Stochastic Löwner Evolution

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Chapitre 1

Introduction

The general aim behind the following lectures is to understand deterministic or probabilistic growth processes coming from physics. While of course these processes naturally occur in the 3D space we will restrict to the planar case where the rich theory of analytic functions is of great help. Already in this case many things remain mysterious but at least there are some results, while the 3D case is totally unknown. Let us start with one of the most famous but also still mysterious DLA-model of stochastic growth. Let K_1 be the closed unit disk. We let a disk of radius 1 walk at random from infinity in the plane and we stop it as soon as it touches K_1. The union of the two disks then forms a compact K_2; we let them start another disk of radius one walk at random from infinity and by induction we obtain in this manner a random sequence of compact sets K_n. What can be said about this sequence; does it converge in law? This is far from being known. How does the diameter scale as n goes to infinity. Numerical simulations suggest that $\operatorname{diam}(K_n) \sim n^{1/d}$ for $d = 1.71...$ Kesten has proved that $d \geq 1.5$ (if it exists). In mathematical terms the set K_{n+1} is obtained from K_n by first choosing a point on ∂K_n with the law given by harmonic measure and then attaching a disk at this point. It may be argued that this is not always possible so we modify the model by using conformal mapping. Define for $\theta \in [0,1[$ and $\delta > 0$ the map $h_{\delta,\theta}$ as the conformal mapping from the complement of the unit disk onto the complement of the unit disk minus the segment $[e^{2i\pi\theta},(1+\delta)e^{2i\pi\theta}]$ with Laurent expansion $az + .., a > 0$ at ∞ (we say that the conformal mapping is normalized): the growth process is then given by a starting cluster K_0 and if K_n id defined by a normalized conformal mapping φ_n then K_{n+1} is defined by its normalized mapping $\varphi_{n+1} = h_{\delta_n,\theta_n}$ with some choice on the constants. For this model to mimic a DLA one must adjust the constants so that the image by φ_n of the segment $[e^{2i\pi\theta},(1+\delta)e^{2i\pi\theta}]$ has a fixed size: however the model makes sense for any choice of the constants involved. We can write

$$\varphi_{n+1}(z) - \varphi_n(z) = z\frac{\varphi_n(h_n(z)) - \varphi_n(z)}{h_n(z) - z}\frac{h_n(z) - z}{z}.$$

If we assume that δ_n is infinitesimal then this formula reads (writing now t instead of n),

$$\frac{\partial\varphi(z)}{\partial t} = z\frac{\partial\varphi(z)}{\partial z}p(z,t)$$

where it is easily seen that $p(.,t)$ is a holomorphic function with negative real part. This equation is known as Löwner equation, which is the equation describing the growth of connected clusters. The importance of Löwner equation lies in the fact that it has a converse: one can start with an equation (1) where $p(.,t)$ is a one parameter family of holomorphic functions with negative real part and it is true that, under some mild regularity conditions, its solutions actually describe a growth process. This fact is going to be the heart of the matter of these notes. In a first part we will develop all the necessary background in complex analysis that is needed to develop the theory of Löwner differential equation and discuss all its classical consequences. The second part (and the most important one will be devoted to the most recent far reaching consequences of this theory. To start the story we recall that Löwner equation is "'driven"' by a one parameter family of holomorphic functions with negative real parts. It is classical that these functions can be described as Cauchy integrals of negative finite measures. An important class of growth processes is the one for which the negative measure is minus a Dirac measure at point ξ_t; this process is called the Löwner process driven by the function ξ. The wonderful idea of Schramm was to take as driving functions a function of the form $\sqrt{\kappa}B_t$ where κ is a nonnegative real number and B_t is a standard Brownian Motion (BM) on the real line. This introduces stochastic calculus into the field and it appears that some knowledge of Itô calculus is necessary to understand SLE_κ (the name of these processes). The second part of these notes will thus be devoted to an introduction to stochastic calculus, and this will be achieved by a non-specialist who begs for indulgence! The third part of these notes will be devoted to applications of the theory. It turns out that SLE_κ describes the scaling limit of many discrete models of statistical mechanics. The most famous one is critical percolation: consider a half-plane tiled with hexagons. Decide to color black the hexagons of the negative real axis and white those of the positive one. We start to walk along the edges of the hexagon at zero, going up: we meet an hexagon and we toss a coin to decide its colour. If it is black we go right and left if it is right; we continue in this way, with the difference that it may happen that the hexagon we arrive on is already coloured. But then we simply use the same rule and it will work since it can be easily seen by induction that the last piece of the path is always between two different colours. Moreover a moment's reflection shows that the path we obtain is simple and goes to infinity. The question we adress is: what happens when the mesh of the lattice converges to 0? The problem here is two fold (not speaking about the delicate problem of what we exactly mean by convergence). It was known that if we let the mesh go to 0 then it must converge to a SLE process if it is conformally invariant; then it has to be the trace of SLE_6 because of the locality property (see below). It has also been shown that SLE_2 is the scaling limit of LERW (loop-erased random walks) and many other similar results of this kind were either proved or conjectured. But perhaps the more spectacular achievment of this theory has been the proof of Mandelbrot conjecture: if U stands for the unbounded component of $\mathbb{C}\backslash B([0,1]$ where B is a planar Brownian Motion then almost surely ∂U has dimension 4/3. One of the tasks of these notes will be to provide a complete proof of this fact.

Chapitre 2

Complements of Complex Analysis

2.1 Simply connected plane domains

An arc in a metric space X is a continuous mapping γ from some interval $[a,b] \subset \mathbb{R}$ in X. Such an arc is said to be closed if $\gamma(a) = \gamma(b)$. Two arcs γ_1,γ_2 defined on the same interval $[a,b]$ and such that $\gamma_1(a) = \gamma_2(a), \gamma_1(b) = \gamma_2(b)$ are said to be homotopic if there exists $\Gamma : [a,b] \times [0,1] \to X$ continuous such that

$$\Gamma(a,t) = \gamma_1(a) \forall t \in [0,1], \Gamma(b,t) = \gamma_1(b) \, \forall t \in [0,1],$$

$$\Gamma(s,0) = \gamma_1(s) \, \forall s \in [a,b], \Gamma(s,1) = \gamma_2(s) \, \forall s \in [a,b].$$

Définition 2.1.1. *: The space X is called simply connected if it is connected and if every closed arc $\gamma : [a,b] \to X$ is homotopic to the constant arc $\gamma_0 : t \in [a,b] \mapsto \gamma(a)$.*

When X is a plane domain we have the following equivalent characterizations of simply connected domains:

Théorème 2.1.1. *: For a connected open subset Ω of $\mathbb{C}$ the following are equivalent:*
(1) Ω is simply connected,
(2) $\overline{\mathbb{C}} \backslash \Omega$ is connected,
(3) For any closed arc γ and any $z \notin \Omega$, $\mathrm{Ind}(z,\gamma)$=0.

We recall that $\mathrm{Ind}(z,\gamma)$ stands for the variation of the argument (mesured in number of turns) of $\gamma(t) - z$ along $[a,b]$. When γ is piecewise C^1 this quantity is also equal to

$$\frac{1}{2i\pi} \int_a^b \frac{\gamma'(s)\mathrm{d}s}{\gamma(s) - z} = \frac{1}{2i\pi} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}.$$

Let f be a holomorphic function defined on a simply connected subdomain Ω of $\mathbb{C}$. By global Cauchy theorem and (3)

$$\int_\gamma f(z)\mathrm{d}z = 0. \tag{2.1}$$

for every closed arc γ of Ω. Fix now $z_0 \in \Omega$. Since Ω is arcwise connected, for every $z \in \Omega$ there exists an arc $\gamma : [a,b] \to \Omega$ such that $\gamma(a) = z_0, \gamma(b) = z$. We define $F(z) = \int_\gamma f(z)\mathrm{d}z$. By (3.3)this definition is independent of the choice of the arc γ as soon as it joins z_0 to z inside Ω. It is easy to check that $F' = f$. We have thus proven that every holomorphic function in a simply connected domain admits a holomorphic anti-derivative.

Théorème 2.1.2. *: A domain Ω in $\mathbb{C}$ is simply-connected $\iff$ every holomorphic function in Ω has an anti-derivative.*

Proof :: We have just proven the $\Rightarrow$ part. To prove the converse let $z_0 \in \mathbb{C}\backslash\Omega$. Then $\frac{1}{z-z_0}$ is holomorphic in Ω and so has an anti-derivative. Cauchy's theorem applied to this function implies that $\mathrm{Ind}(z_0,\gamma) = 0$ for every closed arc in Ω and thus that Ω is simply-connected. Consider in particular a non-vanishing holomorphic function f and let w_0 be any complex number such that $e^{w_0} = f(z_0)$. Define g on Ω as the unique antiderivative of $\frac{f'}{f}$ such that $g(z_0) = w_0$. Then $(fe^{-g})' = (f' - g'f)e^{-g} = 0$ and thus $e^g = f$.We have thus proved that every non-vanishing holomorphic function in a simply-connected domain admits a holomorphic logarithm and thus also a holomorphic determination of its square root. This fact is a key point in the next theorem.

2.2 Riemann Mapping Theorem

Théorème 2.2.1. *(Riemann): Let Ω be a simply-connected proper subdomain of $\mathbb{C}$ and $w \in \mathbb{C}$.Then there exists a unique biholomorphic map $f : \Omega \to \mathbb{D}$ such that $f(w) = 0, f'(w) > 0$. (Here and in the future $\mathbb{D}$ will stand for the unit disk).*

Proof :

1) Uniqueness: Suppose f_1,f_2 do the job; then $h = f_2 \circ f_1^{-1} : \mathbb{D} \to \mathbb{D}, 0 \mapsto 0, h'(0) > 0$ and the same is true for h^{-1}: Schwarz lemma then implies that $h(z) = z, z \in \mathbb{D}$.

