1 easily implies the following, compare [DPU, Lemma 3.4].

Corollary 3.2. *There exist $L \geq 1$ and $\kappa, a > 0$ such that for every interval T intersecting K and its pullback T_j for f^j (i.e. connected component T_j of $f^{-j}(T)$ intersecting K) if $|T| \leq a$, then*

$$|U_j| \leq L^j |U|^\kappa. \tag{3.2}$$

We shall need also the following standard

Lemma 3.3. *Let $f : I \to I$ be a C^2 multimodal map. Then there exists $C_1 > 0$ such that for every interval $T \subset I$, every component T_1 of its pre-image by f^{-1} and for every $x \in T_1$, we have*

$$|T_1|/|T| \leq C_1 |f'(x)|^{-1}.$$

Proof of Theorem C. The fact that ExpShrink implies TCE was proved in the interval case in [NP, Section 1]. We used there the estimate (3.1).

The opposite implication does not have a direct proof in [NP] but one can adapt word by word the proof provided in [P-Holder, Section 4] for the complex case, namely the "telescope method". One applies Lemma 2.10.

The proof of ExpShrink ⇒ Lyapunov is the same as in the complex case, see [PR-LS1, Proposition 4.1]. Also the proof of Lyapunov⇔Negative Pressure is the same. Lyapunov immediately implies UHP. The only difficulty is to prove that UHP (or UHPR) implies ExpShrink. In [NP] the paper [NS] was referred to, but it concerned only the unimodal case. Below is a proof for the generalized multimodal case, following the same strategy for the case of multimodal maps in [R-L], but with a part of the proof being different.

First notice that UHPR implies EC2*(z_0). For this it is sufficient to take an arbitrary $z_0 \in K$ safe and expanding. Its existence follows from Lemma 4.4.

Fix $\epsilon > 0$ such that $B(K, \epsilon) \subset U$, the neighbourhood of K in the definition of (wi), Definition 1.20. Let $R := \delta > 0$ be chosen for $\epsilon > 0$ as in Lemma 2.10. Consider an arbitrary $z_n = w \in f^{-n}(z_0)$ as in the definition of CE2*(z_0), i.e. such that T_n being the pull-back of $B(z_0, R)$ for f^n, containing z_n intersects K. We find a repelling periodic orbit $O(p)$ which follows $f^j(z_n)$ for $j = 0, 1, ..., n$, with period $n(p)$ not much bigger than n and $\log |(f^n)'(z_n)|/n$ is at least (up to a factor close to 1) $\log |(f^{n(p)})'(p)|/n(p)$ for n large, hence positive bounded away from 0 by UHPR. By (wi) $O(p) \subset K$.

We have followed here Proof of Lemma 4.4, where more details are provided. See also, say, [PR-LS1, Lemma 3.1] for a detailed proof.

Now assume that CE2*(z_0) holds. We shall prove ExpShrink. Notice first that there are no indifferent periodic points in K, since otherwise the derivatives of f^{-j} of backward branches at z_0 converging to a indifferent point would not tend to 0 exponentially fast.

Notice that the set of z's for which CE2*(z) holds is dense in K because it is backward invariant and $\bigcup_{n=0}^{\infty} f^{-n}(z_0)$ is dense in K, see Proposition 2.4. One should be careful however because the constant C in the definition of CE2*(z) can depend on $z \in \bigcup_{n=0}^{\infty} f^{-n}(z_0)$.

To prove ExpShrink assume to simplify notation (this does not hurt the generality) that no critical point in K contains in its forward orbit another critical point, see definitionexceptional, item 3.

Case 1. Consider an arbitrary interval T, shorter than $R/2$, whose end points z_0, z_0' satisfy CE2* (remember that they need not be in K) with common constants C and λ. Consider $S \supset T$ its 1/2-scaled neighbourhood in $\mathbb{R}$. Consider consecutive pull-backs T_j of T intersecting K and accompanying pull-backs S_j of S until S_n captures a critical value for the first time, for some $n = n_1$. By bounded distortion property Definition 1.10, writing $T_n = [z_n, z_n']$, we get

$$|T|/|T_n| \geq C(\epsilon)^{-1} \max\{|(f^n)'(z_n)|, |(f^n)'(z_n')|\}. \tag{3.3}$$

Next, for a constant C_1 by Lemma 3.1

$$|T|/|T_{n+1}| \geq C_1{-}1C(\epsilon)^{-1} \max\{|(f^{n+1})'(z_{n+1})|, |(f^{n+1})'(z_{n+1}')|\}. \tag{3.4}$$

Here we also write $T_{n+1} = [z_{n+1}, z_{n+1}']$, but unlike before both z_{n+1} and z_{n+1}' can be f^{n+1} pre-images of the same z or z', i.e. both can be f-pre-images of the same z_n or z_n'.

Next we consider S^1 the 1/2-scaled neighbourhood of $T^1 := T_{n+1}$ and pull-back as before until for $n = n_2$ the pull-back S_n^1 of S^1 captures a critical point.

We continue for an arbitrarily long time m. Notice that each n_s is larger than an arbitrary constant if T is short enough. This follows from Lemma 2.10.

Hence we finish with

$$|T|/|T_m| \geq (C_1 C(\epsilon))^{-\beta m} \max\{|(f^m)'(z_m)|, |(f^m)'(z_m')|\}.$$

for $\beta > 0$ an arbitrarily small constant. Hence

$$|T|/|T_m| \geq (C_1 C(\epsilon))^{-\beta m} C \lambda^m. \tag{3.5}$$

This proves ExpShrink for T with λ_{Exp}' arbitrarily close to λ. Case 1 is done.

Let $\{Q^j\}$ be the family of all open intervals in $\mathbb{R} \setminus K$ with boundary points in K or $\pm\infty$ of length at least $R/6$. Let ∂Q denote the set of all finite boundary points of $Q := \bigcup Q^j$. Denote $D_1 := f(\mathrm{Crit}(f)) \setminus \partial Q$ and $D_2 = f(\partial Q) \setminus \partial Q$.

Let $\rho = \min_{i=1,2} \operatorname{dist}(D_i, \partial Q)$. By definition, $\rho > 0$. Finally let $N \geq 0$ be an integer such that for each $n \geq N$, each pull-back for f^n of every interval of length not larger than R, intersecting K, has length less than $R' := \min\{\rho/2, R/3\}$.

Fix $r : 0 < r < \min\{\rho/2, R/12\}$. Fix $A \subset K$ an arbitrary finite set of points for which CE2* holds and which is r-dense in K. We consider common C in the definition of CE2* for all points in A.

Now consider an arbitrary interval T intersecting K of length at most R. Replace T by $\hat{T}$ being a pull-back of T for f^N. Then $|\hat{T}| \leq R'$ and the same estimate holds for all pull-backs of $\hat{T}$.

The case $\hat{T}$ where is contained in an interval $[z, z']$ contained in a component of $\mathbb{R} \setminus Q$, for $z, z' \in A$, follows from Case 1. (since by the estimates above we can assume $|z - z'| < R/2$). Now we consider the remaining

Case 2. $\hat{T}$ intersects $[b, b']$ for $b \in \partial Q$ and $b' \in A$ the closest to b point in A. We set $z_0 = b'$ and consider a new T by adding $[b, b']$ intersecting $\hat{T}$ to the original $\hat{T}$. Its length is less than $R/12 + R/3 < R/2$.

Let us start to proceed as before by taking pull-backs of S being the 1/2-scaled neighbourhood of the new T. Fix an arbitrary $m > 0$. Let $k : 0 < k \leq m$ be the first time when the pull-back T_k does not intersect any $[b, b']$. Then we first estimate the ratio $|T|/|T_k|$. If k above does not exist we estimate the final $|T|/|T_m|$.

Notice that for all $j \leq k$ the corresponding preimages $z_j \in T_j$ of z_0 exist and belong to $\mathbb{R} \setminus Q$. We apply induction. For all $0 \leq j < k$ we have $|T_j| < R'$ hence $T_j \subset B(\partial Q, r + R')$. Since $r < \rho/2$ and $R' < \rho/2$, $T_j \subset B(\partial Q, \rho)$. Hence, for $T'_j := T_j \setminus \overline{Q}$, by $\rho < D_2$, we obtain $T'_j \cap f(\partial Q) = \emptyset$. Suppose that $z_j \in T'_j$. Hence T'_{j+1}, each pull-back of T'_j for f in T_j, intersects K since $f^{-1}(T_j \setminus T'_j)$ is disjoint from K, since $T_j \setminus T'_j$ is disjoint from K, by f-invariance of K. Hence T'_{j+1} is entirely in $\mathbb{R} \setminus Q$, hence $z_{j+1} \in K$ by the maximality of K.

This consideration finding $z_j \in T_j$ is complete as long as all T_j are disjoint from $\operatorname{Crit}^T(f)$. So suppose that for some $n + 1 \leq k$ the pull-back S_{n+1} captures a critical point c for the first time.

By our definitions T_n intersects an interval $[b, b']$. As before $T_j \subset B(b, \rho)$. Hence, by $\rho < D_1$, we get $f(c) = b$. If T_{n+1} contains a turning critical point c, the f-image of a neighbourhood of c (the "fold") cannot be on the other side of b than b', otherwise c would be isolated in K. Therefore, by $z_n \in T'_n$, the pullback T_{n+1} contains a preimage (even two preimages) of z_n, both in $\mathbb{R} \setminus Q$.

To cope with distortion we deal as in Case 1., pulling back 1/2-scaled neighbourhoods S^t of corresponding T^{n_t} until consecutive capture of a critical point. We conclude with the estimate (3.5) with $m = k$.

(Notice that in fact a capture of a critical point can happen only a finite number of times, since otherwise ∂Q, hence K, would contain an attracting periodic orbit, containing a critical point.)

Still an estimate of $|T_k|/|T_m|$ from below is missing. We are not able to make an estimate depending on $|T_k|$ and will just rely on an estimate of $1/|T_m|$.

We have $T_k \subset [z, z']$ for $z, z' \in A$ and $|z - z'| < R/2$. Hence, by Case 1, for T_s^* denoting the pull-back of $[z, z']$ for f^s containing T_{k+s} we have $|z - z'|/|T_s^*| \geq C_1 C(\epsilon))^{-\beta s} C\lambda^s$. Hence for $\xi > 0$ arbitrarily close to 0 and respective $C(\xi)$ we get $|T_s^*| \leq C(\xi)\lambda^{-s(1-\xi)}$. We have also, by Corollary 3.2, $|T_{k+s}| \leq L^s |T_k|^\kappa$. Summarizing, we have

$$|T_{k+s}| \leq \min\{C(\xi)\lambda^{-s(1-\xi)}, L^s|T_k|^\kappa\} \tag{3.6}$$

Combined with (3.5) in the form $|T_k| \leq C(\xi)\lambda^{-k+\xi}$, it can be calculated that there is $\lambda' > 1$ such that $|T|/|T_m| \geq \mathrm{Const}\,\lambda'^m$. One obtains $\log\lambda'$ arbitrarily close to

$$\frac{\kappa(\log\lambda)^2}{\log L + (1+\kappa)\log\lambda}.$$

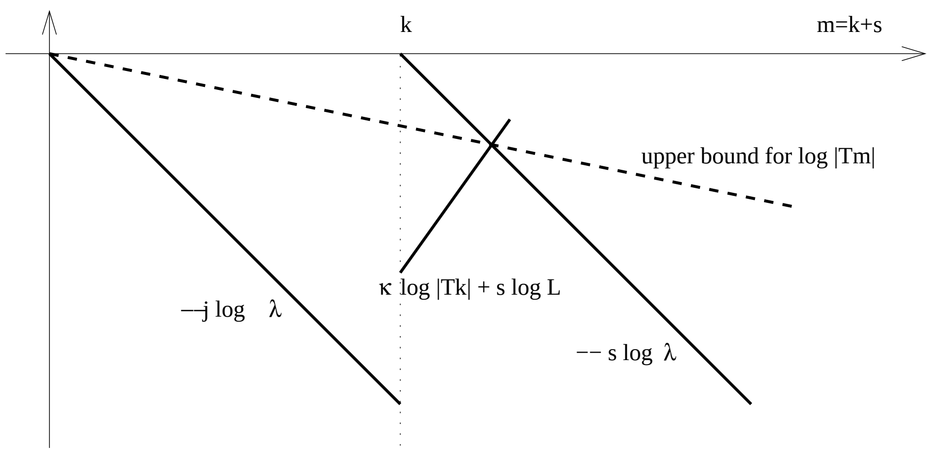

FIGURE 1. Upper bound for $\log|T_m|$

□

Corollary 3.4. *For $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$ satisfying (wi), if $\chi(\mu) = 0$ for μ an f invariant measure on K (K as in Theorem C), then there exists a sequence of repelling periodic points $p_n \in K$ such that $\chi(p_n) \to 0$.*

Proof. If p_n does not exist then by definition UHPR holds. Hence by Theorem C we have $\chi(\mu) > 0$, a contradiction. □

Remark 3.5. Notice that the equivalence of UHP and UHPR has a direct proof omitting the part of Theorem C that EC2*(z_0) implies ExpShrink. (The implication UHPR implies UHP is a special case of the above Corollary.)

Indeed. Suppose there exists a indifferent periodic point $p \in K$. Then it is repelling from one side. Take $z_0 \in K$ safe hyperbolic. It has a backward trajectory $x_j \in f^{-j}(x_0)$ converging to $O(p)$ (from the repelling side). Fix n and find a periodic point $q \in K$ most of whose trajectory "shadows" $x_j : j = 0, ..., n$, as in Lemma 4.4 (or in Proof of Theorem C, the part UHPR implies EC2(z_0)), with $\chi(q) > 0$ arbitrarily close to 0, that contradicts UHPR.

4. Equivalence of the definitions of Geometric Pressure

Let us formulate the main theorem of Katok's theory adapted to the multimodal case, similar to the complex case [PU, Theorem 11.6.1]. Compare also [MiSzlenk].

Theorem 4.1. *Consider $f : U \to \mathbb{R}$ for an open set $U \subset \mathbb{R}$, being a C^2 map with all critical points non-flat. Consider an arbitrary compact f-invariant $X \subset U$. Let μ be an f-invariant ergodic measure on X, with positive Lyapunov exponent. Let $\phi : U \to \mathbb{R}$ be an arbitrary continuous function. Then there exists a sequence X_k, $k = 1, 2, ...$, of compact f- invariant subsets of U, (topologically) Cantor sets or individual periodic orbits*[3]*, such that for every k the restriction $f|_{X_k}$ is an expanding repeller,*

$$\liminf_{k\to\infty} P(f|_{X_k}, \phi) \geq h_\mu(f) + \int \phi \, d\mu,$$

and if μ_k is any ergodic f-invariant measure on X_k, $k = 1, 2, ...$, then the sequence μ_k converges to μ in the weak-topology Moreover*

$$\chi_{\mu_k}(f|_{X_k}) = \int \log |f'| \, d\mu_k \to \int \log |f'| \, d\mu = \chi_\mu(f). \tag{4.1}$$

If X is weakly isolated, see Definition 1.20, then one finds $X_k \subset X$.

If the weak isolation is not assumed, but X is maximal, more precisely if $(f, X, \widehat{I}_X, \mathbf{U}) \in \mathscr{A}_+$ (see Notation 1.4) and if $h_\mu(f) > 0$, then one can find $X_k \subset X$. Moreover one can find $X_k \subset$ interior $\widehat{I}_X$.

Proof. It is the same as in [PU, Theorem 11.6.1] (except the last assertion where, unlike in [PU], we do not assume X is a repeller in its neighbourhood in $\mathbb{R}$, but merely that it is a maximal repeller in $\widehat{I}_K$). It relies on an analogon, adapted to C^2 maps of interval, of [PU, Theorem 11.2.3, Corollary 11.24], saying that for almost every backward trajectory $(x_n, n = 0, -1, ...)$ in the

[3] This case was overlooked in the assertion of [PU, Theorem 11.6.1] (but not in the proof).

Rokhlin's natural extension $(\widetilde{f}, \widetilde{\mu})$ of (f, μ) (the topological inverse limit is denoted by $\widetilde{f}$), there exists a ball (here: interval) $B(x_0, \delta)$ on which all backward branches $f^{-n}, n = 0, 1, ...$ mapping x_0 to x_{-n} exist and distortions are uniformly bounded.

Notice that (except in the last paragraph of the statement of the Theorem) we allow μ to be supported on an individual periodic orbit $O(p)$. Then the proof of the assertion of Theorem 4.1 is immediate: set $X_k := O(p)$. The proof in [PU] covers in fact this case. There is then only one branch of f^{-m}, for m being a period of p, constituting the iterated function system, giving a one-point limit set.

1. If we assume that X is weakly isolated, then all periodic orbits in X_k are contained in X for k large enough. Since they are dense in X_k (see e.g. [Bowen] or [PU, Theorem 4.3.12]), we get $X_k \subset X$.

2. Consider the case where we do not assume X is weakly isolated. So suppose that $(f, X, \widehat{I}_X, \mathbf{U}) \in \mathscr{A}_+$ and $h_\mu(f) > 0$ (in particular μ is infinite). To find $X_k \subset X$ we complement an argument in the proof in [PU, Theorem 11.6.1]. Recall that in [PU] one finds a disc $B = B(x, \beta)$ and a family of inverse branches f_ν^{-m} on B mapping it to $B(x, \frac{5}{6}\beta)$. Denoting the family of points $y_\nu = f_\nu^{-m}(x)$ by D_m one has there

$$\sum_{y \in D_m} S_m\phi(y) \geq \exp(m(h_\mu(f) + \int \phi \, d\mu - \theta)), \tag{4.2}$$

for an arbitrary $\theta > 0$ and $m \geq m_0$ large enough, which easily yields the rest of the proof.

Here we specify better x and β and restrict the family f_ν^{-m} to omit $\partial(\widehat{I}_X)$ by our pull-backs. To this end we repeat briefly the construction, using a method from [FLM].

We consider a set Y of $\widetilde{f}$-trajectories in the inverse limit $\widetilde{X}$ so that $\widetilde{\mu}(Y) > 1/2$, all pull-backs for f^n exist along the trajectory $(x_n) \in Y$ on $B(x_0, \delta)$ for a constant $\delta > 0$ and have uniformly bounded distortion, that is $|(f_\nu^{-n})'(z)/f_\nu^{-n})'(z')| \leq \text{Const}$ for all $z, z' \in B(x_0, \delta)$ for all branches f_ν^{-n} corresponding to the trajectories in Y. We assume also that

$$|S_m\phi(y) - m \int \phi \, d\mu| < m\theta$$

for $y = x_{-m}$ and every $m \geq m_0$, and moreover

$$|S_m \log \mathrm{Jac}_\mu(f)(y) - m h_\mu(f)| < m\theta; \tag{4.3}$$

note that $\int \log \mathrm{Jac}_\mu(f) \, d\mu = h_\mu(f)$ by Rokhlin's formula, see i.e. [PU, Theorem 2.9.7]. Jac is Jacobian in the weak sense, see [PU], compare also Appendix B.

By considering sufficiently many intervals of the form $B(x,\delta/2)$ for $x \in X$ in the support of μ, covering X with multiplicity at most 2 for δ appropriately small, we find x such that given an arbitrary integer $n_0 > 0$

$$\bigcup_{j=0}^{n_0} f^j(\partial(\widehat{I}_X)) \text{ is disjoint from closure of } B(x,\delta/2). \tag{4.4}$$

and $\widetilde{\mu}(Y_x) = C > 0$ for $Y_x := Y \cap \Pi^{-1}(B(x,\delta/2) \cap X)$ where Π denotes the standard projection $(x_k) \mapsto x_0$ from the inverse limit to X. Notice that by definition $x_0 \in B(x,\delta/2)$, hence $B(x_0,\delta)$ being the domain of $f^{-m}_{x_{-m}}$ contains $B(x,\delta/2)$, which is therefore a common domain for all $f^{-m}_{x_{-m}}$ for all $(x_n) \in Y_x$.

By (4.3) by $h_\mu(f) > 0$ for θ small enough for each $(x_n) \in Y_x$ we have

$$\mu(f^{-m}_{x_{-m}}(B(x,\delta/2)) \le \exp(-m(h_\mu(f) - \theta)) \tag{4.5}$$

for $m \ge m_0$, where the subscript x_{-m} means we consider the branch mapping x to x_{-m}. When we consider pull-backs of $B(x,\delta/2)$ along trajectories belonging to Y_x we remove each time the pull-backs whose closures intersect $\partial(\widehat{I}_X)$. Thus we remove each time pull-backs for f of all pull-backs already removed and additionally at most finite $\#\partial(\widehat{I}_X)$ number of pull-backs, so sets of measure $\widetilde{\mu}$ shrinking exponentially to 0 by (4.5), provided $\theta < h_\mu(f)$. If $n_0 + 1$, the time from which removing can start by (4.4), is large enough the measure of the removed part of Y_x is less than, say, $\widehat{\mu}(Y_x)/2$.

Denote the non-removed part by Y'_x. Hence $\widetilde{\mu}(Y'_x) > C/2 > 0$. By (4.5 the number of corresponding branches (pull-backs for f^m) is for each m at least $\mathrm{Const}\exp m(h_\mu(f) - \theta)$. Each branch (except a finite number not depending on m) can be continued by a finite time (also not depending on m) to yield a pull-back of $B(x,\delta/2) \subset B(x_0,\delta)$, to be contained in $B(x, \frac{1}{3}\delta)$ for some m' (greater from m by a constant).

We conclude with (4.2 (with different D_m, θ replaced by 2θ, and m' in place of m). By construction closures of all pull-backs of $B(x,\delta)$ for $f^{m'}$ considered above and their f^j-images for $j = 0,1,...,m'$ are disjoint from $\partial(\widehat{I}_X)$, so since they intersect X they are in $\widehat{I}_X$ and even in the interior of $\widehat{I}_X$.

Therefore the limit set X_k of the constructed Iterated Function System (IFS) is contained in $\widehat{I}_X$. By its forward invariance and the maximality of X in $\widehat{I}_X$, it is contained in X. In fact by construction $X_k \subset \text{interior}\, \widehat{I}_X$.

$\square$

Now we shall prove Theorem B, including Proposition 1.25, except the equalities for P_{Per}, in a sequence of lemmas.

Lemma 4.2. *For every $t \in (-infty, t_+)$ we have $P_{\mathrm{hyp}}(K,t) = P_{\mathrm{varhyp}}(K,t)$. For $t \ge t+$ the inequality $\le$ holds and if in addition (wi) is assumed, then the equality holds.*

Proof. The inequality $P_{\text{hyp}}(K,t) \leq P_{\text{varhyp}}(K,t)$ holds by the variational principle on each hyperbolic isolated subset of K, provided such subsets exist.

The opposite inequality follows from Theorem 4.1 applied to μ on $X = K$ with $\chi_\mu(f) > 0$, such that $h_\mu(f|_K) - t\chi_\mu(f)$ is almost equal to $P_{\text{varhyp}}(K,t)$, and to $\phi \equiv 0$. Then, for all k, for μ_k being the equilibrium on X_k for the potential 0 (i.e. measure with maximal entropy), we obtain

$$\liminf_{k\to\infty} h_{\mu_k}(f|_{X_k}) = \liminf_{k\to\infty} h_{\text{top}}(f|_{X_k}) \geq h_\mu(f|_K).$$

By Theorem 4.1 we have also $\chi_\mu(f) = \lim_{k\to\infty} \chi_{\mu_k}(f)$. Therefore one finds X_k with $P(f|_{X_k}, -t\log|f'|)$ at least $P_{\text{varhyp}}(K,t)$ up to an arbitrarily small positive number.

Notice finally that $X_k \subset K$. This also follows from Theorem 4.1 if $h_\mu(f|_K) > 0$. The only case it is not guaranteed by definition is where the function $\tau \to P_{\text{varhyp}}(K,\tau)$ is linear for $\tau \geq t$.

Then however $t_+ \leq t$. Indeed, in this linear case $P_{\text{varhyp}}(K,t) = -t\inf_{\mu\in\mathscr{M}^+(f,K)}\chi_\mu(f)$. Denote the first τ where $P_{\text{varhyp}}(K,\tau)$ is linear on $[\tau,\infty)$, by t'_+. For $\tau < t_+$ the function $P(\tau) = P_{\text{var}}(\tau)$ is non-linear, in particular strictly decreasing. Hence $P_{\text{varhyp}}(K,\tau) = P(\tau)$ there, so $P_{\text{varhyp}}(K,\tau)$ is non-linear. Hence $t_+ \leq t'_+$ and in the case we consider $t'_+ \leq t$ has been assumed. Hence $t_+ \leq t$. This ends the proof. □

Lemma 4.3. *For $(f,K) \in \mathscr{A}_+$ there exists a hyperbolic isolated f-invariant set $X \subset K \cap \operatorname{interior} \widehat{I}_K$ with $h_{\text{top}}(f|_X) > 0$.*

Proof. By Variational Principle [Walters] and $h_{\text{top}}(f|_K) > 0$ there exists an f-invariant probability measure μ on K with $h_\mu(f) > 0$. Hence, by Theorem 4.1 applied to $\phi \equiv 0$ (as in Proof of Lemma 4.2) there exists a hyperbolic isolated f-invariant set $X_k \subset K \cap \operatorname{interior} \widehat{I}_K$ with $h_{\text{top}}(f|_{X_k}) > 0$. □

Lemma 4.4. *For $(f,K) \in \mathscr{A}_+^{\text{BD}}$ there exists $z_0 \in K$ which is safe, safe forward and expanding. For each such z_0 we have $P_{\text{tree}}(K,z_0,t) \leq P_{\text{hyp}}(K(t))$ for all $t \in \mathbb{R}$. In fact for $t \leq 0$ this estimate holds for all $z_0 \in K$.*

On the other hand for each $(f,K) \in \mathscr{A}_+$ (there is no need to assume BD), $P_{\text{tree}}(K,z_0,t) \geq P_{\text{hyp}}(K(t))$ for all $z_0 \in K$ for $t \geq 0$, and for all $z_0 \in K$ for which there exists a backward not periodic trajectory in K omitting critical points (in particular for all z_0 safe) for $t < 0$.

Notice that if every backward trajectory of z_0 in K meets a critical point then for $t < 0$, $P_{\text{tree}}(K,z_0,t) = -\infty$.

Proof. Due to Lemma 4.3 there exists an f-invariant hyperbolic isolated set $X \subset K \cap \operatorname{interior} \widehat{I}_K$ with μ supported on X such that $h_\mu(f|_X) > 0$ (take just measure of maximal entropy for $f|_X$, existing due to hyperbolicity). By hyperbolicity of $f|_X$, $\chi_\mu(f|_X) > 0$. Hence Hausdorff dimension $HD(X) \geq h_\mu(f|_X)/\chi_\mu(f|_X) > 0$. Choosing δ arbitrarily small in Definition 1.22, we see that the set of points which are not safe has Hausdorff dimension equal to 0. Hence there exists a positive Hausdorff dimension set of safe points z_0, in $X \subset$ interior $\widehat{I}_K$ hence expanding and safe forward.

The proof that $P_{\rm tree}(K, z_0, t) \le P_{\rm hyp}(K(t))$ for $t > 0$ is the same as in the complex case, see e.g. [PU, Theorem 12.5.11] or [PR-LS2]. Briefly: we do not capture neither critical points nor end points of I in the pull-backs for f^j of $B = B(z_0, \exp(-\alpha n))$ for an arbitrarily small $\alpha > 0$, for the time (order of the pull-back) $j \le 2n$.

We "close the loop" in a finite bounded number of backward steps after backward time n, and the forward time n_1 of order not exceeding $\alpha n \log \lambda$, where $\lambda = \lambda_z$, see Definition 1.23.

More precisely: $f^{n_1}(B)$ is an interval of length of order Δ not depending on n. By Remark 2.6 there exists $m(\Delta)$ such that for every $z_1 \in f^{-n}(z_0) \cap K$ there exists $m \le m(\Delta)$ and $z_2 \in \frac{1}{2} f^{n_1}(B) \cap K$ such that $f^m(z_2) = z_1$. By Lemma 2.10 the closure of the pull-back of B for f^{n+m} for n large enough, containing z_2, is contained in $f^{n_1}(B)$. Using all z_1's we obtain an Iterated Function System for backward branches of f^{n+n_1+m}.

We construct an isolated (Cantor) hyperbolic set X as the limit set of the arising IFS, compare Proof of Theorem 4.1. (Notice however that here to know the limit set X is hyperbolic, we need to use bounded distortion, unlike in Theorem 4.1.) Since all pull-backs of B for $f^j, j = 0, 1, ..., n + m$ in the "loops" intersect K and are disjoint from $\partial \widehat{I}_K$ since z is "safe" for the time $2n \ge n + m$ and since $f^j(B) \subset \widehat{I}_K$ for $j = 0, 1, ..., n_1$ (for corrected Δ) by $\mathrm{dist}(f^j(z), \partial \widehat{I}_K)$ bounded away from 0, see Definition 1.23, and by backward Lyapunov stability, see Lemma 2.10 and Remark 2.11, we conclude that $X \subset \widehat{I}_K$. Hence by its invariance and maximality, $X \subset K$. Notice that we need not assume (wi).

The set X is $F := f^{n+n_1+m}$-invariant. To get an f-invariant set one considers $\bigcup_{j=0}^{n+n_1+m} f^j(X)$ and yet extends it, to conclude with an f-invariant isolated set.

To get the asserted inequality between pressures, ones uses for each z_1 as above, the estimate, see [PR-LS2, (2.1)]:

$$|(f^n)'(z_1)|^{-t} \le C|(f^{n+n_1+m})'(x)|^{-t} L^{t(n_1+m)},$$

where $L := \sup |f'|$ and x is an arbitrary point in the pull-back of B for f^{n+m}, containing z_2. To prove this inequality between derivatives we use bounded distortion assumption, the constant C results from it.

Notice however that in this estimate of derivatives we use $t \ge 0$, .

For $t \le 0$ the inequality $P_{\rm tree}(K, t) \le P_{\rm hyp}(K(t))$ needs a separate explanation. Two proofs (in the complex case) can be found in [PR-LS2]. An indirect one, in Theorem A.4, and a direct one, in Section A3. Notice that for $t \le 0$ the function $-t \log |f'|$ is continuous, though logarithm of it attains

$-\infty$ at critical points, and the standard definition of the topological pressure makes sense. The proofs in the interval case are the same. In the indirect proof the inequality [PR-LS2, (A.1)] $P_{\rm tree}(K, z_0, t) \le P(f|_K, -t \log |f'|)$ holds for all $z_0 \in K$. For the existence of large subtrees having well separated branches, used in the proof, see [P-Perron, Lemma 4]. One uses the property that f is C^1. The direct proof in [PR-LS2, Section A3] is a refinement of the proof for $t \ge 0$ (see above) and applicable only for z_0 safe and expanding.

The opposite inequality for each $z_0 \in K$ (with an exception mentioned in the statement of Lemma), follows from the fact that, due to density of preimages property of f on K, see Proposition 2.4, for an arbitrary isolated invariant hyperbolic set $X \subset K$, we can find a backward trajectory $z_0, z_{-1}, ..., z_m$ where $m \le 0$ $\;(f(z_k) = z_{k+1})$ such that $z_m \in K$ is arbitrarily close to X, say close to a point $z' \in X$

Then for each backward trajectory $z'_n, n = 0, -1, ...$ of z' in X there is exactly one backward trajectory $z_{m+n)}$ of z_m so that all $|z'_n - z_{m+n}|$ are small and decrease exponentially to 0. Hence, for $n \ge 0$

$$\sum_{f^n(x)=z',\, x\in X} |(f^n)'(x)|^{-t} \le \text{Const} \sum_{f^n(x)=z_m, x\in K} |(f^n)'(x)|^{-t}, \tag{4.6}$$

hence, $P_{\rm tree}(K, z_m, t) \ge P_{\rm tree}(X, z', t) = P(X, t)$. Finally $P_{\rm tree}(K, z_0, t) \ge \lim_{n\to\infty} \frac{-t \log |(f^{|m|})'(z_m)|}{||m|+n|} + P_{\rm tree}(K, z_m, t) \ge P_{\rm tree}(K, z_m, t)$ for $t \ge 0$ since then the first summand is larger or equal to 0 (0 or ∞) and for $t < 0$ since then the first summand is equal to 0 provided $(f^{|m|})'(z_m) \ne 0$.

□

Remark 4.5. We have proved above that for all $z_0 \in K$ safe and expanding $\limsup$ can be replaced by $\lim$ in the definition of $P_{\rm tree}(K, z_0, t)$, i.e. the limit exists, compare [PU, Remark 12.5.18]. Indeed (4.6) holds for all n hence in the estimates which follow we can consider $\liminf$ in $P_{\rm tree}$.

Lemma 4.6. *For each $(f, K) \in \mathscr{A}_+$, $P_{\rm varhyp}(K, t) = P_{\rm var}(K, t)$ for all $t < t_+$, and assuming (wi) for all $t \ge t_+$.*

Proof. $P_{\rm varhyp}(K, t) \le P_{\rm var}(K, t)$ for all t is obvious.

The opposite inequality is not trivial for $t \ge t_+$ and in the proof we shall apply Theorem C, proved in Section 3.

Suppose there exist $\mu \in \mathscr{M}(f, K)$ with $\chi_\mu(f) = 0$ for which $h_\mu(f) - t\chi_\mu(f)$ are arbitrarily close to $P_{\rm var}(K, t)$. This implies, due to $h_\mu(f) \le 2\chi_\mu(f) = 0$ (Ruelle's inequality), that $P_{\rm var}(K, t) = 0$ and that $t \ge t_+$. Then, by $\chi_\mu(f) = 0$, f is not TCE on K, hence not UHPR, see Theorem C, in particular Corollary 3.4. So there exist repelling periodic points $p \in K$ with $\chi(p)$ arbitrarily close to 0. Thus $P_{\rm varhyp}(K, t)$ is arbitrarily close to 0, hence equal to 0, as well as $P_{\rm var}(K, t)$.

□

We end this Section with commenting on various definitions of $\chi_{\inf}(f,K)$ and $\chi_{\sup}(f,K)$ and therefore the *condensation* and *freezing* phase transition points t_- and t_+, see Introduction.

Proposition 4.7. *(compare* [PR-L2, Proposition 2.3]*, see also* [R-L]*)*
For each $(f,K) \in \mathscr{A}_+^{\mathrm{BD}}$ satisfying (wi), the following holds

1. *Given a repelling periodic point p of f, let m be its period and put $\chi(p) := \frac{1}{m}\log|((f^m)'(p)|$. Then we have*

$$\chi_{\inf \mathrm{Per}} := \inf\{\chi(p) : p \in K \textit{ is a repelling periodic point of } f\} = \chi_{\inf},$$

$$\chi_{\sup \mathrm{Per}} := \sup\{\chi(p) : p \in K \textit{ is a repelling periodic point of } f\} = \chi_{\sup}.$$

2. $$\lim_{n\to+\infty} \frac{1}{n}\log\sup\{|(f^n)'(x)| : x \in K\} = \chi_{\sup}.$$

3. *For each $z \in K$ safe and hyperbolic, we have*

$$\chi_{\inf \mathit{Back}}(z) := \lim_{n\to+\infty} \frac{1}{n}\log\min\{|(f^n)'(w)| : w \in f^{-n}(z)\} = \chi_{\inf}, \tag{4.7}$$

$$\chi_{\sup \mathit{Back}}(z) := \lim_{n\to+\infty} \frac{1}{n}\log\max\{|(f^n)'(w)| : w \in f^{-n}(z)\} = \chi_{\sup}. \tag{4.8}$$

Proof. **1.** The inequalities $\chi_{\inf \mathrm{Per}} \geq \chi_{\inf}$ and $\chi_{\sup \mathrm{Per}} \leq \chi_{\sup}$ follows immediately from the use of the measures equidistributed on periodic orbits involved in $\chi_{\inf \mathrm{Per}}$ and $\chi_{\sup \mathrm{Per}}$.

The inequality $\chi_{\sup \mathrm{Per}} \geq \chi_{\sup}$ follows from Katok's construction of periodic orbits, see Proof of Lemma 4.2 and Lemma 4.1. These periodic orbits are in K by (wi).

It is more difficult to prove $\chi_{\inf \mathrm{Per}} \leq \chi_{\inf}$ in case we do not know *a priori* that the latter number is the limit of a sequence χ_{μ_n} for μ_n hyperbolic measures on K, i.e. if there exists a probability invariant measure on K with $\chi_\mu = 0$. Then the proof of the inequality immediately follows from Theorem C, which in particular says that if such μ exists then UHPR fails, that is there exists a sequence of repelling periodic points $p_n \in K$ such that $\chi(p_n) \to 0$, Corollary 3.4. Compare also Proof of Lemma 4.6. The assumed property (wi) is used as well.