2) Existence: Let $E = \{f : \Omega \to \mathbb{D}\,\text{holomorphic and injective with} f(w) = 0, f'(w) > 0\}$. Let us first prove that E is not empty. To do so, we consider $z_0 \in \mathbb{C}\backslash\Omega$. Then $z \mapsto \frac{1}{z-z_0}$ is a non vanishing holomorphic function in Ω and thus admits a square root we denote g. Since g is open there exists $\varepsilon > 0$ such that $B(g(w),\varepsilon) \subset g(\Omega)$. Then we must have $B(-g(w),\varepsilon) \cap g(\Omega) = \emptyset$ because if not then would exist $\zeta \in \Omega$ such that $g(\zeta) \in B(-g(w),\varepsilon)$. Let $\Omega_1 = g^{-1}(B(g(w),\varepsilon)$; ζ cannot belong to Ω_1 because $B(g(w),\varepsilon) \cap B(-g(w),\varepsilon) = \emptyset$ but on the other hand there exists $\zeta' \in \Omega_1$ such that $g(\zeta') = -g(\zeta) \Rightarrow g(\zeta)^2 = g(\zeta')^2$ thus contradicting the injectivity of g^2. We can then define

$$f(z) = \frac{\varepsilon}{g(z) + g(w)}$$

which sends Ω into $\mathbb{D}$: precomposing with a judicious Möbius transformation, we get an element of E. If $f \in E$ we can consider

$$f^*(z) = f(w + zd(w,\partial\Omega))$$

mapping the unit disk into itself and 0 to 0. Applying Schwarz lemma to f^* we see that

$$f'(w) \leq \frac{1}{d(w,\partial\Omega)}$$

if $f \in E$. Let then $M = \sup\{f'(w), f \in E\}$ and (f_n) a sequence of elements of E such that $f_n'(w) \to M$. It is a normal family so, taking if necessary a subsequence we may assume that (f_n) converges uniformly on compact sets to some map f with $f(w) = 0, f(\Omega) \subset \overline{\mathbb{D}}$. Moreover $f'(w) = M$ so that in particular f is not constant. By Hurwitz theorem it must be injective and open and we have proven that $f \in E$.
To finish the proof it suffices to show that f is onto $\mathbb{D}$.
Suppose not: let $z_0 \in \mathbb{D} \backslash f(\Omega)$. Let

$$h(z) = \frac{z - z_0}{1 - \overline{z_0}z}$$

be an automorphism of the disk such that $h(z_0) = 0$. The mapping hof is one to one and non vanishing; there thus exists g holomorphic and injective in Ω such that $g^2 = h \circ f$. If k is the automorphism of the disk such that $k(g(w)) = 0, k'(g(w))g'(w) > 0$ then $\tilde{f} = k \circ g \in E$. We want to compute $\tilde{f}'(w)$. We compute $|h'(0)| = 1 - |z_0|^2$ and $|h(0)| = |z_0|$. Also

$$|k'(g(w)| = \frac{1}{1 - |g(w)|^2} = \frac{1}{1 - |z_0|}.$$

We have $2g(w)g'(w) = h'(0)f'(w)$ and hence

$$\tilde{f}'(w) = |k'(g(w)||g'(w)| = \frac{(1 + |z_0|)f'(w)}{2\sqrt{|z_0|}}.$$

But $1 + |z_0| > 2\sqrt{|z_0|} \Rightarrow \tilde{f}'(w) > f'(w) = M$, contradicting the maximality of M.

2.3 Boundary Behaviour

We will be often interested by the reciprocal mapping $g = f^{-1}$ and in particular in the behaviour of this function at the boundary, i.e. the unit circle. Our next theorem characterises domains for which g has a continuous extention to the closed disc. Before we study in details this problem let us first notice an easy but useful result:

Proposition 2.3.1. *If U,V are two plane domains and if $f : U \to V$ is an homeomorphism then, if $z_n \in U$ is a sequence converging to ∂U , every limit value of the sequence $f(z_n)$ belongs to ∂V.*

Proof : We may assume WLOG that the sequence $(f(z_n))$ converges to v: if $v \in V$ then $z_n = f^{-1}(f(z_n))$ converges to $f^{-1}(v)$, a contradiction.

Définition 2.3.1. *A compact set $X \subset \mathbb{C}$ is said to be locally connected if*

$$\forall \varepsilon > 0 \exists \delta > 0; \forall x,y \in X, |x-y| < \delta \Rightarrow \exists X_1 \subset X \, connected; x,y \in X_1, diam(X_1) \leq \varepsilon.$$

Théorème 2.3.1. *(Caratheodory): The mapping g has a continuous extension to $\overline{\mathbb{D}}$ if and only if $\partial\Omega$ is locally connected.*

Proof : It is easy to see that the continuous image of a locally connected compact set is again compact and locally connected so the only if part will follow from the fact that if g extends continuoulsly then $\partial\Omega = g(\partial\mathbb{D}$. To prove this last fact consider first $z \in \partial\Omega$: this point is the limit of a sequence (z_n) of points in Ω . But $z_n = g(\omega_n)$ for a sequence $(\omega_n \in \mathbb{D}$ and wlog we may assume that $\omega_n \to \omega \in \partial\mathbb{D}$, from which it follows that $z = g(\omega)$. For the other inclusion suppose that there exists $x \in \partial\mathbb{D}$ such that $g(x) \in \Omega$; then there must exist $\omega \in \mathbb{D}$ such that $g(x) = g(\omega)$ and , if we denote by γ the half-open segment joining ω to x in $\mathbb{D}$, $g(\gamma)$ is a compact subset of Ω . But this is impossible since then $\gamma = f(g(\gamma))$ must be compact in $\mathbb{D}$.

The converse is much harder and will be achieved through a series of lemmas which have their own interest. The first one concerns the continuity at the boundary of the function f itself:

Théorème 2.3.2. *If f maps conformally the domain Ω onto the unit disk and if $\gamma : [0,1] \to \mathbb{C}$ is a curve such that $\gamma(0) \in \partial\Omega, \gamma(]0,1]) \subset \Omega$, then $f \circ \gamma$, which is defined on $(0,1]$, has a continuous extention at 0 and $f \circ \gamma(0) \in \partial\mathbb{D}$. Moreover if we consider two such curves $\gamma_j, j = 1,2$ such that $\gamma_1(0) \neq \gamma_2(0)$ then $f \circ \gamma_1(0) \neq f \circ \gamma_2(0)$.*

Proof : It undergoes the notion of crosscut:

Définition 2.3.2. *A crosscut Γ in a domain Ω is an open Jordan arc such that $\overline{\Gamma} = \Gamma \cup \{a,b\}, a,b \in \partial\Omega$.*

Proposition 2.3.2. *If C is a crosscut of the simply connected domain Ω then $\Omega \backslash C$ has exactly two components.*

Proof : Let $H(z) = z/(1 - |z|)$, so that H is an homeomorphism from $\mathbb{D}$ onto $\mathbb{C}$. Then $H \circ f$ is an homeomorphism from Ω onto $\mathbb{C}$ sending the crosscut C onto a Jordan curve of the Riemann sphere containing the point at infinity by proposition(2.3.1). The proposition then follows from the Jordan curve theorem. We return to the proof. We put $w = \gamma(0)$:

Lemme 2.3.1. *There exists a sequence (r_n) converging to 0 such that $l(f(C(r_n) \cap \Omega) \to 0$ where l denotes length and $C(r) = \partial D(w,r)$.*

Proof : Put $l(r) = l(f(C(r) \cap \Omega))$. By Cauchy-Schwarz,

$$l(r)^2 \leq 2\pi r \int_{t;w+re^{it}\in\Omega} |f'(w+re^{it})|^2 r dt,$$

so that

$$\int_0^{1/4} \frac{l(r)^2}{r} dr \leq 2\pi.$$

The fact that this integral converges implies the lemma.
The lemma implies the existence of a sequence of crosscuts (C_n) separating $\gamma(1)$ from $\gamma(t), t$ small, such that $\mathrm{diam}(C_n) \to 0$. Moreover, if U_n denotes the component of $\Omega \cap C_n$ that does not contain $\gamma(1)$, $\mathrm{diam}(f(U_n)) \to 0$. The fact that f is an homeomorphism then easily imply that $f \circ \gamma$ has a limit at 0. To prove the rest of the theorem we assume that Ω is bounded (the general case needs only a small change): suppose that the images of the two curves have the same end point ζ on the circle. Since Ω is bounded, we may apply a version of the lemma for g. There exists arcs $A_n = \partial D(\zeta, r_n)$ whose image under g have a length converging to 0; this easily implies that the end-points of the two curves have to coïncide.
This last reasonning implies that, whatever the mapping g is we can find for every $z \in \partial\mathbb{D}$ a sequence r_n converging to 0 such that $\gamma_n = g(\mathbb{D} \cap \partial D(z, r_n)$ is a crosscut in Ω of diameter converging to 0 and whose endpoints a_n, b_n converge to a point $\omega \in \partial\Omega$. By the local connectedness assumption there exists a connected subset of $\partial\Omega$ containing a_n, b_n , say L_n, with $\mathrm{diam}(L_n) = \varepsilon_n \to 0$. If $w \in \Omega$, $|w - a_n| > \varepsilon_n$ and if the same is true for z_0 then these two points are separated neither by $C_n \cup L_n$ nor by $\mathbb{C}\backslash\Omega$. We then invoke the following

Théorème 2.3.3. *(Janiszewski) If A,B are closed sets of the complex plane such that $A \cap B$ is connected then, if a,b are two points of the plane which are separated neither by A nor B, then they are not separated by $A \cup B$.*

and conclude that w and z_0 are not separated by $C_n \cap L_n \cap \mathbb{C}\backslash\Omega = C_n \cap \mathbb{C}\backslash\Omega$. It follows that $U_n \subset \{|w - a_n| \leq \varepsilon_n\}$ and consequently that $\mathrm{diam}(U_n) \to 0$. Continuity of g at the point z then easily follows.