2. The inequality $\geq$ follows from Birkhoff Ergodic Theorem. The opposite inequality can be proved via constructing μ as a weak* limit of the measures $\mu_n := \frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(z_n)}$ for z_n involved in the supremum.

3a. The inequality $\chi_{\sup \mathrm{Back}}(z) \leq \chi_{\sup}$ follows immediately from the similar inequality in the previous item.

3b. The inequality $\chi_{\sup \mathrm{Back}}(z) \geq \chi_{\sup \mathrm{Per}}$ (or $\chi_{\sup}$) can be proved by choosing a backward trajectory of z converging to a repelling periodic trajectory $O(p)$ in K with $\chi(p)$ arbitrarily close to $\chi_{\sup \mathrm{Per}}$.

3c. The inequality $\chi_{\inf \mathrm{Back}}(z) \geq \chi_{\inf \mathrm{Per}}$ can be proved by finding a periodic trajectory shadowing an arbitrary piece of backward trajectory of z, compare Proof of Lemma 4.4 Since we have assumed z is safe, the property (wi) is not needed in the proof.

3d. Finally $\chi_{\inf \mathrm{Back}}(z) \leq \chi_{\inf \mathrm{Per}}$ can be proved similarly to 3b.

□

Notice that 3d. in the version $\chi_{\inf \mathrm{Back}}(z) \leq \chi_{\inf}$ our proof is harder because it uses $\chi_{\inf \mathrm{Per}} \leq \chi_{\inf}$ in the item 1.

5. Pressure on periodic orbits

This Section complements Section 4. It is not needed for the proof of Theorem A.

We start with the interval version of a technical fact allowing to estimate the number of "bad" backward trajectories, used in the complex case (in various variants) in several papers on geometric pressure, e.g. [PR-LS1], [PR-LS2], [PR-L1], [PR-L2], [GPR]. (The essence of this fact is just calculating vertices of a graph.)

Definition 5.1. Let $(f, K) \in \mathscr{A}$. Fix n and arbitrary $x_0 \in K$ and $R > 0$. For every backward trajectory of x_0 in K, namely a sequence of points $(x_i \in K, i = 0, 1, ..., n)$ such that $f(x_i) = x_{i-1}$ run the following procedure. Take the smallest $k = k_1 \geq 0$ such that $\mathrm{Comp}_{x_{k_1}} f^{-k_1} B(x_0, R)$ contains a critical point. Next let k_2 be the smallest $k > k_1$ such that $\mathrm{Comp}_{x_{k_2}} f^{-(k_2-k_1)} B(x_{k_1}, R)$ contains a critical point. Etc. until $k = n$. Let the largest $k_j \leq n$ for the sequence (x_i) be denoted by $k((x_i))$ and let the set $\{y : y = x_{k((x_i))}$ for a backward trajectory $(x_i)\}$ be denoted by $N(x_0) = N(x_0, n, R)$.

Lemma 5.2. *(compare e.g.* [PR-LS2, lemma 3.7]*)*

For every $(f, K) \in \mathscr{A}$, for every $\varepsilon > 0$ and for all $R > 0$ small enough and n sufficiently large, for every $x \in K$ it holds $\#N(x, n, R) \leq \exp(\varepsilon n)$.

Proof of Theorem B for P_{Per}. The inequality $P(K, t) \leq P_{\mathrm{Per}}(K, t)$ follows immediately from $P(K, t) = P_{\mathrm{hyp}}(K, t)$. Indeed, for every isolated hyperbolic $X \subset K$ we have Bowen's formula $P(X, t) = P_{\mathrm{Per}}(X, t)$.

For the opposite inequality we can adapt [BMS, proof of Theorem C] for complex polynomials with connected Julia set, or [PR-LS2] for general rational functions. In the latter, a condition H was assumed, saying that for every $\delta > 0$ and n large enough, for any set $P = P_n$ of periodic points of period n such that for all $p, q \in P$ and all $i : 0 \leq i \leq n$ $\mathrm{dist}(f^i(p), f^i(q)) < \exp(-\delta n)$, we have $\#P < \exp(\delta n)$. In [BMS, Lemma 2] this assumption, even a stronger one, has been proved provided there are no indifferent periodic points, i.e. such that $|(f^n)'(p)| = 1$, where n is a period of p.

In our interval C^2 multimodal case the [BMS] condition also holds. Namely

(H*) For every $(f, K) \in \mathscr{A}$, for every $\epsilon > 0$, there is $\rho > 0$ such that for n large enough, for any $p, q \in P \subset K$ as above, satisfying $|f^i(p) - f^i(q)| \leq \rho$, we have $\#(P) < \exp(\epsilon n)$.

Indeed, having $P = P_n$ not satisfying (H*), we can shrink ϵ and assume there is no critical point for f^n in T_n being the convex hull of P_n, and $|T_n| \to 0$ as $n \to \infty$.

Indeed, consider $f^i(T_n^0)$ where T_n^0 is the convex hull of P_n, for $i = 0, ...$ as long as $f^{i_0}(T_n^0)$ contains a critical point c_0 for f (there is only one such critical point, provided ρ is small enough). f^{i_0} is strictly monotone on T_n^0. Then the point $(f|_{T_n^0})^{-i_0}(c_0)$ divides T_n^0 into two subintervals and we define $T_n^{i_0}$ as the convex hull of the points belonging to P in this one which contains the larger number of points belonging to P. Next we consider $i_1 > i_0$ the first $i > i_0$ such that $f^{i_1}(T_n^1)$ contains a critical point c_1. After a consecutive division we continue. We stop with $T_n := T_n^{i_s}$ such that f^{n-i_s} does not contain critical points in $T_n^{i_s}$. Notice that all $i_{t+1} - i_t$ are larger than an arbitrarily large constant for ρ small enough.

Thus there are n arbitrarily large such that there are in T_n an abundance (the number exponentially tending to ∞ as $n \to \infty$) of fixed points for f^n, in particular an abundance of attracting or indifferent periodic orbits for f, all in $\widehat{I}_K$ provided δ is smaller than the gaps between the components of $\widehat{I}_K$, hence in K. This contradicts lack of attracting periodic orbits and the finiteness of the set of indifferent periodic orbits in K, see Remark 1.6.

As in [PR-LS2] and [BMS] we split P_{Per} in two parts.

Definition 5.3. Fix small $r > 0$. We say that a periodic orbit $O \subset K$ of period $n \geq 1$ is *regular* (more precisely: regular with respect to r) if it is not indifferent (hence by BD it is hyperbolic repelling, see Remark 1.11), and there exists $p \in O$ such that f^n is injective on $\mathrm{Comp}_p\, f^{-n}(B(p, r))$. If O is not regular, then we say that O is *singular*.

We denote by Per_n the set of all periodic points of period n in K and denote by Per_n^r the set of all points in $\mathrm{Per}_n \setminus \mathrm{Indiff}(f)$, whose periodic orbits are regular. Denote by Per_n^s the set of all other periodic points in K.

We shall first prove that

$$P_{\mathrm{Per}}^r(K, t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{p \in \mathrm{Per}_n^r} |(f^n)'(p)|^{-t} \leq P_{\mathrm{tree}}(K, t).$$

We choose a finite $r/12$-dense set $A \subset K$ of points, safe (for an arbitrary δ) and hyperbolic with common constants.

Let $O \subset \mathrm{Per}_n^r$ be a regular periodic orbit and let $p_0 \in O$ be such that f^n is injective on $\mathrm{Comp}_{p_0} f^{-n}(B(p_0,r))$. Let $x \in A \cap B(p_0, r/12)$. Consider the (unique) backward orbit $(x_j, j = 0, ..., n)$ such that

$$x_j \in \mathrm{Comp}_{f^{n-j}(p_0)} f^{-j}(B(p_0, r/12)). \tag{5.1}$$

By BD, if r is small enough, then

$$C^{-1}|(f^n)'(x_n)| \le |(f^n)'(p_0)| \le C|(f^n)'(x_n)|$$

for $C := C(11/12)$.

Notice also that

$$\mathrm{Comp}_{x_n} f^{-n}B(x_0, r/2) \subset \mathrm{Comp}_{p_0} f^{-n}B(p_0, 2r/3) \subset B(p_0, r/4) \subset B(x_0, r/3),$$

the middle inclusion by Lemma 2.10 for n large enough. To apply this lemma, if p_0 is too close to $\mathrm{Indiff}(f)$, we replace p_0 by its image under an iterate of f which is far from $\mathrm{Indiff}(f)$. This is possible since $\mathrm{Indiff}(f)$ is finite. If p_0 is close to a indifferent periodic point p', it must be on the repelling side of it, so its forward f-trajectory escapes from a neighbourhood of $O(p')$.

So there is exactly one $q \in \mathrm{Comp}_{x_n} f^{-n}B(x_0, r/2)$ such that $f^n(q) = q$. Hence each $(x_i, i = 0, ..., n)$ above is associated to at most one point in $P^r_{\mathrm{Per}}(K, t)$.

We conclude with

$$\sum_{p \in \mathrm{Per}_n^r} |(f^n)'(p)|^{-t} \le \max\{C^t, C^{-t}\} \sum_{x_n \in f^{-n}(x_0), x_0 \in A} |(f^n)'(z_n)|^{-t},$$

hence

$$\sum_{p \in \mathrm{Per}_n^r} |(f^n)'(p)|^{-t} \le \max\{C^t, C^{-t}\} \# A \sup_{x_0 \in A} \sum_{x_n \in f^{-n}(x_0)} |(f^n)'(x_n)|^{-t}.$$

hence

$$P^r_{\mathrm{Per}}(K, t) \le P_{\mathrm{tree}}(K, t).$$

The next step is to find an upper bound, depending on n, for the number of singular periodic orbits, not indifferent. We follow [PR-LS2], but there is no need to replace p by another point in $O(p)$ except to reach p not too close to $\mathrm{Indiff}(f)$. Given $\rho > 0$, for r small enough all components of $f^{-n}(B(p,r))$ have diameters smaller than ρ by Lemma 2.10. For x_j chosen as above and $p_j = f^{n-j}(p), j = 0, ..., n$ we get $|x_j - p_j| < \rho/2$, hence by (H*) the the number of singular orbits to which the same $(x_0, ..., x_n)$ is assigned is bounded by $\exp(\epsilon n)$.

Now assume $R >> \rho, r$ and apply Lemma 5.2 and preceding notation. We attempt to bound the number S_n of backward trajectories $(x_0, ..., x_n)$ associated to some points in Per_n^s.

Here given singular $O(p)$ of period n we associate with it a backward trajectory $(x_0, ..., x_n)$ as in the regular case, by taking $x_0 \in A$ and x_j satisfying (5.1). Here however even chosen x_0 we do not have uniqueness of x_j.

Denote by X_{k,x_k} the set of all backward trajectories $(x_i, i = 0, 1, ..., n)$ with the same fixed $k = k((x_i)_{i=0,...,n})$ and x_k, associated with singular periodic orbits i.e. belonging to Per^s_n. Repeating the proof in [PR-LS2, Step 4.1] we conclude that $\# X_{k,x_k} \le k \# \mathrm{Crit}(f)$.

(The proof in [BMS] is more elegant at this place. The authors proceed in Definition 5.1 till $k = 2n$, thus incorporating our consideration to bound $\# X_{k,x_k}$ in the bound of $N(x_0, 2n, R)$.)

Thus,

$$S_n \le \#(A) \# \{N(x_0, n, R) : x_0 \in A\} \sup_{k,x_k} \# X_{k,x_k} \le \# A \exp(\epsilon n) n \# \mathrm{Crit}(f).$$

and in consequence

$$\# \mathrm{Per}^s_n \le \exp(\epsilon n) \# A \exp(\epsilon n) n \# \mathrm{Crit}(f) \le \exp(3\epsilon n)$$

for n large enough (this includes also all indifferent periodic orbits in K).

Since for each periodic $p \in K$ and $t \in R$ we have for the probability invariant measure μ_p on $O(p)$,

$$-t\chi(p) = h_{\mu_p}(f) - t\chi_{\mu_p} \le P_{\mathrm{var}}(K, t),$$

we obtain

$$P^s_{\mathrm{Per}}(K, t) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{p\in \mathrm{Per}^s_n} |(f^n)'(p)|^{-t} = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{p\in \mathrm{Per}^s_n} \exp(-tn\chi(p))$$

$$\le \lim_{n\to\infty} \frac{1}{n} \log(\exp(3\epsilon n) \exp(n P_{\mathrm{var}}(K, t)) \le 3\epsilon + P_{\mathrm{var}}(K, t).$$

Finally we obtain

$$P_{\mathrm{Per}}(K, t) \le \max\{P^r_{\mathrm{Per}}(K, t), P^s_{\mathrm{Per}}(K, t)\} \le P(K, t) + 3\epsilon,$$

hence considering $\epsilon > 0$ arbitrarily small and respective r we obtain $P_{\mathrm{Per}}(K, t) \le P(K, t)$. $\square$

6. Nice inducing schemes

6.1. **Nice sets and couples.**

Firstly we adapt to the generalized multimodal interval case the definitions, and review some properties, of nice sets and couples. We follow the complex setting, see e.g. [PR-L2, Chapter 3]. For the interval case for the notion of nice sets see e.g. [BRSS]; in fact they have been introduced and used to consider return maps to them much earlier (see papers by Marco Martens).

Definition 6.1. Let $(f, K, \widehat{I}_K, \mathbf{U}) \in \mathscr{A}$. We call a neighborhood V of the *restricted singular set* $S'(f,K) = \mathrm{Crit}(f) \cup \mathrm{NO}(f,K)$ (see Definition 1.17) in $\mathbf{U}$ a *nice set for* f, if for every $n \geq 1$ we have

$$f^n(\partial V) \cap V = \emptyset, \tag{6.1}$$

and if each connected component of V is an open interval containing precisely one point of $S'(f,K)$. The component of V containing $c \in S'(f,K)$ will be denoted by V^c.

So, let $V = \bigcup_{c \in S'(f,K)} V^c$ be a nice set for f. According to previously used names, Definition 2.8, we call any component W of $f^{-n}(V)$ for $n \geq 0$ intersecting K, a pull-back, or pull-back for f^n, of V. We denote this n by m_W.

Let us recall that all the pull-backs are small for n large, by Lemma 2.10, In particular as by definition they intersect K they are far from $\partial \mathbf{U}$ if all V^c are small enough, see a remark at the end of Definition 2.8.

We have either

$$W \cap V = \emptyset \text{ or } W \subset V.$$

Furthermore, if W and W' are distinct pull-backs of V, then we have either,

$$W \cap W' = \emptyset,\ W \subset W' \text{ or } W' \subset W.$$

For a pull-back W of V we denote by $c(W)$ the point in $S'(f,K) \cap V^c$, where W is an f^{-m_W} pull-back of V^c.

Moreover we put,

$$\mathscr{K}(V) = \{z \in K : \text{ for every } n \geq 0 \text{ we have } f^n(z) \notin V\}.$$

Lemma 6.2. *For all $(f,K) \in \mathscr{A}$ with all periodic orbits in K hyperbolic repelling, the map $f|_K$ is expanding away from singular points, that is expanding on $\mathscr{K}(V)$ for every nice V defined above.*

More generally for every open (in $\mathbb{R}$) set V containing $\mathrm{Crit}(f)$ there exists $\lambda > 1$ and $N = N(V) > 0$ such that if $x, f(x), ..., f^N(x)$ belong to $\mathbf{U} \setminus V$ then $|(f^N)'(x)| > \lambda$ (maybe for a diminished $(\mathbf{U})$)).

Proof. This follows from Mañé's Hyperbolicity Lemma, see e.g. [dMvS, Ch.III, Lemma 5.1]. Indeed we can extend $f|_{\widehat{I}_K}$ to $I \supset \widehat{I}_K$, C^2, as in Lemma A.4, see Remark 2.14, thus getting a C^2 map with all periodic orbits hyperbolic repelling.

Then for $x \in K$ the assertion of our Lemma 6.2 is precisely the assertion of Mañé's Lemma.

For $x \notin K$ close to K we find $y \in K$ close to x, with $f^j(y) \in K \setminus V'$ for V' a neighbourhood of $\mathrm{Crit}(f)$ smaller than V, $j = 0, 1, ..., N(V')$, so that $f^j(x)$ is sufficiently close to $f^j(y)$ to yield $|(f^N)'(x)| > 1 = \frac{\lambda+1}{2} > 1$. This finishes the proof.

(Notice that if we assume additionally bounded distortion Lemma 6.2 follows easily from Lemma 2.10.) $\square$

Lemma 6.3. *For all $(f, K, \mathbf{U}) \in \mathscr{A}_+$ with no indifferent periodic orbits in K, there exist nice sets with all components of arbitrarily small diameters.*

Proof. Let $p \in K$ be an arbitrary periodic point, not in the forward orbit of a point in $S'(f, K)$. It exists by the existence of infinitely many periodic orbits, due to $(f, K) \in \mathscr{A}_+$. Then we use the density of $Q := \bigcup_{n=0}^{\infty} Q_n$ for $Q_n := \sum_{j=0}^{n} f^{-j}(p)$ in K.Notice that this density holds for $Q \cap K$ (see Proposition 2.4, but in the definitions of Q_n and Q we do not restrict f to K.

Fix an arbitrary $c \in S'(f, K)$. If c is an accumulation point of Q from the left hand side side, then for all n large enough Q_n contains points on this side of c. Let $a_{c,n} \in Q_n$ be the point closest to c in Q_n. Similarly, if c is an accumulation point of Q from the right hand side, we define $a'_{c,n} \in Q_n$ to be the point closest to c from the right hand side. If c is not an accumulation point of Q from one side (it is then an accumulation point of Q from the other side since otherwise c is isolated point of K), then there is an interval T adjacent to c from this side with $Q \cap T = \emptyset$. Then $f^k(T) \cap K = \emptyset$ for all positive integer k since otherwise for the first k such that $f^k(T) \cap K \neq \emptyset$ the set $f^k(T)$ is open (since it cannot capture earlier a critical point as $\mathrm{Crit}(f) \subset K$), hence it intersects Q, hence T intersects Q, a contradiction.

(Notice that the case c is not an accumulation point of Q from one side is possible only if $c \in \mathrm{Crit}^I(f) \setminus \mathrm{NO}(f, K)$, in particular c is a critical inflection point. Indeed if c is a turning point then both a_c and a'_c can be defined as f-preimages of the same point in Q_{n-1} closest to $f(c)$ (there points arbitrarily close to $f(c)$ even in K since the "fold" is on the side of K; otherwise $f(c)$ would be isolated in K. Compare the end of Proof of Theorem C in Section 3).)

Thus we can apply Lemma 2.9, and conclude that c is eventually periodic in a repelling periodic orbit $O(z)$, i.e. there exists $k > 0$ such that $f^k(c) \in O(z)$ periodic. Then we define the missing $a_c \in T$ or $a'_c \in T$ as any point in T arbitrarily close to c .

We define $V^c := (a_c, a'_c)$ for all $c \in S'(f, K)$, arbitrarily small for n appropriately large. The property (6.1) follows easily from the definitions. □

Notice that V^c correspond to critical Yoccoz puzzle pieces of n-th generation containing critical points in the complex polynomial setting. The above proof yields the puzzle structure in the generalized multimodal real setting:

Proposition 6.4. *If $(f, K, \mathbf{U}) \in \mathscr{A}_+$ with no indifferent periodic orbits in K, then there exists a partition of a neighbourhood of K in $\mathbb{R}$ into intervals W^k such that $\sup_k \operatorname{diam} W^k$ is arbitrarily small and for all $n > 0$ and k_1, k_2 we have $f^n(\partial W^{k_1}) \cap \operatorname{interior} W^{k_2} = \emptyset$.*

Proof. Consider as above the sets $Q_n := \sum_{j=0}^{n} f^{-j}(p)$ and additionally $\hat{Q}_n := \sum_{j=0}^{n} (f|_{\mathbf{U}})^{-j}(\hat{Q})$, where $\hat{Q}$ consists of a finite family of points, each one being the end point of the open interval adjacent to a respective repelling periodic orbit in K on the side disjoint from K, in the boundary of $\mathbf{U}$, compare the last paragraph in Proof of Lemma 2.10. Now, for each n large enough, we partition a neighbourhood of K into intervals with ends in $Q_n \cup \hat{Q}_n$ $\square$

One can think about Q_n as corresponding to external rays (including their end points) and $\hat{Q}_n$ as corresponding to equipotential lines in Yoccoz' puzzle partition of a neighbourhood of Julia set for a complex polynomial.

Definition 6.5. A *nice couple for* f is a pair $(\widehat{V}, V)$ of nice sets for f such that $\overline{V} \subset \widehat{V}$, and such that for every $n \geq 1$ we have $f^n(\partial V) \cap \widehat{V} = \emptyset$.

If $(\widehat{V}, V)$ is a nice couple for f, then for every pull-back $\widehat{W}$ of $\widehat{V}$ we have either

$$\widehat{W} \cap V = \emptyset \text{ or } \widehat{W} \subset V.$$

Lemma 6.6. *For every $(f, K, \widehat{I}_K) \in \mathscr{A}_+^3$ whose all periodic orbits in K are hyperbolic repelling, for f appropriately modified outside $\widehat{I}_K$ there exist nice couples with all components of arbitrarily small diameters.*

Proof. For the proof see [CaiLi, Lemma 3 and Proposition 5]. To apply it we extend $f|_{\widehat{I}_K}$ in C^3 to $I \supset \widehat{I}_K$ as in Lemma A.4 in Appendix A. Then f is not infinitely renormalizable at critical points in K by weak exactness, so the assumptions of [CaiLi] are satisfied. $\square$

Finally notice that bounded distortion property, Definition 1.10, implies the following

Remark 6.7. If ϵ is such that for a nice couple $(\widehat{V}, V)$ for each $c \in S'(f, K)$ the interval $\widehat{V}^c$ is an ϵ-scaled neighbourhood of V^c, then there is ϵ' such that for each pull-back $(\widehat{W}, W)$ of $(\widehat{V}, V)$ for f^n, such that W intersects K and f^n is a diffeomorphism on $\widehat{W}$, the interval $\widehat{W}$ is an ϵ'-scaled neighbourhood of an interval containing W.

6.2. **Canonical induced map.** The following key definition is an adaptation to the generalized multimodal case of the definition from the complex setting.

Definition 6.8. (Canonical induced map).

Let $(\widehat{V}, V)$ be a nice couple for $(f, K) \in \mathscr{A}$. We say that an integer $m \geq 1$ is a *good time for a point* z in $\mathbf{U}$, if $f^m(z) \in V$, in particular the iteration make sense, and if f^m is a K-diffeomorphism from the pull-back of $\widehat{V}$ for f^m, containing z, to $\widehat{V}$, see Definition 1.13. Let D be the set of all those points in V having a good time and for $z \in D$ denote by $m(z) \geq 1$ the least good time of z. Then the map $F : D \to V$ defined by $F(z) := f^{m(z)}(z)$ is called *the canonical induced map associated to* $(\widehat{V}, V)$. We denote by $K(F) \subset D$ the set where all iterates of F are defined (i.e. the maximal F-invariant set) and by $\mathfrak{D}$ the collection of all the connected components of D. As V is a nice set, it follows that each connected component W of D is a pull-back of V. Moreover, f^{m_W} maps $\widehat{W}$ K-diffeomorphically onto $\widehat{V}^{c(W)}$ and for each $z \in W$ we have $m(z) = m_W$.

Remark 6.9. Notice that for $z \notin V$ if there exists $m \geq 1$ such that $f^m(z) \in V$ then the number $m(z)$ is the time of the first hit of V by the forward trajectory of z by the definition of the nice couple, For $z \in V$ the number $m(z)$ need not be the time of the first return to V.

Later on we shall need the following (compare [PR-L2, Section 3.2].

Lemma 6.10. *If m is a good time for $z \in V$, such that $f^m(z) \in V$, then there exists unique $k \geq 1$ such that $f^m = F^k$ on a neighbourhood of z.*

Proof. If $m(z) = m$ then $f^m = F$ by definition. If $m(z) < m$ then $m - m(z)$ is a good time for $z_1 = f^{m_1}(z)$ for $m_1 = m(z)$ and we consider $m_2 = m(z_1)$. After k steps we get least good times $m_1, ..., m_k$ for $z_1, ..., z_k$ such that $m_1 + ... + m_k = m$. □

Definition 6.11. (Bad pull-backs). Let, as above, $(\widehat{V}, V)$ be a nice couple for $(f, K) \in \mathscr{A}$. A point $y \in f^{-n}(V) \cap V$ is a *bad iterated pre-image of order n* if for every $j \in \{1, \dots, n\}$ such that $f^j(y) \in V$ the map f^j is not a K-diffeomorphism on the pull-back of $\widehat{V}$ for f^j, containing y. In this case every point y' in the pull-back X of V for f^n containing y is a bad iterated pre-image of order n. So it is justified to call X itself a bad iterated pre-image of V of order n. We call a pull-back Y of $\widehat{V}$ for f^n, intersecting K, a *bad pull-back of $\widehat{V}$ of order n*, if it contains a bad iterated pre-image of order n. (Notice that Y can contain several X's.)

Denote by $\mathfrak{L}_V$ the collection of all the pull-backs W of V for f^{-n}, for all $n \geq 1$, intersecting K, such that $f^j(W) \cap V = \emptyset$ for all $j = 0, ..., n-1$.

Denote by $\mathfrak{D}_Y$ the sub-collection of $\mathfrak{D}$ of all those W that are contained in Y. We do not assume that Y is bad in this notation. In particular we can consider $Y = \widehat{V}^c$ with $m_Y = 0$.

So f^{m_W} maps $\widehat{W}$ K-diffeomorphically onto $\widehat{V}^{c(W)}$, $f^{m_Y}(W) \subset V^{c(Y)}$ and $f^{m_Y}(W) \in \mathfrak{D}_{\widehat{V}^{c(Y)}}$. Notice that $f(\mathfrak{D}_{\widehat{V}^{c(Y)}}) \subset \mathfrak{L}_V$ for V small enough (such that $f(V) \cap V = \emptyset$).

From these definitions it easily follows that

Lemma 6.12. *(see* [PR-L1, Lemma 7.4] *and* [PR-L2, Lemma 3.5]*) Let $(\widehat{V}, V)$ be a nice couple for $(f, K) \in \mathscr{A}$ and let $\mathfrak{D}$ be the collection of the connected components of D. Then*

$$\mathfrak{D} = \left(\bigcup_{c \in S'(f,K)} \mathfrak{D}_{\widehat{V}^c} \right) \cup \left(\bigcup_{Y \text{ bad pull-back of } \widehat{V}} \mathfrak{D}_Y \right).$$

Lemma 5.2 yields the following (see [PR-L2, Lemma 3.6] for a more precise version)

Lemma 6.13. *Let $(\widehat{V}, V)$ be a nice couple for $(f, K) \in \mathscr{A}_+$. Then for every $\epsilon > 0$, if the diameters of all $\widehat{V}^c$ are small enough, for all n large enough and for every $x \in V$ the number of bad iterated pre-images of x of order n is at most* $\exp(\varepsilon n)$.

6.3. **Pressure function of the canonical induced map.** Now we shall continue adapting the procedure from the complex case

Let as before $(\widehat{V}, V)$ be a nice couple for $(f, K) \in \mathscr{A}_+$ and let $F : D \to V$ be the canonical induced map associated to $(\widehat{V}, V)$ and $\mathfrak{D}$ the collection of its connected components of D. For each $c \in S'(f, K)$ denote by $\mathfrak{D}^c$ the collection of all elements of $\mathfrak{D}$ contained in V^c, so that $\mathfrak{D} = \bigsqcup_{c \in S'(f,K)} \mathfrak{D}^c$. A word on the alphabet $\mathfrak{D}$ will be called *admissible* if for every pair of consecutive letters $W', W \in \mathfrak{D}$ we have $W \in \mathfrak{D}^{c(W')}$ (that is $W \subset f^{m(W')}(W') = V^{c(W')}$).

We call the 0-1 infinite matrix $A = A_F$ having the entry $a_{W,W'}$ equal to 1 or 0 depending as $W \in \mathfrak{D}^{c(W')}$ or not, the incidence matrix. This matrix in the terminology of [MU, page 3] yields a *graph directed system*, a special case of GDMS.

For a given integer $n \geq 1$ we denote by E^n the collection of all admissible words of length n. Given $W \in \mathfrak{D}$, denote by ϕ_W the the inverse of $F|_W$, on $V^{c(W)}$, and by $\phi_{\widehat{W}}$ its extension to $\widehat{V}^{c(W)}$. The collection ϕ_W is a Graph Directed Markov System, GDMS, see [MU].

For a finite word $\underline{W} = W_1 \dots W_n \in E^n$ put $c(\underline{W}) := c(W_n)$ and $m_{\underline{W}} = m_{W_1} + \dots + m_{W_n}$. Note that the composition

$$\phi_{\underline{W}} := \phi_{W_1} \circ \dots \circ \phi_{W_n}$$

is well defined, extends to $\widehat{V}^{c(\underline{W})}$ and maps it K-diffeomorphically onto $V^{c(\underline{W})}$.

For each $t, p \in \mathbb{R}$ define the *pressure of F for the potential* $\Phi_{t,p} := -t \log |F'| - pm$

$$P(F, -t \log |F'| - pm) := \lim_{n \to +\infty} \frac{1}{n} \log Z_n(t, p) \tag{6.2}$$

where, as before, $m = m(z)$ associates to each point $z \in D$ the least good time of z and for each $n \geq 1$

$$Z_n(t, p) := \sum_{\underline{W} \in E^n} \exp(-m_{\underline{W}} p) \left(\sup \left\{ |\phi'_{\underline{W}}(z)| : z \in V^{c(\underline{W})} \right\} \right)^t.$$

Recall, see e.g. [MU, Section 2.2], that a function $\psi : E^\infty \to \mathbb{C}$ is called Hölder continuous on the associated symbolic space E^∞ of all infinite admissible words, if the sequence of variations

$V_n(\psi) := \sup\{|\psi(\omega_1) - \psi(\omega_2)| : \text{the first } n \text{ letters of } \omega_1 \text{ and } \omega_2 \text{ coincide}\}$ tends to 0 exponentially fast as $n \to \infty$.

Proposition 6.14. *1. For all real p, t, if HBD is assumed, see Definition 2.13, the potential $-t\log|F'| - pm$ is Hölder continuous on the associated symbolic space.*

2. The induced map F is uniformly expanding.

3. The diameters of $W = \phi_{\underline{W}}(V^{c(W)})$ shrink uniformly exponentially to 0 as $n \to \infty$ for n the length of the word $\underline{W}$.

Proof. Item 3. follows from the bounded distortion condition BD, see Definition 1.10 and Remark 6.7, with the exponent of the convergence $(1+2\epsilon')^{-1}$, where for each $c \in S'(f,K)$ the interval $\widehat{V}^c$ is an ϵ-scaled neighbourhood of V^c. Indeed, notice that the BD implies that for every $0 < j \le n$ and $W^j := \phi_{W_1} \circ \cdots \circ \phi_{W_j}(V^{c(W_j)})$ the set W^{j-1} is an ϵ'-scaled neighbourhood of W^j.

Item 2. follows from the fact that $|W| \le (1+2\epsilon')^{-n}$ and $|(\phi'_{\underline{W}}(x)/\phi'_{\underline{W}}(y)| \le C'(\epsilon)$ for $x, y \in V^{c(W)}$. Indeed, these estimates imply

$$|(\phi_{\underline{W}})'(z)| \le |W|C'(\epsilon) \le (1+2\epsilon')^{-n}C'(\epsilon) \le 1/2 < 1$$

for n large enough.

Finally notice that for $x, y \in W$ by the assumption HBD in Definition 2.13 we have $\log|F'(x) - \log|F'(y)| \le C(1+2\epsilon')^{-(n-1)\alpha}$, since V^c containing $F(W)$ is its $(1+2\epsilon')^{-1}$-scaled neighbourhood, by item 3. This expression shrinks exponentially fast to 0 as $n \to \infty$ yielding exponential decay of variations in the symbolic space, hence Hölder continuity of $\log|F'|$, hence Hölder continuity of $-t\log|F'| - pm$. □

This Proposition, Items 2 or 3, allow to define a standard limit set and its coding by the space E^∞ of all admissible infinite sequences

Definition 6.15. Define $\pi : E^\infty \to K(F)$ by

$$\pi(W_1W_2...) = \bigcap_{n=0}^{\infty} \phi_{W_1} \circ ... \circ \phi_{W_n}(V^{c(W_n)}).$$

Notice that due to the *strong separation property*, see [PR-L1, Definition A.1.], holding by the properties of nice couples, π is a homeomorphism.

Notice that due to the BD assumption and due to topological transitivity of $f|_K$ we can replace in the definition of $P(F, -t\log|F'| - pm)$ the expressions $\sup\left\{|\phi'_{\underline{W}}(z)| : z \in V^{c(\underline{W})}\right\}$ summed up over all $\underline{W} \in E^n$, by $|\phi'_{\underline{W}}(z_0)|$ summed up over all F^{-n}-pre-images of an arbitrarily chosen $z_0 \in V$. Compare this with the definition of tree pressure Definition 1.21.

Denote $P(F, -t\log|F'| - pm)$ by $\mathscr{P}(t,p)$. It can be infinite, e.g. at $t \le 0, p = 0$. The finiteness is clearly equivalent to the summability condition, see [MU, Section 2.3],

$$\sum_{W\in\mathfrak{D}} \sup_{z\in W} \Phi_{t,p} = \sum_{W\in\mathfrak{D}} \sup|\phi'_W|^{-t}\exp(-m_W p) < \infty. \tag{6.3}$$

This finiteness clearly holds for all t and $p > P_{\mathrm{tree}}(K,t)$, where it is in fact negative, see [PR-L2, Lemma 3.8]. In Key Lemma we shall prove this finitness on a bigger domain.

We study $\mathscr{P}$ on the domain where it is finite. $\mathscr{P}$ is there a real-analytic function of the pair of variables (t,p), see [MU, Section 2.6] for the analyticity with respect to one real variable. The analyticity follows from the analyticity of the respective transfer (Perron-Frobenius, Ruelle) operator, hence its isolated eigenvalue being exponent of $\mathscr{P}$. The analyticity with respect to (t,p) follows from the complex analyticity with respect to complex t and p and Hartogs' Theorem.

7. Analytic dependence of Geometric Pressure on temperature. Equilibria.

To prove Theorems A and C, in particular to study $P(K,t)$, we shall use the following Key Lemma, whose proof will be postponed to the next section.

Lemma 7.1. *(Key Lemma)*

1. For $(f,K) \in \mathscr{A}_+^{\mathrm{BD}}$, if moreover HBD is assumed, see Definition 2.13, for F and $\mathscr{P}(t,p)$ defined with respect to a nice couple, the domain of finiteness of $\mathscr{P}(t,p)$ contains a neighbourhood of the set $\{(t,p) \in \mathbb{R}^2 : t_- < t < t_+, p = P(K,t)\}$.

2. For every $t : t_- < t < t_+$, $\mathscr{P}(t, P(t)) = 0$.

In this Section we shall prove Theorems A and C using Key Lemma. The proof follows the complex version in [PR-L2, Theorems A and B], but it is simpler (the simplification concerns also the complex case) by omitting the considerations of subconformal measures.

7.1. **The analyticity.** Notice that $\partial \mathscr{P}(t,p)/\partial p < 0$. Indeed, using the the bound $m \geq 1$ for the time of return function $m = m(x)$ giving $F = f^m$, considering arbitrary $p_1 < p_0$ so that (t,p_1) and (t,p_1) belong to the domain of $\mathscr{P}$, we get

$$
\begin{gathered}
P(F, -t\log|F'| - p_1 m) - P(F, -t\log|F'| - p_2 m) \\
= \lim_{n\to\infty} \frac{1}{n} \log Z_n(t,p_1) - \lim_{n\to\infty} \frac{1}{n} \log Z_n(t,p_0) \\
= \lim_{n\to\infty} \frac{1}{n} \log \frac{\sum_{\underline{W}\in E^n} \exp(-m_{\underline{W}} p_1) \left(\sup\left\{|\phi'_{\underline{W}}(z)| : z \in V^{c(\underline{W})}\right\}\right)^t}{\sum_{\underline{W}\in E^n} \exp(-m_{\underline{W}} p_0) \left(\sup\left\{|\phi'_{\underline{W}}(z)| : z \in V^{c(\underline{W})}\right\}\right)^t} \\
\geq \lim_{n\to\infty} \frac{1}{n} n|p_1 - p_0| = |p_1 - p_0| > 0,
\end{gathered}
$$

since the ratio of each corresponding summands in the ratio of the sums above is bounded from below by $\exp m_{\underline{W}}|p_1 - p_0| \geq \exp n|p_1 - p_0|$.

Thus the analyticity of $t \mapsto P(t)$ for $t_- < t < t_+$ follows from the Implicit Function Theorem and the Key Lemma.