For Jordan domains we can precise further the last theorem:

Théorème 2.3.4. *: The domain Ω is a Jordan domain if and only if g extends to a homeomorphism from $\overline{D}$ to $\overline{\Omega}$.*

Proof :

Définition 2.3.3. *A point $\omega \in \partial\Omega$ is called a cut point if $\partial\Omega\backslash\{\omega\}$ is not connected.*

Lemme 2.3.2. *Assume $\partial\Omega$ is locally connected: then g assumes the value $\omega \in \partial\Omega$ exactly once if and only if ω is not a cut point of $\partial\Omega$.*

Proof : If a is the only preimage of ω then $\partial\Omega\backslash\{\omega\} = g(\partial\mathbb{D}\backslash\{a\})$ is connected. Conversely assume that $g(a) = g(a') = \omega$. let l be a crosscut from a to a' in $\mathbb{D}$:

then $g(l)$ is a Jordan curve. By Jordan curve theorem its complement consists of two open components U_1,U_2 and $\partial\Omega\backslash\{\omega\} = (\partial\Omega \cap U_1) \cup (\partial\Omega \cap U_2)$ and thus $\partial\Omega\backslash\{\omega\}$ cannot be connected.

Corollaire 2.3.1. *: If Ω_j,$j = 1$,2 are two Jordan domains then every holomorphic bijection between the two domains extends to a homeomorphism of the closures. Moreover, fixing z_j,$j = 1$,2,3 in this order in the trigonometric sense on $\partial\Omega_1$ and simlarly z'_j,$j = 1$,2,3 on $\partial\Omega_2$ there is a unique holomorphic bijection between Ω_1 and Ω_2 whose extention sends z_j to z'_j,$j = 1$,2,3.*

Proof : Using Riemann mapping theorem and the last one it suffices to prove the corollary for $\Omega_j = \mathbb{D}$,$j = 1$,2 where the result follows from the fact that an automorphism of the disk depends on three parameters.

2.4 Around Koebe Theorem

In this section we study the properties of the class

$$S = \{f : \mathbb{D} \to \mathbb{C} \text{ holomorphic, injective; } f(0) = 0, f'(0) = 1\}$$

As an explicit example, we have the *Koebe function*

$$f(z) = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2} = \frac{1}{4}(\frac{1+z}{1-z})^2 - \frac{1}{4}$$

which is the Riemann mapping of $\mathbb{C}\backslash] - \infty, -\frac{1}{4}]$.

We will also need to study functions defined on $\Delta = \mathbb{C}\backslash\overline{\mathbb{D}}$.

Définition 2.4.1. *: A compact subset K of the plane is called full if $\mathbb{C}\backslash K$ is connected.*

Théorème 2.4.1. *: If the nonempty compact connected set K is full then there exists a unique conformal mapping $F_K : \Delta \to \mathbb{C}\backslash K$ sending ∞ to ∞ and such that*

$$\lim_{z\to\infty} \frac{F_K(z)}{z} > 0$$

.

Proof :

1) Uniqueness: Suppose F_1,F_2 are solutions: then $F = F_2 \circ F_1^{-1} : \Delta \to \Delta$ satisfies

$$\lim_{z\to\infty} \frac{F(z)}{z} > 0.$$

Then $G(z) = \frac{1}{F(\frac{1}{z})}$ extends to a holomorphic function from $\mathbb{D}$ to itself with $G'(0) = 1$ by Schwarz lemma. Schwarz lemma again implies that $G(z) \equiv z$ and thus that

$F_1 = F_2$.
2) Existence: Let $z_0 \in K$ and Ω_0 the image of $\mathbb{C}\backslash K$ under $M : z \mapsto \frac{1}{z-z_0}$. This is a simply connected domain containing 0 so that, by Riemann mapping theorem, there exists a unique $f_0 : \mathbb{D} \to \Omega_0$ holomorphic bijective with $f_0(0) = 0$,$f_0'(0) > 0$. Put then $M_0(z) = \frac{1}{z}$: then $F_K = M^{-1} \circ f_0 \circ M_0$ has the desired properties with

$$\lim_{z\to\infty} \frac{F_K(z)}{z} = \frac{1}{f_0'(0)}.$$

Définition 2.4.2. *: The logarithmic capacity of the compact set K is*

$$\mathrm{cap}(K) = \lim_{z\to\infty} \frac{F_K(z)}{z}.$$

We will use several notions of capacity along this course and an important feature is how they behave under different scalings. We will study more specifically logarithmic capacity in a next paragraph.

Définition 2.4.3. *: We will denote by $\mathcal{K}$ the set of full compact subsets of the plane and by $\mathcal{K}(0)$ the subset of those compacts containing 0. The same symbols with index 1 will denote the same compacts but with the further property of having capacity 1.*

If $K \in \mathcal{K}_1$ then for $|z| > 1$,

$$F_K(z) = z + b_0 + \sum_{n\geq 1} \frac{b_n}{z^n}.$$

Théorème 2.4.2. *(Area Theorem): If $K \in \mathcal{H}_1^*$ then ($|.|$ stands for Lebesgue measure)*

$$|K| = \pi(1 - \sum_{n\geq 1} n|b_n|^2).$$

Proof : If γ is a smooth curve surrounding a region A then an immediate application of Stokes formula shows that

$$|A| = \frac{1}{2i}\int_\gamma \overline{z}dz.$$

We apply this to $\gamma = F_K(r\partial\mathbb{D})$ with $r > 1$:

$$\frac{1}{2i}\int_\gamma \overline{z}dz = \frac{1}{2i}\int_0^{2\pi} \overline{F}_K(re^{i\theta})ire^{i\theta}F_K'(re^{i\theta})d\theta = \pi(r^2 - \sum_{n=1}^{\infty} n|b_n|^2)$$

and the result follows by letting r decreasing to 1.

Lemme 2.4.1. *: If $f \in S$ then there exists an odd function $h \in S$ such that for $z \in \mathbb{D}$*

$$h(z)^2 = f(z^2).$$

As an example, if f is the Koebe function then h is the Riemann mapping onto the plane minus the two slits $[i,+i\infty[, [-i,-i\infty[$.
Proof : The function $z \mapsto \frac{f(z)}{z}$ does not vanish in $\mathbb{D}$ and thus possesses a square root g. Put $h(z) = zg(z^2)$. it is clearly odd and $h(z)^2 = f(z^2)$. If $h(z_1) = h(z_2)$ then $z_1^2 = z_2^2$ and thus $z_1 = z_2$ since h is odd. Finally $g(z^2) = 1 + .. \Rightarrow h(z) = z + ...$

Théorème 2.4.3. *(Koebe): If $f \in S$ then, if $f(z) = z + a_2 z^2 + ..$ we have $|a_2| \leq 2$*
Proof : Let h be as above and

$$g(z) = \frac{1}{h(\frac{1}{z})} = z - \frac{a_2}{2z} + ...$$

An application of the area theorem finishes the proof.

Théorème 2.4.4. *(Koebe): If $f \in S$ then $f(\mathbb{D}) \supset B(0,\frac{1}{4})$.*
Proof : Let $z_0 \notin f(\mathbb{D})$. The function

$$\tilde{f}(z) = \frac{z_0 f(z)}{z_0 - f(z)} = z + (a_2 + \frac{1}{z_0})z^2 + ..$$

is in S; by the preceeding theorem $|a_2 + \frac{1}{z_0}| \leq 2$ which in turn implies, since already $|a_2| \leq 2$, that $\frac{1}{|z_0|} \leq 4$.

Corollaire 2.4.1. *: If $f : \Omega \to \Omega'$ is holomorphic and bijective and if $f(z) = z'$,$d = d(z,\partial\Omega)$,$d' = d(z,\partial\Omega')$ then*

$$\frac{1}{4}d' \leq d|f'(z)| \leq 4d'.$$

Proof : We may assume $z = z' = 0$. The function $\tilde{f}(w) = \frac{f(dw)}{df'(0)}$ belongs to the class S; an application of the preceeding theorem shows that $\tilde{f}(\mathbb{D}) \supset B(0,1/4) \Rightarrow d' \geq \frac{1}{4}|f'(0)|d$.
Théorème 2.4.5. *(Koebe): If f is holomorphic and injective in the unit disk then for every $z \in \mathbb{D}$*

$$|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}| \leq \frac{4|z|}{1-|z|^2}. \tag{2.2}$$

Proof : Put $T_z(w) = \frac{w+z}{1+\bar{z}w}$. It is an automorphism of $\mathbb{D}$ satisfying $T_z(0) = w, T_z'(0) = 1 - |z|^2$. Then

$$\tilde{f}(w) = \frac{f(T_z(w)) - f(z)}{f'(z)(1-|z|^2)} = w + (\frac{f''(z)(1-|z|^2)}{2f'(z)} - \overline{z})w^2 + ..$$

so that $\tilde{f} \in S$. By Koebe's theorem

$$|\frac{f''(z)(1-|z|^2)}{2f'(z)} - \overline{z}| \leq 2$$

and the result follows by multiplication by $\frac{2z}{1-|z|^2}$.