Remark 7.2. In [PR-L2, Subsection 4.4] an indirect argument was provided, expressing the derivative $\partial \mathscr{P}/\partial p$ by a Lyapunov exponent for the equilibrium existing by the last item of Theorem A. The above elementary calculation is general, working also in the complex case.

7.2. **From the induced map to the original map. Conformal measure.** We shall call here any ϕ-conformal measure for f on K for $\phi := (\exp p)|f'|^t$ a (t,p)-*conformal measure* for f, see the definition in 1.4. We call μ on $K(F)$ a (t,p)-*conformal measure* for an induced map F, as constructed in Subsection 6.2, if it is ϕ-conformal for F and for $\phi(x) := (\exp pm(x))|F'(x)|^t$.

A $(t, P(t))$-conformal measure $\mu_{F,t}$ for F, supported on $K(F)$, exists by Key Lemma and by [MU, Theorems 3.2.3]. It is non-atomic and supported $K_{\mathrm{con}}(f)$ since the latter set contains $K(F)$ by definitions.

Similarly as in [PR-L1, Proposition B.2] $\mu_{F,t}$ can be extended to the union of $\mathfrak{L}_V$, see Definition 6.11, by

$$\tilde{\mu_t}(A) := \int_{f^{m(W)}(A)} (\exp - P(t)m(W))|(f^{-m(W)})'(x)|^t \, d\mu_{F,t}(x)$$

for each $W \in \mathfrak{L}_V$ and every Borel set $A \subset W$ on which $f^{m(W)}$ is injective. The conformality of this measure for f, i.e. 1.4, holds for all $A \subset W \in \mathfrak{L}_V$ and $A \subset K(F)$ by the same proof as in [PR-L1]. Finally normalize $\tilde{\mu_t}$ by setting $\mu_t := \tilde{\mu_t}/\tilde{\mu_t}(K(F)$. (Sometimes we omit tilde over not normalized μ to simplify notation.)

Notice that $\mu = \mu_t$ is finite and non-atomic, as a countable union of non-atomic pull-backs of $\mu_{F,t}$ to $W \in \mathfrak{L}_V$. The summability $\sum_{W \in \mathfrak{L}_V} \mu(W) < \infty$ follows from BD (bounded distortion) and (8.1).

Denote $\hat{K}(F) := \bigcup_{W \in \mathfrak{L}_V} f^{-m(W)}(K(F))$. Let $x \in V \setminus (K(F) \cup S'(f,K))$. We prove that

$$y = f(x) \notin \hat{K}(F). \tag{7.1}$$

Suppose this is not the case. For V small enough, $y \notin V$. Denote $z = f^{m(y)}(y)$ where $m(y)$ is the time of the first return of y to V, see Remark 6.9. Then there exists an infinite word $\underline{W} = W_1 W_2 ... \in E^\infty$ such that $z = f^{m(y)}(y) = \pi(\underline{W})$. Let $n_0 \geq 1$ be the least n such that Y_n being the pull-back of $\widehat{W_n} = \phi_{W_1} \circ ... \circ \phi_{W_n}(V^{c(W_n)}$ of order $m(y)+1$ containing x is disjoint from $S'(f,K)$. It exists since $|\widehat{W_n}| \to 0$ as $n \to \infty$. Hence $|Y_n| \to 0$, hence $Y_n \cap S'(f,K) = \emptyset$ as $x \notin S'(f,K)$. In consequence

$$m = m(y) + 1 + m_{W_1} + ... + m_{W_{n_0}}$$

is a good time for x. Hence, by Lemma 6.10, there exists $k \geq 1$ such that $f^m(x) = F^k(x)$. In consequence $x \in K(F)$, a contradiction.

(More precisely $x = \pi(\underline{W'})$, where either $\underline{W'} = W_0 W_1 W_2...$ for W_0 being a pull-back of V for $f^{m(y)+1}$, containing x in the case $m(y)+1$ is a good time for x, or $\underline{W'} = W_0' W_{n_0+1}...$ otherwise, where $W_0' \ni x$ and $m_{W_0'} = m$.)

Suppose now that $A \subset V \setminus K(F)$. Then by definition $\mu(A) = 0$. We shall prove that also $\mu(f(A)) = 0$. Indeed, we have just proved that $f(A \setminus S'(f,K)) \cap \widehat{K}(F) = \emptyset$, hence $f(A \setminus S'(f,K))$ is disjoint from the set $\widehat{K}(F)$ of full measure μ. Notice also that $\mu(f(S'(f,K))) = 0$ since μ does not have atoms.

Thus the proof of existence of a $(\exp P(t))|f'|^t$-conformal measure μ for f on K as asserted in Theorem A, is finished. This measure is non-atomic and supported on $K_{\mathrm{con}}(f)$ by construction. It is zero on exceptional sets since such sets are finite and μ is non-atomic. It is positive on open sets in K by Lemma C.2.

If there is another $(\exp p)|f'|^t$-conformal measure ν, with $p \le P(t)$, then it is positive on open sets by Lemma C.2, $p = P(t)$ and $\mu \ll \nu$. In consequence both measures are proportional. The proof is the same as in [PR-L2, Subsection 4.1]. Also the proof of ergodicity is the same.

7.3. **Equilibrium states.** The proof of this part of Theorem A is the same as in the complex case, so we only sketch it.

1. *Existence*

As in [PR-L2, Lemma 4.4] one deduces from Key Lemma 7.1 that the return time function $m(z)$ is μ_t integrable, where μ_t is the unique probability $(t, P(t))$-conformal measure, for every $t_- < t < t_+$. Moreover the exponential tail inequality holds

$$\sum_{W \in \mathfrak{D}, m_W \ge n} \mu_t(W) \le \exp(-\epsilon n), \tag{7.2}$$

for a constant $\epsilon > 0$ (depending only on t) and all n large enough.

Due to the summability (finiteness of $\mathscr{P}(t, P(t))$ 6.3 and Hölder continuity of the potential function $\Phi_{t,P(t)}$ on E^∞, see Proposition 6.14, there exists an F invariant probability measure $\rho_{F,t}$ absolutely continuous with respect to the conformal measure $\mu_{F,t}$ on $K(F)$ being the restriction of μ_t to this set (so $\mu_{F,t}(K(F)) < 1$ usually).

Indeed, consider the Gibbs state on E^∞ given by [MU, Theorem 2.2.4]. Project it by π_* to $\rho_{F,t}$ on $K(F)$. Its equivalence to $\mu_{F,t}$ on $K(F)$ follows from Gibbs property [MU, (2.3)] holding for both measures.

Moreover there exists $C_t > 0$ such that $C_t \le d\rho_{F,t}/d\mu_{F,t} \le C_t^{-1}$ i.e the density is bounded and bounded away from 0.

One has used here also, while applying [MU], in particular concluding the ergodicity of $\rho_{F,t}$, the fact that the incidence matrix A_F is finitely irreducible, which follows from the topological transitivity of f. The argument for this irreducibility is as follows, compare [PR-L1, Proof of Lemma 4.1].

1a. *Proof of the irreducibility*

Consider an arbitrary repelling periodic point $p \in K \setminus \bigcup_{n\geq 0} f^n(S'(f,K))$. For every $c \in S'(f,K)$ choose a backward trajectory $\gamma_1(c)$ converging to the periodic orbit $O(p)$, existing by Proposition 2.5. It omits $S'(f,K)$ by our assumption $f^n(S'(f,k)) \cap S'(f,K) = \emptyset$ for all $n > 0$. Therefore for V small enough all these trajectories, hence the pull-backs of $\widehat{V}$ along them, are disjoint from V.

Next notice that for each $c \in S'(f,K)$ one can choose a finite backward trajectory $\gamma_2(c) = (p = z(c)_0, z(c)_{-1}, ..., z(c)_{-N(c)})$ such that $z(c)_{-N(c)} \in V^c$, by the density of preimages property 2.4.

Consider now any two $c, c' \in S'(f,K)$. Consider a pull-back of $\widehat{V}$ going along $\gamma_1(c)$ for the time $M(c)$ so long that

(a) the diameter of the associated pull-back $\widehat{W}(c)$ of $\widehat{V}^c$ is small enough that its consecutive pull-backs for f^j along $\gamma(c')$ have diameters small enough to be disjoint from $S'(f,K)$, for all $j = 0, ..., -N(c')$

(b) the distance of $\widehat{W}(c)$ from $O(p)$ is small enough that $\widehat{W}(c,c')$ being the pull-back of $\widehat{W}(c)$ is in $V^{c'}$.

Then, for $x = x(c,c')$ being the $f^{-(M(c)+N(c'))}$-pre-image of c in $\widehat{W}(c,c')$, the time $m = m(c,c') := M(c) + N(c')$ is a good time, see Definition 6.8. Hence by Lemma 6.10 there exists $k = k(c,c')$ such that $f^m = F^k$ on $W(c,c')$.

We conclude that for every two $W', W \in \mathfrak{D}$, if $W \subset V^c$ and $f^{m_{W'}} = V^{c'}$, then there is an admissible word of length $k(c,c') + 2$ on the alphabet $\mathfrak{D}$ starting with W' and ending with W.

Existence. Continuation

Now, using the F-invariant measure $\rho_{F,t}$, one defines an f-invariant measure in the standard way:

$$\rho_t := \sum_{W\in\mathfrak{D}} \sum_{n=0}^{m_W-1} f_*^n \rho_{F,t}|_W. \tag{7.3}$$

Finiteness of ρ_t follows from the integrability of $m(x)$ with respect to $\rho_{F,t}$ which follows from the integrability with respect to $\mu_{F,t}$ and the boundness of $\frac{d\mu_{F,t}}{d\rho_{F,t}}$.

The absolute continuity of ρ_t with respect to the conformal measure μ_t follows easily from this definition, see e.g. [PR-L1, p.165-166]. Indeed. For each $W \in \mathfrak{D}$ and $n : 0 \leq n < m_W$ write $J_{W,n} := |(f^n)'|^{-t}$ on $f^n(W)$, and 0 on the rest of $\mathbf{U}$. Replace ρ_t in 7.3 by the auxiliary $\mu'_t = \sum_{W\in\mathfrak{D}} \sum_{n=0}^{m_W-1} f_*^n \mu_{F,t}|_W$. Then, using 7.3 and the boundness of

$d\mu_t'/d\rho_t$, we get

$$+\infty > \mu_t'(K) = \sum_{W,n} \int J_{W,n}\, d\mu_t = \int \sum_{W,n} J_{W,n}\, d\mu_t,$$

by Lebesgue Monotone Convergence Theorem. Moreover for every continuous function $u : K \to \mathbb{R}$

$$\int u d\mu_t' = \sum_{W,n} \int u J_{W,n}\, d\mu_t = \int u(\sum_{W,n} J_{W,n})\, d\mu_t.$$

Thus, it follows that μ_t', hence ρ_t is absolutely continuous with respect to μ_t.

1b. *The density function*

The density $d\rho_t/d\mu_t$ is bounded away from 0. Indeed, let $N(V)$ be such that $\bigcup_{n=0}^{N}(V \cap K) = K$. Then for μ_t almost every $z \in K$ there exists $y \in K(F)$ and $N \le N(V)$ such that $f^N(y) = z$. We get

$$\frac{d\rho_t}{d\mu_t}(z) \ge \frac{d\rho_{F,t}}{d\mu_{F,t}}(y)|(f^N)'|^{-t} e^{-NP(t)} \ge C_t(\max_{n=0,1,\dots,N(V)} \sup_K |(f^n)'|^t e^{nP(t)})^{-1},$$

compare again [PR-L1, page 166].

1c. *Equilibrium*

The proof is the same as in [PR-L2, Lemma 4.4]. Denote $\rho := \rho_{F,t}$. Consider its normalized extension given by (7.3), by $\rho' := \rho_t/\rho_t(K)$. One uses the fact that ρ is the equilibrium state for the shift map on E^∞ (identified with $K(F)$) and the potential $\Phi_{t,P(t)}$, i.e. $P(F, -t\log|F'| - P(t)m) = h_\rho(F) - \int (t\log|F'| + P(t)m)\, d\rho$, which is equal to 0 by hypothesis. Thus, using generalized Abramov's formula, see [Zwe, Theorem 5.1], we get

$$h_{\rho'}(f) = (\rho_t(K))^{-1} h_\rho(F) = (\rho_t(K))^{-1} \left(\int (t\log|F'| + P(t)m)\, d\rho \right) \tag{7.4}$$

$$= (\rho_t(K))^{-1} t \int \log|f'| d\rho_t + P(t) = t \int \log|f'|\, d\rho' + P(t).$$

This shows that ρ' is an equilibrium state of f for the potential $-t\log|f'|$.

2. *Uniqueness*

Let $t_- < t < t_+$ and ν be an ergodic equilibrium measure for $f|_K$ and the potential $-t\log|f'|$. Since $P(t) > -t\chi_\nu$, it follows that $h_\nu(f) > 0$. Hence by Ruelle's inequality $\chi_\nu(f) > 0$. Now one can refer, as in the complex case, to [Dobbs2] adapted to the interval case. In our existence of nice couples case there is however a simpler proof using existence of nice couples and inducing method, omitting Dobbs (and Ledrappier) method of using canonical systems of conditional measures on the measurable partition into local unstable manifolds in Rokhlin natural extension. See Appendix B.

3. *Mixing and statistical properties*

To prove these properties in theorem A we refer, as in [PR-L1] and [PR-L2, Subsection 8.2], to Lai-Sang Young's results, [Young].

In the case when there is only one critical point in K one can apply these results directly, and in the general case one fixes an appropriate $c_0 \in S'(f,K)$ and considers $\widehat{F} : V^{c_0} \cap K(F) \to V^{c_0} \cap K(F)$, the first return map for iteration of F. This makes sense since $\rho_{F,t}(V^{c_0} \cap K(F)) > 0$ as μ_t is positive on open sets in $K(F)$.

$\widehat{F}$ is an infinite one-sided Bernoulli map, with $\mathfrak{D}$ replaced by $\hat{\mathfrak{D}}$ being the joining of $\mathfrak{D}$ and its appropriate F^j-preimages, see the next paragraph. Now we refer to [Young] considering Young's tower for $\widehat{F}$ and the integer-valued function $\hat{m}$ on $V^{c_0} \cap K(F)$ defined $\rho_{F,t}$-a.e. by $\widehat{F}(x) = f^{\hat{m}}(x)$, more precisely $\hat{m}(x) = \sum_{j=0,\dots,m_F-1} m(F^j)(x)$, where $F^{m_F} = \widehat{F}$.

The tower $T_{\widehat{F},\hat{m}}$ is the disjoint union of pairs $(U,n) = (\phi_{\underline{W}}(V^{c(\underline{W})}), n)$, with $W_1 \subset V^{c_0}$, $c(W_{m_F} = c_0$ and $c(W_i) \neq c_0$ for $i = 1, \dots, m_F - 1$ (the length of $\underline{W}$ being m_F) and $n = 0, 1, \dots, m_U - 1$.

On each (U,n) define the measure $\hat{\rho}_{F,t}$ as the image of $\rho_{F,t}$ on $U = (U,0)$ under the mapping $(x,0) \mapsto (x,n)$. It is easy to see that $(x,n) \mapsto f^n(x)$ projects this measure to ρ_t on K.

Then by [Young], see also [PR-L1, Subsection 8.2], to prove mixing, exponential mixing and CLT it is sufficient to check the following.

1) The greatest common divisor of the values of m_U is equal to 1.

2) The tail estimate 7.2 $\sum_{U, m_U \geq n} \rho_t(U) \leq \exp(-\varepsilon n)$ for a constant $\varepsilon > 0$, for $U \in \hat{\mathfrak{D}}$.

To prove 1) we proceed as in 1a. (the proof of irreducibility) with some refinements, compare [PR-L1, Lemma 4.1]. First notice that provided topological exactness of $f|_K$ there are in K periodic orbits of all periods large enough. To prove this it is sufficient to be more careful in Proof of Lemma 2.11 while "closing the loop". Given a pull-back T for f^n, of $B = B(z_0, \exp(-\alpha n))$ and large $S = f^m(B)$ such that $f^m : B \to S$ is a diffeomorphism and $m << n$, then we find a diffeomorphic pull-back from T into S of an arbitrary length not exceeding n.

Now choose an arbitrary $O(p)$ as in 1a. For every $c \in S'(f,K)$ choose a backward trajectory $\gamma_1(c)$ converging to $O(p)$. Choose $\gamma_2 = (p, z_{-1}, z_{-2}, \dots z_{-N})$, a backward trajectory of p, such that $z_{-N} \in V$. Let $n \leq N$ be the least $n \leq N$ such that $z_{-n} \in V$. Let $c_0 \in S'(f,K)$ be such that $z_{-n} \in V^{c_0}$. Thus for each c, in particular for c_0, we find, as in 1a., a backward trajectory $c_0, x_{-1}, \dots, x_{-m}$ of c_0 such that $x_{-m} \in V^{c_0}$ no $f^{-j}(c_0)$ is in V for $j = 1, 2, \dots, m-1$ (on its way going several times along $O(p)$). Next find a point $y \in V^{c_0}$ shadowing this trajectory, hence $F(y) = f^m(y) \in V^{c_0}$. Hence $\widehat{F} = F = f^m$ at y, with the return time m being an arbitrary integer of the form $a + km_p$, where m_p is a period of p.

Considering now $\#S'(K,f)+1$ number of periodic points in $K \setminus \bigcup_{n\geq 0} f^n(S'(f,K))$ with pairwise mutually prime periods, we find two, having the same c_0. Then the condition on the greatest common divisor of return times being 1 is satisfied for appropriate choices of k. For details see [PR-L1, Lemma 4.1].

To prove 2) one uses 7.2 for F, i.e. for $W \in \mathfrak{D}$. One repeats roughly the estimates in [PR-L1, the top of p. 167]. The key point is the finiteness of the sum of the middle factors in the decomposition $\widehat{F}'(x) = F'(x) \cdot (F^{m-2})'(F(x)) \cdot F'(F^{m-1}(x))$, for $F^m = \widehat{F}$, namely, writing $m_j(y) := m(y) + m(F(y)) + ... + m(F^{j-1}(y))$,

$$\sum_{j\geq 1} \sum_y{}^{*} |(F^j)'(y)|^{-t} \exp -(m_j(y))(P(t)-\varepsilon) < \infty \tag{7.5}$$

for a positive ε. Here the star * means that we consider only y one for each word $\underline{W}$ (of length j), such that no $F^i(y)$ belongs to V^{c_0} for $i = 0,1,...,j$. This summability holds because the pressure of the subsystem of the system in 6.2 where we omit symbols in V^{c_0} is strictly less than the full $P(F, -t\log|F'| - P(t)m = 0$ by Key Lemma 7.1, hence negative. So we have a room to subtract ε in 7.5 and preserve convergence.

8. Proof of Key Lemma. Induced pressure

Again the proof repeats the proof in [PR-L2], so we only sketch it (making some simplification due to real dimension 1).

Part 1.

1.1. First notice that for every $t \in \mathbb{R}$ and every p close enough to $P(t)$

$$\sum_{W \in \mathfrak{L}_V} \exp(-pm_W) \operatorname{diam}(W)^t < +\infty. \tag{8.1}$$

The proof is the same as in [PR-L2, Lemma 6.2] and uses the fact that f is expanding on $\mathscr{K}(V)$, i.e. outside V, see notation in Definition 6.1. Briefly: $\mathscr{K}(V)$ can be extended to an isolated hyperbolic subset $\mathscr{K}'(V)$ of K (even better than in [PR-L2] where we can guarantee only the existence of Markov partition, compare [PU, Remark 4.5.3]). This set can be extended to an even larger isolated hyperbolic set $\mathscr{K}''(V) \subset K \setminus S'(f,K)$. Then $P(f|_{\mathscr{K}'(V)}, -t\log|f'|) < P(f|_{\mathscr{K}''(V)}, -t\log|f'|) \leq P(K,t)$.

1.2. Let $\mathscr{W}_{n,k}$ be the family of all components of the set

$$A_{n,k} := \{x \in \mathbf{U} : 2^{-(k+1)} \leq \operatorname{dist}(x, S'_n) \leq 2^{-k}\}$$

for $S'_n := \bigcup_{j=1,\dots,n} f^j(S'(f,K)))$, with the exception that if such a component is shorter than $2^{-(k+1)}$ we add it to an adjacent component of $A_{n,k+1}$. Clearly for each integer k

$$\#sW_{n,k} \le 2n\#S'(f,K), \tag{8.2}$$

in particular the bound is independent of k. The coefficient 2 appears because given k the intervals of $\mathscr{W}_{n,k}$ can lie on both sides of each point in S'_n.

Denote $\mathscr{W}_n := \bigcup \mathscr{W}_{n,k}$.

(For each n the family $\mathscr{W}_n$ is a Whitney decomposition of the complement of S'_n, similarly to the Riemann sphere case in [PR-L2].)

1.3. Clearly for every $a > 1$ there exists $b > 1$ such that for every $L \in \mathscr{W}_n$ and an interval T intersecting L, if aT is disjoint from S'_n, then $T \subset bL$. We denote by aT the interval T rescaled by a with the same center, similarly bL.

We apply this for $T = W \in \mathfrak{L}_V$ such that W is disjoint from S'_n. Hence a as above exists by Remark 6.7.

For each $L \in \mathscr{W}_n$ let $m(L)$ be the least integer m such that $f^j(bL)$ is disjoint from $S'(f,K)$ for all $j = 0, 1, ..., m$. Hence $f^{m(L)+1}$ is a diffeomorphism on bL. Notice also that $f^{m(L)+1}(bL)$ is large since it joins a point $c \in S'(f,K)$ to ∂V^c, since L intersects a set $W \in \mathfrak{L}$ (the family $\mathfrak{L}_V$ is dense in K by the topological transitivity) hence $W \subset bL$. Therefore $f^{m(L)+1}(L)$ is also large, since otherwise L would be small in bL see Definition 1.10.

For every Y being a bad pull-back of $\widehat{V}^c$ or $Y = \widehat{V}^c$ in the decomposition in Lemma 6.12, denote by $\mathscr{W}(Y)$ the set of all components L_Y of $f^{-(m_Y+1)}(L)$ intersecting Y for all $L \in \mathscr{W}_{m_Y}$, such that there exists $W \in \mathfrak{D}$ intersecting L_Y. Notice that for each L as above and its pull-back $L_Y \in \mathscr{W}(Y)$ the mapping $f^{m_Y+1} : L_Y \to L$ is a K-diffeomorphism.

For each $L_Y \in \mathscr{W}(Y)$, using distortion bounds, we get

$$\begin{aligned} &\sum_{W \in \mathfrak{D}_Y,\, W \cap L_Y \neq \emptyset} \exp(-pm_W)|W|^t \\ &\le \mathrm{Const} \exp(-p(m_Y + 1 + m(L))\Big(\frac{|L_Y|}{|f^{m(L)}(L)|}\Big)^t \sum_{W \in \mathfrak{L}_V} \exp(-pm_W)|W|^t. \end{aligned} \tag{8.3}$$

The latter sum is finite by 8.1. Since $f^{m(L)}$ is large it contains a safe hyperbolic point z from a finite *a priori* chosen set $Z \subset K$. Let z_{L_Y} be its $f^{m_Y+1+m(L)}$-preimage in L_Y. Hence, in 8.3, the term $\left(\frac{|L_Y|}{|f^{m(L)}(L_Y)|}\right)^t$ can be replaced up to a constant related to distortion, by $|(f^{m_Y+1+m(L)})'(z_{L_Y})|^{-t}$ which can be upper bounded by $\mathrm{Const}\, \gamma^{m_Y+1+m(L)}$ for any γ satisfying

$$\exp\left(-\tfrac{1}{2}(P(t) - \max\{-t\chi_{\inf}, -t\chi_{\sup}\})\right) < \gamma < 1.$$

For $t \leq 0$ this holds due to $\lim_{n\to+\infty} \frac{1}{n} \log \sup\{|(f^n)'(z)| : z \in K\} = \chi_{\sup}$, compare [PR-L2, Proposition 2.3, item 2]. For $t > 0$ we use the same with $\chi_{\sup}$ replaced by $\chi_{\inf}$, but we use the assumption that $z \in Z$ is safe and expanding, compare [PR-L2, Proposition 2.3, item 2].

In conclusion

$$\sum_{W \in \mathfrak{D}_Y,\, W \cap L_Y \neq \emptyset} \exp(-pm_W)|W|^t \leq \mathrm{Const}\, \gamma^{m_Y+1+m(L)}. \tag{8.4}$$

1.4. Now we collect above estimates and use Lemma 6.13 to prove the summability 6.3. which yields the finiteness of $\mathscr{P}(t,p)$.

$$\begin{aligned} \sum_{W\in\mathfrak{D}} & \exp(-pm_W)|W|^t \\ & \leq \mathrm{Const} \sum_Y \sum_{L_Y} \sum_{W \in \mathfrak{D}_Y,\, W\cap L_Y \neq \emptyset} \exp(-pm_W)|W|^t \\ & \leq \mathrm{Const} \sum_{m,k} \sum_{Y: m_Y = m} \sum_{L \in \mathscr{W}_{n,k}} \sum_{L_Y} \gamma^{m+1+m(L)} \\ & \leq \mathrm{Const} \sum_{m,k} (\exp m\epsilon) 2m \# S'(f,K) \gamma^m \gamma^{\alpha k} < \infty, \end{aligned} \tag{8.5}$$

provided $\gamma \exp \epsilon < 1$, where $\alpha := \frac{\inf_{n,k} \inf_{L\in\mathscr{W}_{n,k}} m(L)}{k}$ is positive since f is Lipschitz continuous.

Part 2.

The main point is to prove that $p(t)$, the zero of $\mathscr{P}(t,p)$, is not less than $P(t)$. For this end we prove that for all $p > p(t)$ we have $P(f|_K, -t\log|f'|) \leq p$ which is equivalent to $P(f|_K, -t\log|f'| - p) \leq 0$. For this it is sufficient to prove that the summability of the "truncated series"

$$T_F(w) := \sum_{k=1}^{+\infty} \sum_{y\in F^{-k}(w)} \exp(-pm_k(y))|(F^k)'(y)|^{-t},$$

where $m_k(y) := \sum_{j=0,\dots,k-1} m(F^j(y))$, for an arbitrary $w \in K(F)$ implies the summability of the "full series"

$$\sum_{n=1}^{+\infty} \exp(-pn) \sum_{y\in f^{-n}(z_0)} |(f^n)'(y)|^{-t},$$

for an arbitrary safe expanding $w \in K$. The proof repeats word by word the proof in [PR-L2, subsection 7.2, page 694] and will be omitted.

The idea is that fixed a safe expanding point $w \in V$, for every $y \in f^{-n}(w) \cap K$ we consider $s : 0 \leq s \leq n$ the time of the first hit of V by the forward trajectory of y. Next let $m : s \leq m \leq n$ be the largest positive time so that $y' := f^s(y)$ is a good pre-image of $f^m(y)$. If such m does not exist (i.e. y' is a bad pre-image of w) then we set $m = s$. By Lemma 6.10

$f^{m-s}(y') = F^k(y')$. The sum over each of three kind of blocks, to the first hit of V, of F^k-type, and over bad blocks, is finite.

Appendix A. More on generalized multimodal maps

Instead of starting from a multimodal triple (or quadruple) defined in Introduction with the use of the notion of the maximality one can act from another end. Let $K \subset \mathbb{R}$ be a compact subset of the real line, U be its open neighbourhood being the union of a finite number of open pairwise disjoint intervals $U^j, j = 1, ..., m(U)$, and let $f : U \to \mathbb{R}$ be a C^2 mapping having a finite number of critical points, all non-flat, and such that $f(K) \subset K$.

Definition A.1. We say that the triple (f, K, U) satisfies Darboux property if for every interval $T \subset U$ there exists an interval $T' \subset \mathbb{R}$ (open, with one end, or with both ends) such that $f(T \cap K) = T' \cap f(K)$, compare [MiSzlenk, page 49].

Call each set $L \subset K$ equal to $T \cap K$ for an interval T a *K-interval.* Then Darboux property means that the f-image of each K-interval in U is a K-interval.

Finally notice that Darboux property for f implies Darboux property for all its forward iterates on their domains.

Proposition A.2. *Let (f, K, U) be a triple as above. Assume that $f|_K$ is topologically transitive. Assume also that the triple satisfies Darboux property. Then we can replace U by a smaller open neighbourhood $\mathbf{U}$ of K such that f restricted to the union $\widehat{I}_K$ of convex hulls $\widehat{I}^j_K$ of $\mathbf{U}^j \cap K$, where as in the notation above $\mathbf{U}$ is the union of a finite number of open pairwise disjoint intervals $\mathbf{U}^j, j = 1, ..., m(\mathbf{U})$, gives a reduced generalized multimodal quadruple $(f, K, \widehat{I}_K, \mathbf{U})$ in the sense of Definition 1.3, in particular K is the maximal repeller in $\mathbf{U}$.*

2. The converse also holds. The maximal repeller $K = K(f)$ for $(f, K, \widehat{I}) \in \mathscr{A}$ satisfies Darboux property on each $\widehat{I}^j \cap K$.

Proof of Proposition A.2. For each $j : 1 \leq j \leq m(U)$ denote by $\widehat{K}^j$ the convex hull of $U^j \cap K$ and $\widehat{K} = \widehat{K}_U := \bigcup_{j=1,...,m(U)} \widehat{K}^j$.

Let V be a bounded connected component of $\widehat{K}^{j_0} \setminus K$ for an integer j_0. Since all the critical points of f are contained in K the map f maps V diffeomorphically to $f(V)$. Then Darboux property with T equal to the closure of V implies that either $f(V)$ is disjoint from $\widehat{K}$ or $f(V)$ is a bounded connected component of $\widehat{K}^{j_1} \setminus K$ for an integer j_1. We say a bounded connected component of $\widehat{K}^j \setminus K$ is *small* if for every integer $n \geq 1$ the map

f^n is defined on V and if $f^n(V)$ is contained in $\widehat{K}$. A small component is *periodic* if there is an integer $k \geq 1$ such that $f^k(V) = V$.

Extend f to a C^2 multimodal map $g : I \to I$ of a closed interval containing U with all critical points non-flat. Then by [dMvS, Ch. IV, Theorem A] there is no wandering interval for g, in particular either V is eventually periodic or V is uniformly attracted to a periodic orbit $O(p)$. Notice that in the latter case all $f^n(V), j = 0, 1, ...$ are pairwise disjoint since otherwise a boundary point of one of them, hence a point in K would belong to another $f^n(V)$ that contradicts the disjointness of all $f^n(V)$ from K.

The latter case however cannot happen. Indeed, $O(p)$ is in K by compactness of K. However attracting periodic orbits cannot be in K, as $f|_K$ is topologically transitive and K is infinite, compare arguments in Definition 1.5.

If $O(p)$ is indifferent and V is not eventually periodic then for $x \in K$ being a boundary point of V we have $f^n(x) \to O(p)$ and moreover $f^n(W) \to O(p)$ for W a neighbourhood of x. Indeed denoting by m a period of p we can assume that f^m is strictly monotone in a neighbourhood D of p since $\mathrm{Crit}(f)$ is finite, and all the points $f^n(x)$ belong to D for n large enough. Then by the monotonicity the intervals between them are also attracted to $O(p)$. Hence $f^n(W)$ converge to $O(p)$. (In particular $f^n(W)$ belong to $B_0(O(p))$ and $O(p)$ is attracting from at least one side.) This again contradicts topological transitivity, as there is no return of W to a neighbourhood of x.

Thus we can assume that V is a periodic small component. By [dMvS, Ch. IV,Theorem B] all such components have minimal periods bounded by a constant. Notice that in each period m there can be only a finite number of them. Otherwise in a limit of such components we have a periodic point p of minimal period m in a limit of such components V_n, so for $p_n \in \partial V_n$ we have $p = \lim p_n$. All p_n for n large enough belong to an interval where f^m is defined and has no critical points. Consider all V_n on the same side of p. Then the orbits of the interiors H_n of the convex hulls of $p_{2n}, p_{2(n+1)}$ are pairwise disjoint and each H_n intersects K, at p_{2n+1}. This contradicts topological transitivity of $f|_K$.

Now we end the proof by removing from $\widehat{K}$ the family of all small periodic components. Due to the just proved finiteness of this family we obtain the decomposition of this new set, denoted by $\widehat{I}_K$, into a finite family of closed intervals $\widehat{I}^1, ..., \widehat{I}^m$ (where m can be larger than the original $m(U)$). Finally we take $\mathbf{U}$, a small neighbourhood of $\widehat{I}_K$ as in Definition 1.3, contained in U and consider the original f on it.

This completes the proof of the part 1 of the Proposition.

The part 2 says that the maximal repeller $K(f)$ in $\widehat{I}$ satisfies Darboux property. To see this notice that if $x \in K(f)$ i.e. its forward trajectory stays in $\widehat{I}$, then obviously the forward trajectory of its every preimage y in $\widehat{I}$ stays in $\widehat{I}$, hence $y \in K(f)$, which yields Darboux property for $K(f)$.

$\square$

In particular we have proved

Corollary A.3. *For each triple (f, U, K) as above, f being C^2, Darboux, topologically transitive, there exists a reduced triple $(f, K, \mathbf{U})$ with no attracting periodic orbits in $\widehat{I}_K$ and with at most finite number of indifferent periodic orbits in $\widehat{I}_K$, all of them in K.*

This Corollary explains a part of Remark 1.6 stated in Introduction. If one assumes from the beginning the maximality of K then by the topological transitivity of f on K there are no attracting periodic orbits in K, hence in $\widehat{I}_K$.

If $O(p)$ is a indifferent periodic orbit in $\widehat{I}_K$, hence similarly, by the maximality of K this orbit is in K. Since it is not repelling from at least one side and there are no attracting periodic orbits in $\widehat{I}_K$, it is attracting from this side. One proves there is at most finite number of such orbits by repeating arguments of Proof of Corollary A.3 above, using [dMvS, Ch. IV, Theorem B]. See also Proposition A.5 below. Notice however that for this finiteness in $\widehat{I}_K$ (or K) we do not need to assume *bounded distortion.*

Finally we prove the following.

Lemma A.4. *Given $(f, K) \in \mathscr{A}^r$ for $r \geq 2$ there exists an extension of $f|_{\widehat{I}_K}$ to a C^r-mapping $g : I \to I$ for a closed interval I containing a neighbourhood of $\widehat{I}_K$, such that all periodic orbits for g in $I \setminus K$ are hyperbolic repelling.*

Proof. We already know that there are no periodic orbits in $\widehat{I}_K \setminus K$. Extend f to C^r $g : I \to I$ so that the extension has only non-flat critical points and $g(\partial I) \subset \partial I$. Using [dMvS, Ch. IV, Theorem B] we can correct g outside of $\widehat{I}_K$, so that there is only finite number of attracting or one-side attracting periodic orbits, compare Proof of Proposition A.2 above. By Kupka-Smale Theorem we can smoothly perturb g, outside $\widehat{I}_K$, so that outside K all repelling periodic orbits are hyperbolic and there are no indifferent periodic orbits.

Finally let $B_0(p)$ be the immediate basin of attraction for an attracting periodic point $p \in I \setminus \widehat{I}_K$ for g. Then, since no $B_0(f^j(p))$ for $f^j(p) \in \widehat{I}_K$ contains critical points (as we assumed that all critical points in $\widehat{I}_K$ are in K) the map f maps $B_0(f^j(p))$ injectively onto $B_0(f^{j+1}(p))$. Notice that it does not matter whether we write here f or g.

We replace, if necessary, g on each $B_0(f^i(p))$ for i such that $B_0(f^i(p)) \subset I \setminus \widehat{I}_K$, by a mapping whose all monotonicity branches end at $\partial B_0(f^{i+1}(p))$ (Chebyshev-like). Then for an appropriate choice of g all periodic orbits in $B_0(O(p))$ are hyperbolic repelling.

If a periodic orbit $O(p)$ is one side attracting we make it attracting by a saddle-node bifurcation thus reducing the construction to the previous case. $\square$

Here is the proof of another fact (a standard one, put here for completeness) mentioned in Introduction:

Proposition A.5. *For every* $(f, K, \mathbf{U}) \in \mathscr{A}^{\mathrm{BD}}$*, the set of periodic points in* $\mathbf{U}$ *that are not hyperbolic repelling is finite, if we count an interval of periodic points as one point. Such a non-degenerate interval cannot intersect* K*.*

Proof. By BD, for each periodic point $p \in \mathbf{U}$ attracting or indifferent attracting from one side, the immediate basin of attraction $B_0(O(p)))$ contains a critical point or its boundary contains a point belonging to $\partial\mathbf{U}$, and there is only a finite number of such points. Otherwise for g being the branch of $f^{-m(p)}$ such that $g(p) = p$ and $m(p)$ is the minimal period of p, the ratio $|(g^n)'(x)/(g^n)'(p)|$ would be arbitrarily large for n large and appropriate x such that the branch g^n exists on $B(p, 2|x - p|)$. If $O(p)$ is repelling but $|(f^{m(p)})'| = 1$, then BD also fails for backward branches of iterates of g by a simple calculation.