Théorème 2.4.6. *(Distortion Theorem): If $f \in S$ then, for $z \in \mathbb{D}$*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3} \tag{2.3}$$

Proof : Put $h = \log f'$. It suffices to prove the theorem for $z = x \in (0{,}1)$. We can write

$$x\Re(h'(x)) = \Re(\frac{xf''(x)}{f'(x})$$

from which it follows, using the preceeding theorem, that

$$\frac{2x-4}{1-x^2} \leq \Re(h'(x)) \leq \frac{4+2x}{1-x^2}$$

and the result follows by integration.

Théorème 2.4.7. *(Second distortion theorem): If $f \in S$ $z \in \mathbb{D}$ then*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2} \tag{2.4}$$

Proof : The upper bound follows easily from the preceeding theorem:

$$\forall x \in (0{,}1)\, |f(x)| \leq \int_0^x \frac{1+t}{(1-t)^3} dt = \frac{x}{(1-x)^3}.$$

The lower bound is obvious if $|f(z)| \geq 1/4$ since $\frac{r}{(1+r)^2}$is always less than $\frac{1}{4}$. Assume now that $f(z) = x \in (0{,}1/4)$: then by Koebe theorem $[0{,}x] \subset \Omega$. Let $C = f^{-1}([0{,}x]) \subset \mathbb{D}$ then

$$f(z) = \int_C f'(\zeta)d\zeta = \int_C |f'(\zeta)||d\zeta| \geq \int_C \frac{1-|\zeta|}{(1+|\zeta|)^3}|d\zeta| \geq \int_0^x \frac{1-r}{(1+r)^3}dr = \frac{r}{(1+r)^2}.$$

2.5 Harmonic measure and Beurling theorem

If Ω is a bounded plane domain then , under some mild regularity conditions, one can solve Dirichlet problem in Ω: for every function f continuous on $\partial\Omega$ one can find a function u continuous on $\overline{\Omega}$ such that $u|\Omega$ is harmonic and such that $u|\partial\Omega = f$. The maximum principle implies that, if $z \in \Omega$ the application $f \mapsto u(z)$ is a continuous linear form on the space $C(\partial\Omega)$. By the Riesz representation theorem, there exists a unique probability measure μ on $\partial\Omega$ such that for any $f \in C(\partial\Omega)$,$u(z) = \int_{\partial\Omega} f d\mu$.

Définition 2.5.1. *This measure is called the harmonic measure at point z in Ω, and written as $\omega(z,\Omega,.)$.*

There is an equivalent probalistic definition of harmonic measure. Let B_t^z be a standard Brownian motion started at point z and let $\tau_\Omega = \inf\{t \geq 0; B_t \notin \Omega\}$. Then, for a Borel subset E of $\partial\Omega$,

$$\omega(z,\Omega;E) = P[B^z_{\tau_\Omega} \in E]$$

and this formula implies the more general, and useful following one:

$$\forall f \in C(\partial\Omega), u(z) = E[f(B^z_{\tau_\Omega}].$$

We now investigate the size of the harmonic measure of a set E, in terms of its diameter and distance to z.

Théorème 2.5.1. *Let λ be a full continuum joining 0 and the boundary of the unit disk. Then if $z \in \mathbb{D}\backslash\lambda$ then the probability that a Brownian motion starting from z will hit the boundary of the circle before hitting λ is smaller than $c\sqrt{|z|}$ In other terms,*

$$\omega(z,\mathbb{D}\backslash\lambda;\partial\mathbb{D}) \leq c\sqrt{|z|}.$$

Here is an intuitive proof of this result: if $|z| = r$ then this probability is $\geq$ than the same probability in the case $z = -r, \lambda = [0,1]$ where the result follows by an explicit computation.

Corollaire 2.5.1. *: Let Ω be a simply connected domain with $\infty \in \partial\Omega$, $z_0 \in \partial\Omega$ and $z \in \Omega; \mathrm{d}(z,\partial\Omega) > r$: then the probability that a Brownian motion starting from z will hit $\partial B(z_0,r)$ before $\partial\Omega\backslash B(z_0,r)$ is smaller than $c\sqrt{\frac{r}{|z-z_0|}}$.*

The corollary follows from the theorem by an inversion about z_0.
To prove then (3.5) we apply the preceeding corollary with the help of (2.7.1).

2.6 Capacity

If $K \in \mathcal{K}(\prime)$ we have defined in the last section the mapping $F_K : \Delta \to \mathbb{C}\backslash K$ and defined the logarithmic capacity of K as

$$\lim_{z\to\infty} \frac{F_K(z)}{z}$$

. We have also used the mapping $f_K : \mathbb{D} \to \mathbb{C}$ defined by $f_K(w) = \frac{1}{F_K(\frac{1}{w})}$ and satisfying $f'_K(0) = \frac{1}{\mathrm{cap}(K)}$. For any closed $E \subset \mathbb{C}$ we define $\mathrm{rad}(E)$ as the radius of the smallest disk centered at 0 and containing E. We can restrict the study of logarithmic capacity to compact sets containing 0 because of the obvious

Proposition 2.6.1. *: The logarithmic capacity is translation and rotation invariant. If h is an homotethy of amplitude λ then $\mathrm{cap}(h(K)) = \lambda\mathrm{cap}(K)$.*

As a consequence we observe that the (logarithmic) capacity of a disc of radius r is equal to r and since the map $z \mapsto z + 1/z$ sends Δ onto $\mathbb{C}\backslash[-2,2]$ we see that the capacity of a line segment of length l is $l/4$.

Proposition 2.6.2. *: If K,$K' \in \mathcal{K}(0)\, K \subset K'$ then* $\operatorname{cap}(K) \subset \operatorname{cap}(K')$ *with equality if and only if $K = K'$.*

Proof : Put $h = f_K^{-1} \circ f_{K'}$ taking $\mathbb{D}$ into $\mathbb{D}$ and sending 0 to 0. The proposition follows by applying Schwarz lemma to h.

Proposition 2.6.3. *: If $K \subset \mathcal{K}(0)_1$ then* $1 \leq \operatorname{rad}(K) \leq 4$

If $K \in \mathcal{K}(0)_1$ we have $F_K(w) = w + b_0 + \frac{b_1}{w} + ..$ and $f_K(z) = z + a_2 z^2 + ..$ and an easy computation shows that $b_0 = -a_2$. We have thus proven the

Proposition 2.6.4. *: If $K \subset \mathcal{K}_1$ then $|b_0| \leq 2$.*

Proposition 2.6.5. *There exists a constant $c > 0$ such that if $K \subset \mathcal{K}(0)_1$ then for $|z| > 1$,*

$$|F_K(z) - z| \leq c \tag{2.5}$$

Proof : It obviously suffices to find c such that the inequality is satisfied for $|z| \geq 2$ since then $|F(z)| \leq c + 2$ if $|z| \leq 2$. Suppose thus that $|z| \geq 2$ and put $w = \frac{1}{z}$. We have

$$|F_K(z) - z| = \frac{|w - f_K(w)|}{|w||f_K(w)|} \tag{2.6}$$

Lemme 2.6.1. *: For $r < 1$ there exists $c_r > 0$ such that if $f \in S$ and $|z| \leq r$ then*

$$|f(z) - z| \leq c_r |z|^2$$

Proof : By (2.2)

$$|f''(z)| \leq (\frac{4}{1-r^2} + \frac{2r}{1-r^2})\frac{1+r}{(1-r)^3} = A_r$$

if $|z| \leq r$.We deduce from this that $|f'(z) - 1| \leq A_r|z|$ if $|z| \leq r$ and the result by integration.

We return to the proof of the proposition. By the distortion theorem $|f_K(w)| \geq \frac{4}{9}|w|$ if $|w| \leq \frac{1}{2}$. The result then follows by combining this inequality with the lemma and (2.6).

The *Green Function* associated with the full compact K is the function $\Phi_K(z) = \ln|g_K(z)|$ (we recall that $g_K = f_K^{-1}$). The Green function is the unique harmonic function outside K with boundary values 0 on ∂K such that $\Phi_K(z) \sim \ln|z|$ at infinity.

Proposition 2.6.6. *: There exists a constant $c > 0$ such that if $K \in \mathcal{K}(0)$ and $|z| > 4\operatorname{cap}(K)$ then*

$$|\Phi_K(z) - \log|z| + \ln(\operatorname{cap}(K))| \leq \operatorname{cap}(K)\frac{1}{|z|} \tag{2.7}$$

Proof : By scaling, it suffices to prove the proposition for $K \in \mathcal{K}(0)_1$. If $|z| > 4$ then automatically $z \in \mathbb{C} \backslash K$ and the preceeding proposition implies that $|g_K(z) - z| \leq C'$. The result follows then by observing that $|\Phi_K(z) - \log|z|| = |\log(1 + \frac{|g_K(z)| - |z|}{|z|})|$

The rest of this paragraph consists in a probabilistic interpretation of the preceeding notions. This point of view has the advantage to allow a generalization of the notion of logarithmic capacity to not necessarily connected compact sets K. The non-specialist reader may postpone its reading until he has read chapter 3.

Corollaire 2.6.1. *: There exists a constant $c > 0$ such that if $K \in \mathcal{H}$,if B is a Brownian motion and τ is the first time B reaches K and $|z| > 4\mathrm{cap}(K)$ then*

$$|E^z(\log|B_\tau|) - \ln(\mathrm{cap}(K))| \leq c\frac{\mathrm{cap}(K)}{|z|}. \tag{2.8}$$

Proof : $h(z) = \log|z| - \Phi_K(z)$ is a bounded harmonic function; by the optional sampling theorem $h(z) = E^z(h(B_\tau))$ and the corollary follows by application of the last proposition.