In the remaining case there would be attracting or indifferent attracting from one side periodic points (with the same or doubled period) arbitrarily close to p, contradicting the fact already proved that there is only a finite number of them.

If an interval T of periodic points contains $x \in K$ then $x \in \partial T$ since otherwise it has a neighbourhood $W \subset K$ on which f^m is identity for a period m. Then by topological transitivity x is isolated in K, which is not possible by Lemma 2.1. If $x \in \partial T \cap K$, then $(\mathrm{interior}\, T) \cap K = \emptyset$. Then x must be a limit of non-repelling periodic orbits from the side T' different from T since otherwise it either attracts from the side of T' so it is isolated in K which contradicts topological transitivity, or it repells from that side that contradicts BD (similarly to the non-hyperbolic repelling case above). But this contradicts the finiteness of $A(f) \cup \mathrm{Indiff}(f)$ which we have already proved.

□

Remark A.6. Notice that Darboux property allows to consider the factor of $f|_{\mathrm{conv}(\widehat{I} \cap K)}$, where "conv" means "convex hull", by contracting to points all closures of the connected components of the $\mathrm{conv}(\widehat{I} \cap K) \setminus \bigcup_{j=1,\dots,m} \mathrm{conv}(\widehat{I}^j \cap K)$ and their $f|_{\mathrm{conv}(\widehat{I} \cap K)}$-preimages. The resulting map g is a piecewise monotone map of interval $g : I \to I$. More precisely for the points that arise from contracted intervals $\widehat{I}^j$, the turning critical points, and the most left and most right end points of $\widehat{I}_K$ denoted according the order in $\mathbb{R}$ by $a_0 < a_1 < ... < a_{m'}$, where $m' \geq m$, the mapping g is strictly monotone and continuous on each $(a_j, a_{j+1 (\mathrm{mod}\, m')})$. It can have discontinuities at the points that arose from the contracted intervals $\widehat{I}^j$. If (f, K, U) is already reduced, then there is no interval whose each g^k-image belongs entirely to some $(a_{j_k}, a_{j_k + 1 (\mathrm{mod}\, m)})$. Such maps are called regular, see e.g. [HU]. Of course we loose the smoothness of the original f so this factor is useful only to study topological dynamics of f.

Now we prove the fact promised in Introduction, Remark 1.16, that allows in our theorems to assume only *topological transitivity* and to use in proofs formally stronger *weak exactness*.

Lemma A.7. *For every $(f, K) \in \mathscr{A}_+$ the mapping $f|_K$ is weakly topologically exact.*

Proof. (mostly due to M. Misiurewicz)

Step 1. Density of preimages

For $(f, K) \in \mathscr{A}$ (i.e. not assuming positive entropy) we prove that for all $x \in K$ the set $A_\infty(x) := \bigcup_{n\geq 0}(f|_K)^{-n}(x)$ is dense in K. This proves Proposition 2.4

Indeed, let T be an open interval intersecting K. Denote $\widehat{f} := f|_{\widehat{I}_K}$. Let

$$W := \bigcup_{n\geq 0} \widehat{f}^n(T),$$

while iterating we each time act with $\widehat{f}$ on the previous image intersected with $\widehat{I}_K$, the domain of $\widehat{f}$. The set $W \cap K$ is dense in K. Indeed, this set coincides with $\bigcup_{n\geq 0}(f|_K)^n(T)$ since if a trajectory $x_0, ..., x_n = x$ is in $\widehat{I}_K$, then by the maximality of K this trajectory is in K, so $\widehat{f}$ is equal to $f|_K$ on it. Hence the density of $W \cap K$ follows from the topological transitivity of $f|_K$.

The components of $f(W)\cap\widehat{I}_K$ are contained in the components of $W\cap\widehat{I}_K$. There is a finite number (at most the number of the boundary points of $\widehat{I}_K$) of components of W touching $\partial\widehat{I}_K$. We call them "boundary components". This number is positive and the $\widehat{f}$-forward orbit of each of them touches or intersects $\partial\widehat{I}_K$ (i.e. a "cut" happens) in finite time, i.e. a "boundary component" returns to a "boundary component". Otherwise there would be a wandering component, or a periodic component W' with $\bigcup_n \widehat{f}^n(W')$ not having points of $\partial\widehat{I}$ in its closure, which is not possible by the topological transitivity of $f|_K$. So there is only a finite number of components of W (bounded by the sum of the times of the returns). This together with the density of W proves that W covers K except at most a finite set, hence the density of each $A(x)$ in K for x not belonging to a finite set $\mathscr{E} := \widehat{I}_K \setminus W$.

In fact $A(x)$ is dense in K for all $x \in K$. Indeed. If $\widehat{f}(y) = x \in \mathscr{E}$ then $y \in \mathscr{E}$, since W is forward invariant. Hence by the finiteness of $\mathscr{E}$ the point x is periodic. The orbit $O(x)$ is repelling or indifferent repelling from one side, since on the side it is not repelling it cannot be an accumulation point of K, and it cannot be isolated in K. If x has an f-preimage $z \in \widehat{I} \setminus O(x)$ then z, as non-periodic, belongs to W, hence $x \in W$, a contradiction. So there exits an open neighbourhood U of $O(x)$ such that the compact set $\widehat{f}(\widehat{I}_K \setminus U)$ is disjoint from $O(x)$. Denote $U' := \widehat{I}_K \setminus \widehat{f}(\widehat{I}_K \setminus U)$.

Thus, there exists $S \subset U'$, an open interval intersecting K, its forward orbit leaves U after some time m, with the sets $f^j(S)$ disjoint from S for $j < m$, and never again returns to U', hence never intersect S, contrary to the topological transitivity.

Now we prove weak exactness, provided $(f, K) \in \mathscr{A}_+$, that is $h_{\rm top}(f|_K) > 0$. The proof above does not yield this, since we do not have N (for interval exchange maps such N does not exist). We did not even proved that in $K = \bigcup_{n=0}^{\infty} f^n(W \cap K)$ one can replace ∞ by a finite number.

Step 2. Semi-conjugacy to maps of slope β. First replace $f|_{\widehat{I}_K}$ by its factor g as in Remark A.6. The map g is piecewise strictly monotone, piece continuous, with m' pieces $I_j = (a_j, a_{j+1}), j = 1, ..., m'$. Denote $\bigcup_j \{a_j\}$ by a^h.

If g is continuous, then by Milnor-Thurston Theorem, see [MT] or [ALM, Theorem 4.6.8], g is semi-conjugate to some h, piecewise continuous, with constant slope, $\log |h'| = \beta = h_{\rm top}(f|_K) > 0$, via a non-decreasing continuous map. In fact this semi-conjugacy is a conjugacy via a homeomorphism since g is topologically transitive, since $f|_K$ is. Milnor-Thurston Theorem holds also for g piecewise continuous, piecewise monotone, which is our case, with the proof as in [ALM], [Misiurewicz].

Step 3. Growth to a large size. As before (for f) we consider an open interval T and act by h. More precisely for $T = T_0$ in one of the intervals (a_j, a_{j+1}) define inductively T_{n+1} as a longest component of $h(T_n) \cap I_j$, $j = 1, ..., m'$. As long as T_n are short, $h(T_n)$ capture a_j (i.e the "cuts" happen) rarely. So, due to expansion by the factor $\beta > 1$ the interval T_n is larger than a constant d not depending on T, for n large enough. Fix n such that T_n contains a point in a^h in its boundary. It exists since otherwise $|T_n|$ would grow to ∞. Denote this n by $n(T)$.

Step 4. Cover except a finite set. We go back to f and denote the semi-conjugacy from $f|_K$ to h (via g) by π. We consider now f piecewise strictly monotone, that is $\widehat{I}_K$ is cut additionally at turning critical points. We keep the same notation f and $\widehat{I}_K = \bigcup \widehat{I}^j$ except that now $j = 1, ..., m'$ rather than m, compare Remark A.6. We write $\widehat{I}^j = [a^f_{j,0}, a^f_{j,1}]$. It is useful now to consider $\breve{f} = f|_{\bigcup_j (a^f_{j,0}, a^f_{j,1})}$, i.e. restriction to the union of the open monotonicity intervals. Denote $\bigcup_{j,\nu} \{a^f_{j,\nu}\}$ by a^f. Finally let $d' > 0$ be such a constant that if $|x - y| < d'$ then $|\pi(x) - \pi(y)| < d$.

Consider open intervals $T^{j,\nu} \subset \widehat{I}^j$ of length d' adjacent to $a_{j,\nu}$. Set, compare Step 1,

$$\breve{W}^{j,\nu} := \bigcup_{n \geq 0} \breve{f}^n(T^{j,\nu}) \setminus a^f, \tag{A.1}$$

subtracting each time a^f while iterating $\breve{f}$.

By Step 1, $\breve{W}^{j,\nu}$ covers K except at most a finite set $\breve{\mathscr{E}} \subset K$. (It does not matter in Step 1 whether we deal with $\widehat{f}$ or $\breve{f}$, but now we prefer $\breve{f}$ to deal with open sets.)

Step 5. Doubling. Now we shall prove that in the union in (A.1) one can consider in fact a finite set of n's and if we do not subtract a^f and act with f it covers K.

Notice that $\breve{\mathscr{E}} \supset a^f$, by the definitions. Double each point of $\breve{\mathscr{E}}$ which is the both sides limit (accumulation) of K. More precisely, consider the disjoint union $\widehat{K}$ of the compact sets of the form $S \cap K$, covering K, where S's are closed intervals with ends in $\breve{\mathscr{E}}$ and pairwise disjoint interiors. Denote the projection from $\widehat{K}$ to K gluing together the doubled points by $\hat{\pi}$. Denote $\hat{\pi}^{-1}(\breve{\mathscr{E}})$ by $\hat{\mathscr{E}}$ and more generally $\hat{\pi}^{-1}(X)$ by $\hat{X}$ for any $X \subset K$. Denote the lift of $f|_K$ to $\widehat{K}$ by F. Notice that maybe it is not uniquely defined at points not doubled, whose f-images are doubled. We treat F as 2-valued there.

F maps $\widehat{K}$ onto $\widehat{K}$ by topological transitivity of $f|_K$, similarly to $f|_K$ mapping K onto K see Lemma 2.1. Suppose $x \in \hat{\mathscr{E}}$ and $F(y) = x$. If $y \notin \hat{\mathscr{E}}$, then $y \in \widehat{W \cap K}$, where $W := \breve{W}^{j,\nu}$. Then $x \in F(\widehat{W \cap K})$, moreover x is in the interior of this set in $\widehat{K}$ since $f|_K$ is open in a neighbourhood of $\hat{\pi}(y)$.

If $y \in \hat{\mathscr{E}}$, then consider $z \in \widehat{K}$ such that $F(z) = y$. If $z \notin \hat{\mathscr{E}}$ then as above y is in the interior of $F(\widehat{W \cap K})$. Then x is in the interior of $F^2(\widehat{W \cap K})$, but for this we use the fact that F is an open map in a neighbourhood of y due to doubling at $\hat{\pi}(x)$. We continue and if we have an F-backward trajectory of x in $\hat{\mathscr{E}}$, then it is periodic and we arrive at a contradiction with topological transitivity of $f|_K$ as in Step 1.

Finally by the compactness of $\widehat{K}$ we can choose finite subcovers from open covers, hence, after projecting $\widehat{K}$ back to K we can write

$$K = \bigcup_{n=0}^{N(j,\nu)} (f|_K)^n (T^{j,\nu} \cap K).$$

We end the proof by setting $N = \max_{j,\nu} N(j,\nu)$.

Since $f^{n(T)}(T)$ contains an interval of the form $T^{j,\nu}$ by the definitions of d' and d, then

$$K = \bigcup_{i=0}^{N} (f|_K)^{n(T)+i}(T \cap K).$$

□

Therefore the following holds

Proposition A.8. *If $(f,K) \in \mathscr{A}$ then the properties: $h_{\mathrm{top}}(f|_K) > 0$ (i.e. $(f,K) \in \mathscr{A}_+$) is equivalent to weak topological transitivity of $f|_K$.*

This follows immediately from Lemma A.7 and the following general fact easily following from the definition of topological entropy.

Lemma A.9. *Let $f : K \to K$ be a continuous map of a metric compact non one-point space K. Then weak topological exactness implies positive topological entropy.*

Appendix B. Uniqueness of equilibrium via inducing

We shall prove here uniqueness of equilibrium state, asserted in Theorem A, using our inducing construction.

Given a nice couple $(\widehat{V}, V)$ and the induced map F as before, consider the following subsets of K: $\mathscr{K}(V)$, $K(F)$, defined before, and

$$Q(F) := \{z \in V \cap K : \exists \text{ infinitely many returns to } K, \text{ but no good returns}\}.$$

We have $\nu(\mathscr{K}(V)) = 0$ since otherwise, by forward invariance of $\mathscr{K}(V)$ and ergodicity, ν is supported on $\mathscr{K}(V)$ and $h_\nu(f) - t\chi_\nu(f) = P(f,t) > P(f|_{\mathscr{K}(V)}, -\log|f'|)$, the latter inequality by Part 1.1 of Proof of Lemma 7.1, compare [PR-L2, Lemma 6.2]. This contradicts the opposite Variational Principle inequality $h_\nu(f) - t\chi_\nu(f) \le P(f|_{\mathscr{K}(V)}, -\log|f'|)$.

We conclude for the "basin" $\mathscr{B}\mathscr{K}(V) := \bigcup_{j=0}^\infty f^{-j}(\mathscr{K}(V))$, using the f-invariance of ν, that $\nu(\mathscr{B}\mathscr{K}(V)) = 0$.

We prove now that also $\nu(Q(F)) = 0$. Denote $\psi := \mathrm{Jac}_\nu(f)$, Jacobian in the weak sense, that is a ν integrable function such that (1.4) holds after removal from K a set Y of zero ν measure (i.e. 1.4 holds for A satisfying additionally $A \cap Y = \emptyset$). Such ψ exists under our assumptions and moreover Rokhlin entropy formula $h_\nu(f) = \int \log\psi d\nu$ holds, see e.g. [PU, Proposition 2.9.5 and 2.9.7]. (One sided, countable, even finite, generator exists by weak exactness and the finiteness of the number of maximal monotonicity intervals of f in the definition of generalized multimodal maps.)

By Birkhoff Ergodic Theorem for every $a, b > 0$ there exists a set $K_{\nu,b} \subset K$ such that $\nu(K_{\nu,b}) > 1 - b$ and for all n large enough and all $y \in K_{\nu,b}$

$$\Big|\frac{1}{n}\sum_{j=0}^{n-1} \log\psi(f^j(y) - \int \log\psi\, d\nu\Big| < a.$$

For each integer $n \ge 0$ denote $Q_n := \{y \in K \cap V : x = f^n(y) \in V$, and y is a bad iterated preimage of x of order $n\}$. By Lemma 6.13 the set Q_n is covered by at most $\exp(\epsilon n)$ pull-backs of V of order n. Hence

$$\nu(Q_n \cap K_{\nu,b}) \le \exp(\epsilon n)\exp(-(h_\nu(f) - a)n).$$

We have $Q(F) = \bigcap_{m=0}^\infty \bigcup_{n \ge m} Q_n$ hence $\nu(Q(F) \cap K_{\nu,b}) = 0$ if $\epsilon + a < (h_\nu(f)$. Since b can be taken arbitrarily close to 0 we obtain $\nu(Q(F)) = 0$.

Let now $z \in V \cap K$ has infinitely many returns to V but only a finite number of good return times. Let n_0 be the largest one. Assume it is positive. Denote $y := f^{n_0}(z)$. If $n_1 < n_2 < ...$ are consecutive times of return of z to V bigger than n_0, then either for their subsequence n_{j_k}, for each k the point y is a bad iterated preimage of order $n_{j_k} - n_0$, hence it belongs to $Q(F)$, or there is an arbitrarily large n_j such that $n_j - n_0$ is a good time for y. Since n_j is not good for z we conclude that for $\widehat{W}$ being the pull-back of $\widehat{V}$ for f^{n_j} one of the sets $f^i(\widehat{W}), i = 0, 1, ..., n_0 - 1$ intersects $S'(f, K)$. Since diameters of $\widehat{W}$ tend to 0 as $j \to \infty$ we conclude that $z \in \bigcup_{i=0}^{n_0-1} f^{-i}(S'(f, K))$. Compare the respective reasoning in Subsection 7.2.

Thus, for the "basins" $\mathscr{B}Q(F) := \bigcup_{n=0}^{\infty} f^{-n}Q(F)$ and $\mathscr{B}S'(f, K) := \bigcup_{n=0}^{\infty} f^{-n}S'(f, K)$

$$\nu(\mathscr{B}Q(F) \setminus \mathscr{B}S'(f, K)) = 0.$$

Notice finally that ν has no atoms in $S'(f, K)$. Indeed, by the invariance of ν for every $z \in K$ it holds $\nu(\{f(z)\}) \geq \nu(\{z\})$. Hence if ν had an atom in $S'(f, K)$ then $\bigcup_{n=0}^{\infty} f^n(S'(f, K)$ would be finite and in consequence due to ergodicity it is supported on a periodic orbit. Then $h_\nu(f) = 0$ what contradicts the assumption it is an equilibrium for $t_- < t < t_+$ (see the beginning of the proof of Uniqueness).

We conclude that the "basin" $\mathscr{B}K(F) := \bigcup_{n=0}^{\infty} f^{-n}K(F)$ has full measure ν, hence $\nu(K(F)) > 0$.

Consider the inverse limit (natural extension) $(\widetilde{K}, \widetilde{f}, \widetilde{\nu})$. Consider also the inverse limit $(\widetilde{K}(F), \widetilde{F})$ (just topological). Define Π the projection of $\widetilde{K}$ to K defined for every f-trajectory $y = (y_j)_{j \in Z}$, by $\Pi(y) = y_0$. Denote by ι the embedding of $\widetilde{K}(F)$ in $\widetilde{K}$ defined by the completing of each F-trajectory to f-trajectory. Notice that $m = m(\Pi(y))$ for $y \in \widetilde{K}(F)$ is time of the first return to $\iota\widetilde{K}(F)$ for the mapping $\widetilde{f}$. Indeed, m is by definition the least good return time of $\Pi(y)$ to V. So suppose there is $j : 0 < j < m$ a time of return of y to $\widetilde{K}(F)$. Consider $k < j$ such that $j - k$ is a good time for y_k. We have $k > 0$ since otherwise j is a good time for $\Pi(y) = y_0$, due to $f^j \circ f^{-k} = f^{j-k}$. But by $\widetilde{f}^j(y) \in \widetilde{K}(F)$ there are infinitely many $k < j$ such that $j - k$ is a good time for y_k, a contradiction.