Corollaire 2.6.2. *: Let $K \in \mathcal{K}$ and $r > \mathrm{cap}(K)$. If B is a Brownian motion whose starting point follows a uniform law on $\partial B(0,r)$ then*

$$E(\log|B_\tau|) = \mathrm{cap}(K) \tag{2.9}$$

Proof : Let $|z| > r$ and τ_1 the first time that B^z reaches the circle of radius r. Then

$$E^z(\log|B_\tau|) = \int_{\partial B(0,r)} E^\zeta(\log|B_\tau|)\omega_z(|d\zeta|)$$

where ω_z stands for harmonic measure at point z in $\mathbb{C}\backslash\overline{B}(0,r)$. Since

$$\int_{\partial B(0,r)} \omega_z(|d\zeta|) = |d\zeta|$$

it follows that $M(r)$, the average of $E^z(\log|B_\tau|)$ over the circle centered at 0 and of radius r is independent of r: the result then follows from last corollary.

2.7 Half-plane capacity

We denote by $\mathbb{H}$ the upper-half plane $\{y > 0\}$.

Définition 2.7.1. *A bounded set $A \subset \mathbb{H}$ is called a compact $\mathbb{H}$-hull if $A = \overline{A} \cap \mathbb{H}$ and $\mathbb{H} \backslash A$ is simply connected.*

We will denote by $\mathcal{Q}$ the set of compact hulls: if $A \in \mathcal{Q}$ we define $A^* = \overline{\{z \in A \quad \text{or} \quad \overline{z} \in A\}}$. If A is connected then $A^* \in \mathcal{K}$ (the set of full compacts).

Proposition 2.7.1. *: If $A \in \mathcal{Q}$ there exists a unique holomorphic and bijective map $g_A : \mathbb{H}\backslash A \to \mathbb{H}$ such that*

$$\lim_{z\to\infty} g_A(z) - z = 0 \tag{2.10}$$

Proof : By Riemann mapping theorem there is a holomorphic and injective mapping $g : \mathbb{H}\backslash A \to \mathbb{H}$ such that $\lim_{z\to\infty} g(z) = \infty$. By compactness of the hull, this map g is holomorphic and injective in $\mathbb{H} \cap \mathbb{C}\backslash\overline{D}(0,R)$ for some $R > 0$ and thus extends to a holomorphic injective map on $\mathbb{C}\backslash\overline{D}(0,R)$. We thus have $g(z) = az + b_0 + \frac{b_1}{z} + ..,z \to \infty$ and $a,b_0,b_1 \in \mathbb{R}$. Replacing g by $g_A = \lambda g + b$ with some $\lambda > 0,b \in \mathbb{R}$ we may assume that $\lim_{z\to\infty} g_A(z) - z = 0$.
This proposition allows us to state the

Définition 2.7.2. $\mathrm{hcap}(A) = \lim_{z\to\infty} z(g_A(z) - z)$.

As for the logarithmic capacity let us investigate how hcap scales. First of all it is obviously invariant by real translation. Concerning homotheties let r be a positive real number: we have $g_{rA}(z) = rg_A(z/r)$, from which it follows easily that

$$\mathrm{hcap}(rA) = r^2\mathrm{hcap}(A) \tag{2.11}$$

.

Exemples :
1) $A = \overline{\mathbb{D}} \cap \mathbb{H}$. In this case $g_A(z) = z + \frac{1}{z} \Rightarrow \mathrm{hcap}(A) = 1$.
2) $A = (0,i]$, then $g_A(z) = \sqrt{z^2 + 1} = z + \frac{1}{2z} + .. \Rightarrow \mathrm{hcap}(A) = \frac{1}{2}$.
3) We leave as an exercise to the reader to show that $\mathrm{hcap}(A) = \frac{\alpha(1-\alpha)}{2}$ if $A = (0,\alpha^\alpha(1-\alpha)^{(1-\alpha)}e^{i\alpha\pi}]$.

An important property of this new capacity is that it is positive. The fact that it is real follows by symetry: positivity follows from the observation that the function $v(z) = \Im(z - g_A(z))$ is bounded in $\mathbb{H}$ and tends to 0 at ∞. By maximum principle, it is nonnegative since it is so on the boundary. Positivity of hcap then follows by the definition: $\mathrm{hcap}(A) = \lim_{y\to+\infty} iy.iv(iy)$.

The next proposition is the probabilistic translation of this, allowing generalization of the definition to sets A for which $\mathbb{H}\backslash A$ is still a domain, but not necessarily simply connected:

Proposition 2.7.2. *: Suppose $A \in \mathcal{Q}, B$ is a Brownian motion and τ is the smallest time such that $B_\tau \in \mathbb{R} \cup A$. Then for every $z \in \mathbb{H}\backslash A$*

$$\Im z = \Im g_A(z) + E^z(\Im B_\tau) \tag{2.12}$$

and

$$\mathrm{hcap}(A) = \lim_{y\to\infty} yE^{iy}[\Im B_\tau]. \tag{2.13}$$

Moreover, if $\mathrm{rad}(A) < 1$*,*

$$\mathrm{hcap}(A) = \frac{2}{\pi}\int_0^\pi E^{e^{i\theta}}(\Im B_\tau)\sin\theta \mathrm{d}\theta \tag{2.14}$$

The next proposition shows monotonicity:

Proposition 2.7.3. *: If* A,$A' \in \mathcal{Q}$,$A \subset A'$ *then*

$$\mathrm{hcap}(A') = \mathrm{hcap}(A) + \mathrm{hcap}(g_A(A'\backslash A) \tag{2.15}$$

Proof : We have the equality

$$g_{A'} = g_{g_A(A'\backslash A)} \circ g_A$$

from which me may deduce

$$g_{A'}(z) = g_{g_A(A'\backslash A)}(g_A(z) - z + z) \sim g_{g_A(A'\backslash A)}(z + \frac{\mathrm{hcap}(A)}{z}) \sim z + \frac{\mathrm{hcap}(A)}{z} + \frac{\mathrm{hcap}(g_A(A'\backslash A))}{z}.$$

This monotonicity happens to be strict:

Proposition 2.7.4. *If* A *is a hull such that hcap*$(A) = 0$*, then* $A = \emptyset$*.*

Proof : By the preceeding proposition, we may assume that A is connected. Recall that A^* is the symmetrized compact around the real axis. Let us call F the Riemann mapping $F : \Delta \to \mathbb{C}\backslash A^*$: we have

$$F(z) = \mathrm{cap}(A^*)\,[z + b_0 + b_1/z + ..]$$

and $|b_1| \leq 1$ by area theorem. On the other hand, if we denote by φ the map $\varphi : z \mapsto z + 1/z$ then $G_A = \varphi \circ F^{-1} : \mathbb{H}\backslash A \to \mathbb{H}$ is an holomorphic bijection. But

$$G_A(z) = \frac{z}{\mathrm{cap}(A^*)} - \frac{b_0}{\mathrm{cap}(A^*)} + \frac{\mathrm{cap}(A^*) - b_1\mathrm{cap}(A^*)}{z}$$

and we easily deduce the next proposition, which has its own interest:

Proposition 2.7.5. $\mathrm{hcap}(A) = \mathrm{cap}(A^*)^2(1 - b_1)$.

The conclusion now follows from Area theorem: if $\mathrm{hcap}(A) = 0$ then either $A^* = \emptyset$ or $\mathrm{hcap}(A^*) = 1$, in which case $F(z) = \mathrm{cap}(A^*)\,[z + b_0 + 1/z]$ and $A = \emptyset$.

Proposition 2.7.6. *: If* A_1,$A_2 \in \mathcal{Q}$ *then*

$$\mathrm{hcap}(A_1) + \mathrm{hcap}(A_2) \geq \mathrm{hcap}(A_1 \cup A_2) + \mathrm{hcap}(A_1 \cap A_2).$$

Proof : Write $\tau_j = \tau_{\mathbb{H}\backslash A_j}, j = 1{,}2$ (hitting time), $\tau = \tau_{\mathbb{H}\backslash A_1\cup A_2}, \eta = \tau_{\mathbb{H}\backslash A_1\cap A_2}$. We have $\tau = \tau_1 \wedge \tau 2 = \tau_1\chi_{\{\tau_1\leq\tau_2\}} + \tau_2\chi_{\{\tau_2<\tau_1\}}$ so that

$$E^{iy}[\Im(B_{\tau_1})] + E^{iy}[\Im(B_{\tau_2})] = E^{iy}[\Im(B_\tau)] + \int_{\{\tau_2<\tau_1\}} \Im(B_{\tau_1}) + \int_{\{\tau_1\leq\tau_2\}} \Im(B_{\tau_2})$$

and we conclude by noticing that $B_{\tau_j} \geq B_\eta$ so that

$$\int_{\{\tau_2<\tau_1\}} \Im(B_{\tau_1}) + \int_{\{\tau_1\leq\tau_2\}} \Im(B_{\tau_2}) \geq E^{iy}[\Im(B_\eta)].$$

Proposition 2.7.7. *: If $x > \mathrm{rad}(A)$,*

$$g_A(x) = \lim_{y\to\infty} \pi y[\frac{1}{2} - \omega(iy,\mathbb{H}; [x, +\infty))] \tag{2.16}$$

while if $x < -\mathrm{rad}(A)$,

$$g_A(x) = \lim_{y\to\infty} \pi y[-\frac{1}{2} + \omega(iy,\mathbb{H}; (-\infty,x])]; \tag{2.17}$$

Proof : (in the case $x > \mathrm{rad}(A)$): we consider first the case $A = \emptyset$. Then

$$\lim_{y\to\infty} \pi y[\frac{1}{2} - \omega(iy,\mathbb{H}; [x,+\infty))] = \lim_{y\to\infty} \pi y\omega(iy,\mathbb{H}; [0,x]) = \lim_{y\to\infty} \pi y \int_0^\infty \frac{1}{\pi(1+s^2)}\mathrm{d}s = x.$$

In the case $A \neq \emptyset$ we write $g_A = u_A + iv_A$ and we use conformal invariance of the Brownian:

$$= \omega(iy,\mathbb{H}\backslash A; [x,+\infty)) = \omega(g_A(iy),\mathbb{H}; [g_A(x),+\infty)) = \omega(v_A(iy),\mathbb{H}; [g_A(x)-u_A(iy),+\infty)). \tag{2.18}$$

But as $y \to \infty$, $\quad u_A(iy) \to 0 \quad v_A(iy) \sim y$ and the result follows.

Corollaire 2.7.1. *: For any $A \in \mathcal{Q}$,$|g_A(z) - z| \leq 3\mathrm{rad}(A)$.*

Proof : The preceeding proposition implies in particular that g_A is increasing as a function of A. By the example above we deduce that if $\mathrm{rad}(A) \leq 1$ we have, for $x > 1$, $\quad x \leq g_A(x) \leq x + \frac{1}{x}$. If then $\mathrm{rad}(A) = 1$ then $-2 \leq g_A(-1) \leq g_A(1) \leq 2$ and it follows that $|g_A(z) - z| \leq 3$ for $z \in \partial(\mathbb{H}\backslash A)$ and thus everywhere by maximum principle. The general case follows by scaling.

Théorème 2.7.1. *: There exists a constant $c > 0$ such that $\forall A \in \mathcal{Q} \quad \forall z$; $|z| \geq \mathrm{rad}(A)$*

$$|z - g_A(z) + \frac{\mathrm{hcap}(A)}{z}| \leq c\frac{\mathrm{rad}(A)\mathrm{hcap}(A)}{|z|^2} \tag{2.19}$$

Proof : By scaling it suffices to prove it for $\mathrm{rad}(A) = 1$. For this purpose we introduce $h(z) = z - g_A(z) + \frac{\mathrm{hcap}(A)}{z}$. Then

$$v(z) = \Im(h(z) = \Im(z - g_A(z)) - \frac{\Im z}{|z|^2}\mathrm{hcap}(A).$$

We can write

$$Im(z - g_A(z)) = \int_{\mathbb{D}\cap\mathbb{H}} \Im(u)\omega(z,\mathbb{H}\backslash A; du) = \int_0^\pi \sin(\theta)p(z,e^{i\theta})d\theta$$

where

$$p(z,e^{i\theta}) = \frac{\Im z}{|z|^2}\frac{2}{\pi}\sin\theta[1 + O(\frac{1}{|z|}].$$

We can already deduce from this that

$$|v(z)| \le \frac{\Im z}{|z|^3}\mathrm{hcap}(A)$$

for $|z| \ge 2$. From this it follows that $|\partial_x v(z)| \le c\frac{\mathrm{hcap}(A)}{|z|^3}$, $\quad |\partial_y v(z)| \le c\frac{\mathrm{hcap}(A)}{|z|^3}$ and thus that $|h'(z)| \le \ c\frac{\mathrm{hcap}(A)}{|z|^3}$. Integrating from iy to ∞ we deduce that $|h(iy)| \le c\frac{\mathrm{hcap}(A)}{|z|^2}$ and finally we get the result, using integration over the circle of radius r:

$$|h(re^{i\theta}| \le |h(ir)| + c\frac{\mathrm{hcap}(A)}{r^2}$$

2.8 Caratheodory convergence

Définition 2.8.1. *: Let (Ω_n) be a sequence of domains in $\mathcal{A}$ and $f_n : \mathbb{D} \to \Omega_n; \quad f_n(0) = 0, f_n'(0) > 0$ the corresponding Riemann mappings. We say that:*

- $\Omega_n \xrightarrow{\text{(Cara)}} \mathbb{C}$ *if* $f_n'(0) \to \infty$,
- $\Omega_n \xrightarrow{\text{(Cara)}} \{0\}$ *if* $f_n'(0) \to 0$,
- $\Omega_n \xrightarrow{\text{(Cara)}} \Omega$ *if* $f_n \to f$ *uniformly on compact sets, where* $f : \mathbb{D} \to \Omega$ *is holomorphic, bijective and satisfies* $f(0) = 0, f'(0) > 0$.

We will say more generally, if (Ω_n) is a sequence of domains containing points z_n, that (Ω_n,z_n) converges in the sense of Caratheodory to (Ω,z) if $(\Omega_n - z_n) \xrightarrow{\text{(Cara)}} (\Omega - z)$ in the sense of the preceeding definition.

Définition 2.8.2. *The kernel of a sequence (Ω_n) in $\mathcal{A}$ is the largest domain U containing 0 such that for all compact set $K \subset U$ we have $K \subset \Omega_n$ for n large enough.*

Théorème 2.8.1. $\Omega_n \xrightarrow{\text{(Cara)}} \Omega$ *if and only if the kernel of every subsequence is* Ω.

Proof : Suppose first that $\Omega_n \xrightarrow{\text{(Cara)}} \Omega$. Then the sequence of Riemann mappings f_n converges to the Riemann mapping f. Let K be a compact subset of Ω. Then $K \subset f((1-2r)\overline{\mathbb{D}})$ for $r > 0$ small enough and thus $K \subset f_n((1-r)\overline{\mathbb{D}}) \subset \Omega_n$ for n large enough. It follows that the kernel of every subsequence contains Ω. Let $\tilde{\Omega}$ be the kernel of some subsequence: we consider an increasing sequence of compact sets K_n such that $\tilde{\Omega} = \cup K_n$. By hypothesis, $\forall m, \exists N_m; K_m \subset \Omega_n , n \ge N_m$.

Then (f_n^{-1}),$n \geq N_m$ is a uniformly bounded sequence of univalent functions on K_m. By diagonal process we can extract a subsequence converging to $\tilde{g}$ converging uniformly on compact sets of $\tilde{\Omega}$ and necessarily $\tilde{g} = f_{-1}$ on Ω and is univalent by Hurwitz theorem. But $\tilde{g}(\tilde{\Omega}) = \mathbb{D} = f^{-1}(\Omega)$ and necessarily $\tilde{\Omega} = \Omega$.
Suppose conversely that the kernel of every subsequence is $\Omega \in \mathcal{A}$. Choose $r_j, j = 1{,}2; 0 < r_1 < d(0{,}\partial\Omega) < r_2$. By Koebe theorem, $f_n'(0)$ cannot converge to 0 or ∞, nor any subsequence. So there exists a subsequence converging uniformly on compact sets to $f : \mathbb{D} \to f(\mathbb{D})$. Arguing as in the first part we see that necessarily $f(\mathbb{D}) = \Omega$.

Chapitre 3

Löwner Differential Equation

3.1 Radial Löwner Processes

3.1.1 Definition and first properties

Let $(K_t)_{t\geq 0}$ be a (strictly) increasing family of compact sets in $\mathcal{K}(0)$, i.e. a growing family of full compact sets containing 0. We denote by (Ω_t) the complement of K_t and by f_t the Riemann map of Ω_t, i.e. the unique holomorphic bijection from Δ onto Ω_t such that

$$f_t(\infty) = \infty \text{ and } \lim_{z\to\infty} \frac{f_t(z)}{z} > 0.$$

We may then write $f_t(z) = c(t)z + ...$ where $c(t) = \mathrm{cap} K_t$ is the logarithmic capacity.

We make the following assumptions:

1) The family (Ω_t) is continuous in the sense of Caratheodory convergence.
This property implies in particular that the function $t \mapsto c(t)$ is continuous and stricly increasing.

2) $\lim_{t\to\infty} c(t) = +\infty$ and $c(0) = 1$.

Frequently we will assume that $K_0 = \mathbb{D}$.
If these conditions are satisfied one may perform a time-change and assume that $c(t) = e^t$.

We start the study of such growth processes by observing that if $s \leq t$,$f_t(\Delta) \subset f_s(\Delta)$ so that $h_{s,t}(z) = f_s^{-1} \circ f_t$ is a well-defined map from Δ into itself fixing ∞.

Définition 3.1.1. *If f,g : $\Delta \to \mathbb{C}$, are two holomorphic functions we say that f is subordinate to g (and denote this by $f \prec g$) if there exists $\varphi : \Delta \to \Delta$ holomorphic and fixing ∞ such that $f = g \circ \varphi$.*

Notice that, by Schwarz lemma, $|\varphi(z)| \geq |z|$ so that not only $f(\Delta) \subset \Delta$ but also, for every $r > 1$, $f(\{|z| > r\}) \subset g(\{|z| > r\})$.

Définition 3.1.2. *The family $(f_t)_{t\geq 0}$ of holomorphic and injective mappings from*

Δ into $\mathbb{C}$ is called a Löwner chain if
1) $f_t(z) = e^t z + ...,$
2) $f_t \prec f_s$ if $0 \le s \le t$.

We have seen a way to produce Löwner chains. The next proposition shows that this is the only way to produce such chains. It shows indeed that a Löwner chain has to be continuous for the topology of convergence on compact sets: by Caratheodory kernel theorem this implies that the family (Ω_t) is continuous for Caratheodory topology.