Denote the normalized restriction of $\widetilde{\nu}$ to $\widetilde{K}(F)$ by $\widetilde{\nu}*$ Hence $m \circ \Pi$ is $\widetilde{\nu}*$-integrable on $\iota\widetilde{K}(F)$, more precisely $\int_{\iota\widetilde{K}(F)} m \, d\widetilde{\nu}* = 1$ by ergodicity.

Define $\nu = \iota_*^{-1}\Pi_*(\widetilde{\nu}*)$. By above, the function m is $\nu*$-integrable. Notice that $\nu*$ is an equilibrium measure for F and the potential $-t \log |F'| - mP(t)$. This follows from the fact that $\widetilde{\nu}$ is an equilibrium for f and $-t \log |f'|$ and the calculation (7.4) done in a different order, for $\nu, \nu*$ playing the role of ρ', ρ.

Now we refer to [MU, Theorem 2.2.9] saying in particular that the equilibrium for F and our Φ is unique. Thus $\nu* = \rho$, hence $\nu = \rho'$, due to the formula (7.3) for both measures. Notice that (7.3) gives ν because it makes

$\widetilde{\nu}$ out of $\widetilde{\nu}*$ because m is the time of the first return map for $\widetilde{F}$. The proof of Uniqueness is finished.

Appendix C. Conformal pressures

Here we shall define and discuss various versions of conformal pressures announced in Remark 1.30 and compare them to $P(K,t)$, thus complementing Theorem B.

Definition C.1. Recall after Introduction that similarly to the complex case [PR-L2] and [P-conical] define *conformal pressure* for $t \in \mathbb{R}$ by

$$P_{\mathrm{conf}}(K,t) := \log \lambda(t),$$

where, see 1.7,

$$\lambda(t) = \inf\{\lambda > 0 : \exists \mu \text{ on } K \text{ which is } \lambda |f'|^t - \text{conformal}\}. \tag{C.1}$$

For $\phi = \lambda |f'|^t$ as above, we call also μ a (λ,t)-conformal measure for f on K.

It is immediate, see the end of Proof of Lemma C.3 that

Lemma C.2. *For all t real and λ positive, if μ is a (λ,t)-conformal measure on an f-invariant set $K \subset U$ for $f : U \to \mathbb{R}$ a generalized multimodal map, with $f|_K$ being weakly exact, then μ is positive on all open subsets of K.*

If the equation in 1.4 holds only up to a constant ϵ, namely

$$|\mu(f(A)) - \int_A \phi \, d\mu| \le \epsilon$$

for every Borel A where f is injective, we say the measure μ is ϵ-(λ,t)-conformal.

In the interval case, it is profitable to assume only the inequality

$$\mu(f(x)) \ge \lambda |f'|^t \mu(x) \tag{C.2}$$

for all $x \in \mathrm{NO}(f,K)$, i.e. at which $f|_K$ is not open, see Definition 1.17. and 1.4 only for all $A \subset K$ disjoint from $\mathrm{NO}(f,K)$.

We call such μ a (λ,t)-*conformal* measure.*

Assume $(f,K) \in \mathscr{A}$. Then the set $\mathrm{NO}(f,K)$, hence the set where only the inequality C.2 holds instead of the equality, is finite, see Lemma 2.2.

In particular a (λ,t)-*conformal* measure* can have atoms at f-images of turning critical points, but not at inflection critical points that are not end points, for $t > 0$. (Compare the notion of almost conformal measures in [HU] or compare [DU]).

Unfortunately we do not know whether there always exist at least one measure with Jacobian $\lambda |f'|^t$, to define $\lambda(t)$ in 1.7. So we do not know whether always $P_{\mathrm{conf}}(K,t)$ makes sense. (We can prove however this existence for $t_- < t < t_+$, see Corollary 1.29 and Section 7.)

We define the *conformal star-pressure* $P^*_{\rm conf}(K,t)$ as in 1.7 but allowing the measures μ to be (λ,t)-conformal*. This always makes sense. Indeed, as we shall prove in Proposition C.4 below, the set of measures in 1.7 defining $P^*_{\rm conf}(K,t)$ is non-empty.

Lemma C.3. *Let $(f,K)\in\mathscr{A}_+$Assume $t\in\mathbb{R}$. Suppose that μ is a (λ,t)-conformal* measure on K. Then either μ is positive on all open (in K) subsets of K or μ is supported on a finite weakly S'-exceptional set $E\subset K$ or S'-exceptional in case there are no singular connections (see Definition 1.17).*

If μ is (λ,t)-conformal on K then it is positive on all open sets in K.

Proof. (compare [MS, Lemma 3.5]) Suppose that there exists open $U\subset K$ with $\mu(U)=0$. Then, by the weak exactness of f on K, see Lemma A.7, implying that there is $n>0$ such that $\bigcup_{j=0}^n f^j(U)=K$, we conclude that μ is supported by the finite set $\Sigma:=\bigcup_{j=1}^n\bigcup_{t=0}^{j-1} f^{j-t}(f^t(U)\cap {\rm NO}(f,K))$.

Σ is contained in a weak S'-exceptional set E. Otherwise there would exist $x\in\Sigma$ and its infinite backward trajectory $G\subset K$ omitting ${\rm NO}(f,K)$, hence there exists $z\in G$ such that $A(z):=\bigcup_{j=0}^\infty (f|_K)^{-j}(z)$ omits ${\rm NO}(f,K)$ (Exercise). It is dense in K hence it hits U. In consequence by 1.4 μ would have atoms in U, so $\mu(U)>0$, a contradiction.

(Instead of finding $A(z)$ omitting ${\rm NO}(f,K)$, one can deal as in Proof of Proposition 2.7 finding $z\in G$ such that $A_n(z)$ omits ${\rm NO}(f,K)$ for $n=n(\epsilon)+N$ for U being of the form $B(x,\epsilon)\cap K$.)

If there are no singular connections, then by the (λ,t)-conformality* of μ by the above consideration Σ is contained in an f-forward invariant weak S'-exceptional set, hence by definition in an S'-exceptional set.

If μ is conformal and $\mu(U)=0$, then $\mu(K)=0$ since $\mu(f^j(U))=0$ for all $j\ge 0$. Indeed, by the conformality of μ and $t\ge 0$, there is no way to get positive measure (in particular atoms) in $f^j(U)$. $\square$

Proposition C.4. *For every $(f,K)\in\mathscr{A}_+^{\rm BD}$, for all $t\ge 0$ and $\lambda=\exp P(K,t)$ there exists a (λ,t)-conformal* measure and the inequalities $P^*_{\rm conf}(K,t)\le P(K,t)\le P_{\rm conf}(K,t)$ hold, provided the latter pressure makes sense. If (f,K) is not weakly S'-exceptional, or not S'-exceptional in case there are no singular connections, then the equality $P(K,t)=P^*_{\rm conf}(K,t)$ holds.*

Proof. The strategy of the proof is the same as, say, in [PU, Proof of Theorem 12.5.11, Parts 2 and 3] in the complex case.

First we prove the existence of a $\lambda,t)$-conformal* measure for $\lambda=\exp P_{\rm tree}(K,z_0,t)$ for an arbitrary safe and expanding point $z_0\in K$ hence $P^*_{\rm conf}(K,t)\le P(K,t)$. Consider a sequence of measures

$$\mu_n=\sum_{k=0}^\infty\sum_{x\in(f|_K)^{-k}(z_0)} D_x\cdot\phi_k\cdot\lambda_n^{-k}|(f^k)'(x)|^{-t}/\Sigma_n,$$

where $\lambda_n\searrow\lambda$ for $\lambda=\exp P_{\rm tree}(K,z_0,t)$, where each D_x denotes Dirac measure at x. Each Σ_n is the normalizing denominator such that μ_n is a probability measures. The numbers ϕ_k are chosen so that $\phi_{k+1}/\phi_k \to 1$ as $k \to \infty$ and $\Sigma_n \to \infty$ as $n \to \infty$. Compare [PU, Lemma 12.5.5]

All μ_n are ϵ-(λ_n, t)-conformal for an arbitrary $\epsilon > 0$ and all n respectively large. Define μ as a weak* limit. This procedure to get μ is called Patterson-Sullivan method.

Then μ has Jacobian $\lambda|f'|^t$ for $\lambda = P_{\text{tree}}(K, z_0, t)$ in the above sense for all $A \subset K$ not containing points belonging to $\mathrm{NO}(f, K)$, in particular turning critical points. For each critical point $c \in K$ we obtain for all n and r small enough $\mu_n(f(B(c,r)) \leq 2 \sup \lambda_k (2r)^t + \epsilon$, where $\epsilon > 0$ is arbitrarily close to 0, compare [PU, Remark 12.5.6]. Therefore for $t > 0$ we have a uniform bound for $\mu_n(f(B(c,r))$, arbitrarily small if r and ϵ are small. If c is an inflection point for f then f is open at c, i.e. $f(B(c,r))$ is a neighbourhood of $f(c)$, therefore $\mu(f(c)) = 0$ yielding the conformality of μ also at c. If c is a turning point then $f(B(c,r))$ need not be a neighbourhood of $f(c)$ in K and the mass $\mu(f(c))$ can be positive, coming from the other side of $f(c)$ than the 'fold' $f(B(c,r))$. Therefore μ satisfies merely C.2 at c. The same concerns end points belonging to $\mathrm{NO}(f, K)$. Thus we end with P^* rather than P.

To prove $P(K,t) \leq P^*_{\text{conf}}(K,t)$, or merely $P(K,t) \leq P_{\text{conf}}(K,t)$ in the general case, we consider the version $P_{\text{hyp}}(K,t)$ of $P(K,t)$. For every (λ, t)-conformal* measure μ on K we consider a small $U \subset K$ open in K intersecting a hyperbolic isolated $X \subset K$, where $P_{\text{hyp}}(K,t)$ is almost attained by $P(f|_X, -t \log |f'|)$, see Definition 1.18. Then, for $\mu(U) > 0$, see Lemma C.3 the rest of the proof is identical as in [PU, Proof of Theorem 12.5.11, Parts 2], giving $P(f|_X, -t\log|f'|) \leq \log \lambda$. So if (f, K) is not weakly S'-exceptional (or S'-exceptional and there are no singular connections) the proof of $P(K,t) \leq P^*_{\text{conf}}(K,t)$ is finished.

In the exceptional case the proof of $P(K,t) \leq P_{\text{conf}}(K,t)$ is the same, using $\mu(U) > 0$.

□

For $t < 0$ and conformal pressures we follow [PR-LS2, Appendix A.2]. Notice again (compare Proof of Lemma 4.4 the case $t \leq 0$) that the pressure $P_{\text{tree}}(K,t)$ is the classical pressure since $|f'|^{-t}$ is continuous, i.e.

$$P(K,t) = P(f_| K, -t \log |f'|).$$

To see this consider $P_{\text{var}}(K,t)$. See [PR-LS2, Theorem A.7] and [Keller, Theorem 4.4.1] for the variational principle for potential functions with range in $\mathbb{R} \cup -\infty$.

For $t < 0$ it is natural to consider a different definition of conformal pressure, called in [PR-LS2] backward conformal pressure. In the complex setting each conformal measure is an eigenmeasure for the operator dual to transfer operator (Ruelle operator, with weight function $|(f')|^{-t}$. In the complex setting the transfer operator is a bounded operator on the space of continuous functions since f is open. Unfortunately for the interval multimodal maps this is not so; the transfer operator on continuous functions can have the range not in continuous functions, since f need not be an open map. The discontinuity of the image cannot be caused by critical values, since at critical points the weight function is 0 (remember that $t < 0$). It can be caused by noncritical end points not mapped to end points.

Definition C.5. For $t < 0$ define the (λ, t)-backward conformal pressure by

$$P_{\mathrm{Bconf}}(K, t) := \log \lambda(t),$$

where

(C.3)

$$\lambda(t) := \inf\{\lambda > 0 : \exists \mu \text{ backward conformal on } K \text{ with Jacobian } \lambda^{-1}|f'|^{-t}\}$$

and μ backward conformal means here that for every Borel set $A \subset K$ on which f is injective

$$\mu(A) = \int_{f(A)} \lambda^{-1} |f'(f|_A)^{-1}(x))|^{-t} d\mu(x). \tag{C.4}$$

For $t < 0$, analogously to the case $t \geq 0$, we define also *backward conformal* pressure* by

$$P^*_{\mathrm{Bconf}}(K, t) := \log \lambda(t),$$

where in C.3 we allow measures satisfying C.4 for A not containing points of $\mathrm{NO}(f, K)$ and satisfying merely C.2 at these points. This can be written in the form

$$\mu(x) \leq \lambda^{-1}|(f')|^{-t}\mu(f(x)). \tag{C.5}$$

So in particular for x a critical point $\mu(x) = 0$ due to $t < 0$, hence we get in fact the equality in C.5. However there is no reason to this equality at the non-critical points of $\partial(\widehat{I}_K)$ where f is not open on K.

We call such measures (λ, t)-*backward conformal**.

Remark C.6. Formally the difference between P^*_{conf} and P^*_{Bconf} is only that C.5 rewritten in the form of C.2, that is $\mu(f(x)) \geq \lambda |f'|^t \mu(x)$, does not make sense for $t < 0$ as we have the undefined product $0 \cdot \infty$ on the right hand side there. Therefore the definition of P^*_{Bconf} includes more measures.

Similarly to the case $t \geq 0$, we prove for $t < 0$ the following

Lemma C.7. *Let $(f, K) \in \mathscr{A}_+^{\mathrm{BD}}$. Assume $t < 0$. Suppose that μ is a (λ, t)-backward conformal* measure on K. Then either μ is positive on all open (in K) subsets of K or μ is supported on a finite weakly S'-exceptional set $E \subset K$, or S'-exceptional in case there are no singular connections (see Definition 1.17).*

Hence, analogously to Proposition C.4, we obtain

Proposition C.8. *For every $(f,K) \in \mathscr{A}_+^{\mathrm{BD}}$, for all $t < 0$ and $\lambda = \exp P(K,t)$ there exists a (λ,t)-backward conformal* measure and the inequality $P^*_{\mathrm{Bconf}}(K,t) \le P(K,t)$ holds. If (f,K) is not weakly S'-exceptional, or not S'-exceptional in case there are no singular connections, then the equality $P(K,t) = P^*_{\mathrm{conf}}(K,t)$ holds. Moreover infimum in C.3 can be omitted since all λ's coincide.*

Proof. The proof of $P_{\mathrm{hyp}}(K,t) \le P^*_{\mathrm{Bconf}}(K,t)$ for f on K non-exceptional is the same as the proof of PropositionC.4 and relies on $\mu(U) > 0$ asserted in Lemma C.7.

The proof of $P^*_{\mathrm{Bconf}}(K,t) \le P_{\mathrm{tree}}(K,z_0,t)$ is also the same as in the proof of PropositionC.4 via the construction of an appropriate μ. It holds also for exceptional f.

To see all λ's coincide one repeats the calculation in [PR-LS2, A.3] for every (λ,t)-backward conformal* measure μ on K. Namely

$$\int \sum_{x \in (f^n|_K)^{-1}(z)} \lambda^{-n} |(f^n)'(x)|^{-t} \, d\mu(z) \ge \mu(K) = 1.$$

For $\delta > 0$ arbitrarily small, for every z under the integral and n large enough,

$$\exp n(P(f|_K, -t\log|f'|) + \delta) \ge \sup_{z\in K}\Big(\sum_{x\in(f^n|_K)^{-1}(z)} |(f^n)'(x)|^{-t},$$

see [P-Perron, Lemma 4], compare Lemma 4.4, hence we get
$\log\lambda \le P(f|_K, -t\log|f'|)$. □

The next proposition, mainly summarizing the previous ones, says that there are conformal* measures positive on open sets, provided only there are no singular connections. In Section 7 we shall have proved for $t_- < t < t_+$ stronger Corollary 1.29, where in particular the assumption no singular connections was not needed (though for simplicity used in the proof), using more advanced techniques.

Proposition C.9. *For $(f,K) \in \mathscr{A}_+^{\mathrm{BD}}$ not S'-exceptional for all real t, or without singular connections (allowing the S'-exceptional case) for $t : t_- < t < t_+$, there exists on K a (λ,t)-conformal* measure for $t \ge 0$ or (λ,t)-backward conformal* measure for $t < 0$, positive on open sets, zero on all S'-exceptional sets, or weakly S'-exceptional if there are no singular connections, with $\log\lambda = P(K,t)$.*

Proof. The non-exceptional case has been already dealt with. In the exceptional case consider μ constructed by Patterson-Sullivan method as in Proofs of Proposition C.4 and Proposition C.8. Hence the resulting measure μ is (λ,t)-conformal* or (λ,t) backward conformal* for λ satisfying $\log\lambda = P(K,t)$. If $\mu(U) > 0$ fails, then μ is supported in an S'-exceptional set E by Lemmas C.7 and C.7.

Now if the exceptional set E has positive measure μ, by the forward invariance and finiteness of $E_{\max}$ there is a periodic orbit $O(p) \subset E_{\max}$ with

$\mu(O(p)) > 0$. Let m be its period. Hence $\lambda^m |(f^m)'(p)|^t \geq 1$. Therefore $\log\lambda \geq -t\chi(p)$, where $\chi(p) := \frac{1}{m}\log|(f^m)'(p)|$.

In fact the equality $\log\lambda \geq -t\chi(p)$ holds, since by Lemma 2.2 $O(p)$ is disjoint from $S'(f,K)$ so μ is conformal.

Thus

$$P(t,K) = -t\chi(p)$$

hence $t \leq t_-$ or $t \geq t_+$. (Compare the proof of [MS, Lemma 3.5].)

□

Remark C.10. For $t < 0$ for exceptional maps to have the equality of pressures it is sometimes appropriate to consider supremum instead of infimum in the definition C.3. See [PR-LS2, Remark A.8]

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† Feliks Przytycki, Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00956 Warszawa, Poland.

E-mail address: `feliksp@impan.gov.pl`

‡ Juan Rivera-Letelier, Facultad de Matemáticas, Campus San Joaquín, P. Universidad Católica de Chile, Avenida Vicuña Mackenna 4860, Santiago, Chile.

E-mail address: `riveraletelier@mat.puc.cl`