Proposition 3.1.1. *If (f_t) is a Löwner chain then for $0 \le s \le t$,*

$$\forall z \in \Delta, |f_t(z) - f_s(z)| \le 2^{12}(e^{t-s} - 1)e^{4t}\frac{(|z|+1)^5}{(|z|-1)^3}.$$

Proof : Writing $f_t(z) = f_s(h_{s,t}(z))$ we have the inequality

$$|f_t(z) - f_s(z)| \le |h_{s,t}(z) - z| \sup_{u \in [h_{s,t}(z), z]} |f_s'(u)|.$$

Lemme 3.1.1. *Let $\Phi : \Delta \to \mathbb{C}$ be a Riemann mapping of a domain not containing 0, $\Phi(z) = cz + .., c > 0$, then $|\Phi(z)| \le 4c|z|$,$z \in \Delta$.*

Proof : It merely consists in applying the second distortion theorem to the map $\varphi(\omega) = c/\Phi(1/\omega)$ which belongs to the class S.
If $\Phi = f_s$ we write F_s for the corresponding φ we get, for $u \in \Delta$,$|f_s'(u)| \le |F_s'(1/u)||f_s(u)|^2$. Applying the lemma together with the first distortion theorem for F_s we obtain

$$|f_s'(u)| \le 16e^s\frac{(|u|+1)^3}{(|u|-1)^2}.$$

Similarly, a direct application of the lemma to the function $h_{s,t}$ gives $|h_{s,t}(z)| \le 4e^{t-s}|z|$.
We now come to the estimation of $|h_{s,t}(z) - z|$. First of all, by Schwarz lemma, $|h_{s,t}(z)| \ge |z|, z \in \Delta$. It follows that the function defined by

$$p_t(z) = \frac{e^{t-s}+1}{e^{t-s}-1}\frac{h_{s,t}(z) - z}{h_{s,t}(z) + z}$$

belongs to the class $\mathcal{P}(\Delta)$ of holomorphic functions in Δ with value 1 at ∞ and with positive real part.

Lemme 3.1.2. *If $p \in \mathcal{P}(\Delta)$ then*

$$\frac{|z|-1}{|z|+1} \le |p(z)| \le \frac{|z|+1}{|z|-1}$$

and this shows in particular that $\mathcal{P}(\Delta)$ is a normal family.

Proof : The mapping
$$\zeta \mapsto \frac{\zeta+1}{\zeta-1}$$
maps $\{x > 0\}$ onto Δ. Thus the map
$$z \mapsto \frac{p(z)+1}{p(z)-1}$$
sends Δ into itself and fixes ∞. By Schwarz lemma
$$\frac{p(z)+1}{p(z)-1} = u$$
with $|u| \geq |z|$. Since then
$$p(z) = \frac{1+u}{u-1}$$
we must have
$$\frac{|z|-1}{|z|+1} = \inf_{|u|\geq|z|} |\frac{u+1}{u-1}| \leq |p(z)| \leq \sup_{|u|\geq|z|} |\frac{u+1}{u-1}| = \frac{|z|+1}{|z|-1}.$$
Applying the last lemma we get
$$|h_{s,t}(z) - z| \leq (e^{t-s}-1)|h_{s,t}(z)+z|\frac{|z|+1}{|z|-1} \leq (e^{t-s}-1)2|h_{s,t}(z)|\frac{|z|+1}{|z|-1}.$$
Combining all the estimates we can conclude the proof of the proposition.
For further purposes let us notice that we have actually proved, on the way, the following

Corollaire 3.1.1. $\lim_{t\to s} h_{s,t}(z) = z$ *and the convergence is uniform on compact sets.*

3.1.2 Löwner differential Equation.

We come to the heart of the matter:

Théorème 3.1.1. *The family (f_t) is a Löwner chain if and only if*
(1) For each t, f_t is holomorphic in Δ and for each $z \in \Delta$, $t \mapsto f_t(z)$ is absolutely continuous. Moreover, f_0 is injective in Δ and $\forall t \geq 0, f_t(z) = e^t z + ..$ at ∞.
(2) There exists a family (p_t) of functions in $\mathcal{P}(\Delta)$, measurable in t, such that for almost $t \in [0, +\infty[$,
$$\forall z \in \Delta\,, \frac{\partial f_t(z)}{\partial t} = z\frac{\partial f_t(z)}{\partial z} p_t(z). \tag{3.1}$$

Suppose first that (f_t) is a Löwner chain: We can then write $f_t(z) = f_s(h_{s,t}(z))$ and thus

$$\frac{f_t(z)-f_s(z)}{t-s} = \frac{f_s(h_{s,t}(z))-f_s(z)}{h_{s,t}(z)-z} = \frac{h_{s,t}(z)-z}{h_{s,t}(z)+z}\frac{e^{t-s}+1}{e^{t-s}-1}\frac{e^{t-s}-1}{t-s}\frac{h_{s,t}(z)+z}{e^{t-s}+1}.$$

Lemme 3.1.3. *There exists a negligible subset E of $\mathbb{R}$ such that if $s \notin E$ then $t \mapsto f_t(z)$ is differentiable at s for every $z \in \Delta$.*

Proof : it is a direct consequence of the theorem of Vitali.

Take $s \in E$: since $\mathcal{P}(\Delta)$ is a normal family we may choose a sequence (t_n) converging to s such that $p_{s,t_n} \to p_s \in \mathcal{P}(\Delta)$ where

$$p_{s,t}(z) = \frac{h_{s,t}(z)-z}{h_{s,t}(z)+z}\frac{e^{t-s}+1}{e^{t-s}-1}.$$

Letting $n \to \infty$ we then obtain (3.1). The absolute continuity of f_t in t follows from proposition (3.1.1).

We come to the converse. Before starting the proof let us first notice that if (f_t) is a Löwner chain then the function $h_{s,t}(z) = f_s^{-1} \circ f_t(z)$, as a fuction of s, is a solution of the differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -wp_t(w), w(t) = z. \tag{3.2}$$

We reverse the point of view and consider the differential equation (3.2). Since

$$\frac{\mathrm{d}|w|^2}{\mathrm{d}t} = -2|w|^2\Re p_t(w)$$

the modulus of a solution is decreasing. It follows that the equation (3.2) has a solution $s \mapsto w(s;t,z)$ defined on $[0,t]$. By Cauchy-Lipschitz theorem, this function is injective in z. Moreover

$$\frac{\partial}{\partial s}(f_s(w(s;t,z))) = f_s'(w)\frac{\partial w}{\partial s}(s;t,z) + \frac{\partial f}{\partial s}(w(s;t,z)) = 0.$$

Since $w(t;t,z) = z$, it implies that $\forall s \leq t$, $f_s(w(s;t,z)) = f_t$. Taking $s = 0$ shows that all the $f_t's$ are injective and morover $f_t \prec f_s$. Since the Löwner partial differential equation obviously imply that $c(t) = e^t$, the proof is complete.

3.1.3 The Case of Slit Domains

We start with a very specific growing family, historically the one that Löwner has considered in his work in 1923. In consists in taking $K_t = \overline{\mathbb{D}} \cup \gamma([0,t])$ where $\gamma : [0,+\infty[\to \Delta$ is continuous and injective with $|\gamma(0)| = 1$,$\gamma(t) \in \Delta, t > 0$ and

$\gamma(t) \to \infty, t \to +\infty$.
The complement of a simple arc being connected, it is obvious that the corresponding family of domains

$$\Omega_t = \overline{\mathbb{C}} \backslash K_t$$

is continuous in the Caratheodory topology.
It follows that $t \mapsto c(t)$ increases continuously from 1 to $+\infty$ and we may as well assume that $c(t) = e^t$, so that the process is well described by a Löwner chain $(f_t)_{t \geq 0}$.

We wish to identify the family (p_t) of functions in $\mathcal{P}(\Delta)$ characterizing this process. To this end we define $g_t = f_t^{-1}$: since $\gamma(t)$ is not a cut point of $\gamma([0,t]) \cup \mathbb{R}$, the function g_t extends continuously to the point $\gamma(t)$ and we can define, for $t \geq 0$,

$$\lambda(t) = g_t(\gamma(t)) \in \partial\Delta.$$

Théorème 3.1.2. *The function λ is continuous on $[0, +\infty[$ and*

$$\forall t \geq 0 \, , p_t(z) = \frac{z + \lambda(t)}{z - \lambda(t)}.$$

In other words p_t is for all $t \geq 0$ the Poisson integral of the Dirac mass at $\lambda(t)$.
Proof : For $0 \leq s \leq t$ we may define $\delta(s,t) = g_t(\gamma([s,t])) \subset \partial\Delta$ and $S_{s,t} = g_s(\gamma([s,t])) \subset \overline{\Delta}$. The map $h_{s,t} = g_s \circ f_t$ can be extended, by Schwarz reflection, to a conformal isomorphism between the sphere minus $\delta(s,t)$ onto the sphere minus $S_{s,t} \cup S_{s,t}^*$, the reflection of $S_{s,t}$ into the unit circle. Notice that, by Beurling estimate, $\mathrm{diam}\delta(s,t) \to 0$ as s increases to t while $\mathrm{diam} S_{s,t} \to 0$ as t decreases to s by Caratheodory theorem.
Let us prove first that $S_{s,t}$ approaches $\lambda(t)$ as $s \nearrow t$. We now that for every $\varepsilon > 0, \delta(s,t)$ lies inside the disk C_ε of center $\lambda(t)$ and radius ε for s close enough to t. We know that $h_{s,t}$ converges uniformly on compact subsets of Δ to the identity. Using Cauchy formula with the contour consisting of the circle centered at 0 and radius 2 and $C_{\varepsilon'}$ we actually see, letting $\varepsilon' \to 0$, that this convergence occur on the compacts of the sphere minus $\lambda(t)$, and in particular on C_ε. The claim easily follows.
Using the same type of arguments with $h_{s,t}^{-1}$ (which also converges uniformly on compact subsets of Δ towards identity) one can see that $\delta(s,t)$ approaches $\lambda(s)$ as $t \searrow s$. The continuity of λ follows.
The previous proof, which is due to Ahlfors, has the advantage of being concise and elegant. Here is an alternative one, more quantitative, based on the following lemma that we will need for different puposes:
Proposition 3.1.2. *Let K be a compact set such that $\overline{\mathbb{D}} \subset K \subset \overline{\mathbb{D}} \cup \overline{D}(1,\varepsilon), \Omega$ its complementary domain and $g_K : \Omega \to \Delta$ its Riemann map. Then*

$$\forall z \in \partial\Delta \cap \partial\Omega \, , |g_K(z) - z| \leq C\varepsilon$$

for a universal constant C.

Proof : We write as usual $f_K = g_K^{-1}$ which maps Δ into itself so that $|f_K(z) \geq |z|$ by Schwarz lemma, which in turn implies that

$$p(z) = \frac{e^t+1}{e^t-1}\frac{f_K(z)-z}{f_K(z)+z}$$

where e^t =cap(K) , defines a function in $\mathcal{P}$. It follows that

$$|p(z)| \leq \frac{|z|+1}{|z|-1}$$

and, together with Koebe theorem we obtain that

$$|f_K(z)-z| \leq Ct$$

on $|z| = 2$. By using the same argument as in the first proof we also obtain

$$|f_K(-1)+1| \leq Ct.$$

We now define Δ_ε as being the image of the upper half plane intersected with $|z| > \varepsilon$ under the homography (sending the whole upper half plane onto Δ)

$$\zeta \mapsto \frac{1-i\zeta}{1+i\zeta}$$

and g_ε its Riemann map which can be computed explicitely: in particular

$$g_\varepsilon(z) = \frac{1-\varepsilon^2}{1+\varepsilon^2}z + ..,$$

showing that cap$(K) \leq 1 + C\varepsilon^2$. Let now fix $\theta \in [\varepsilon,\pi[$ and define $e^{i\theta'} = g_K(e^{i\theta})$. Then the harmonic measure seen from ∞ in Ω of the arc from $e^{i\theta}$ to $f_K(-1)$ is

$$\frac{\pi-\theta'}{2};$$

comparing this harmonic measure with the harmonic measure of the same arc in the domains $\Delta, \Delta_\varepsilon$ we get the desired result.

Corollaire 3.1.2. *Aternative proof of the continuity of λ.*

We keep the notations of the first proof. We may write

$$|g_t(\gamma(t)) - g_s(\gamma(s)| \leq |g_t(\gamma(t)) - g_s(\gamma(t))| + |g_s(\gamma(t)) - g_s(\gamma(s))|.$$

The second term in this sum is clearly bounded from above by Cd where d =diam$(S_{s,t})$. The first term may be written as

$$|\lambda(t) - h_{s,t}(\lambda(t))|$$

which is also bounded from above by Cd. But d is comparable with the harmonic measure seen from ∞ in $\Delta\backslash S_{s,t}$ of $S_{s,t}$ which is also equal to the harmonic measure seen from ∞ in Ω_t of $\gamma([s,t])$. By Beurling theorem, this is bounded from above by $C\sqrt{\text{diam}\gamma([s,t])}$. Continuity follows.

To derive the Löwner equation we make use of the following formula which is a variant of Cauchy's

$$\log(\frac{h_{s,t}(z)}{z}) = \frac{1}{2\pi}\int_{\delta_{s,t}} \frac{z+e^{i\theta}}{z-e^{i\theta}} \log|h_{s,t}(e^{i\theta})|d\theta \tag{3.3}$$

If we let $z \to \infty$ in this equality we get

$$t-s = \frac{1}{2\pi}\int_{\delta_{s,t}} \log|h_{s,t}(e^{i\theta})|d\theta$$

and we put $z = g_t(\omega)$ to obtain

$$\log(\frac{g_t(\omega)}{g_s(\omega)}) = \frac{1}{2\pi}\int_{\delta_{s,t}} \frac{g_t(\omega)+e^{i\theta}}{g_t(\omega)-e^{i\theta}} \log|h_{s,t}(e^{i\theta})|d\theta.$$

Putting everything together we get

$$\frac{\partial}{\partial t}(\log(g_t(\omega))) = \frac{g_t(\omega)+\lambda(t)}{g_t(\omega)-\lambda(t)}$$

from which the equation for f_t follows as in the general case.

3.1.4 Löwner Proof of the Bieberbach conjecture for $n = 3$.

In this paragraph we develop the result for which Löwner has invented (or discovered) his equation. Let us recall the Bieberbach conjecture: if $f(z) = z + a_2z^2 + a_3z^3 + ..$ belongs to the class S then $\forall n \geq 2, |a_n| \leq n$. The case $n = 2$ is of course covered by Koebe theorem and generally it sufffices to prove that $|a_n| \leq Ia_1|$ for any map $f : \mathbb{D} \to \mathbb{C}$ injective with $f(z) = a_1z + a_2z^2 + ...$ Moreover we may as well, replacing $f(z)$ by $f(rz)$, assume that f is analytic across the unit disc. So in order to prove Bieberbach conjecture for $n = 3$ we may assume that $\partial f(\mathbb{D})$ is a Jordan curve Γ lying inside the unit disc. using Caratheodory convergence theorem we approximate f by the Löwner chain f_t obtained from the slit starting at 1 along the radius towards 0 until it reaches Γ and then following Γ until it reaches its starting point.
To be more precise we need here a modification of the radial processes we have introduced in the last paragraph, because we work in the disc instead of Δ. But

we pass from one process to the other by the rule $F_t(z) = 1/f_t(1/z)$. To be concrete, if F_t is a Löwner chain in Δ with equation

$$\frac{\partial F_t}{\partial t} = z\frac{\partial F_t}{\partial z}p_t(z)$$

then f_t satisfies the equation

$$\frac{\partial f_t}{\partial t} = -z\frac{\partial f_t}{\partial z}p_t(1/z).$$

In our case we must then have

$$\frac{\partial f_t}{\partial t} = -z\frac{\partial f_t}{\partial z}\frac{\lambda(t)+z}{\lambda(t)-z}$$

for some continuous function $\lambda : [0,t_0] \to \partial\mathbb{D}$.
Now we can write, developping at 0 these expressions starting from $f_t(z) = e^{-t}(z + a_2z^2 + a_3z^3 + ..)$,

$$\frac{\partial f_t}{\partial t}(z) = -e^{-t}(z + (a_2 - a_2')z^2 + (a_3 - a_3')z^3 + ..),$$

$$\frac{\partial f_t}{\partial z}(z) = e^{-t}(1 + 2a_2z + 3a_3z^2 + ..),$$

$$\frac{\lambda(t)+z}{\lambda(t)-z} = 1 + 2\frac{z}{\lambda} + 2\frac{z^2}{\lambda^2} + ..$$

from which it follows by identification that

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_2e^t) = -2\frac{e^t}{\lambda}$$

(notice that gives a new proof of Koebe theorem), and

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_3e^{2t}) = e^{2t}(-4\frac{a_2}{\lambda} - \frac{2}{\lambda^2})$$

from which we easily deduce that

$$a_3e^{2t} = 4\left(\int_0^t \frac{e^s}{\lambda(s)}ds\right)^2 - 2\int_0^t \frac{e^{2s}}{\lambda(s)^2}ds.$$

Writing $\lambda(s) = e^{i\theta(s)}$ and taking the real part of the last expression we get

$$\Re(a_3e^{2t}) = 4\left(\int_0^t e^s\cos(\theta(s))ds\right)^2 - 4\left(\int_0^t e^s\sin(\theta(s))ds\right)^2 - 2\int_0^t e^{2s}\cos(2\theta(s))ds.$$

By Cauchy-Schwarz inequality,

$$\left(\int_0^t e^s\cos(\theta(s))ds\right)^2 \le e^t\int_0^t e^s\cos(\theta(s))^2ds$$

and

$$\Re(a_3e^{2t}) \le 4\int_0^t e^s(e^t - e^s)ds + e^{2t} - 1 < 3e^{2t}.$$

3.1.5 Chains generated by Curves

We have studied so far the case of a growth process of the form $K_t = \gamma([0,t])$ where γ is a Jordan arc, and shown that this Löwner process is driven by a continuous function U_t with values in the unit circle; we will call such a function a driving function. Conversely, if $t \mapsto U_t$ is a driving function, what can be said about the corresponding Löwner process? The next theorem precisely describe those Löwner processes starting at $\overline{\mathbb{D}}$ that are driven by a continuous function:

Théorème 3.1.3. *The process $(\Omega_t)_{t\geq 0}$ has a driving function if and only if for every $T \geq 0\,, \varepsilon > 0$ there exists $\delta > 0$ such that for every $t \leq T$ there exists a connected subset S of Ω_t of diameter less than ε disconnecting $K_{t+\delta}$ from ∞.*

The if part: (to be done)
To prove the converse we observe that it suffices to prove that the diameter of $g_t(K_{t+\delta}\backslash K_t)$ goes to 0 with δ. This set thus corresponds to an interval $\delta_{s,t} = I$ of the unit disk which is small. Let z_I be the point in Δ whose distance to $\partial\Delta$ is $|I|/2$ and is attained at the middle of the interval I. Then it is known that there exits a crosscut of Δ passing through z_I and separating I from ∞ such that its image has length $\leq C\mathrm{dist}(f_{t+\delta}(z_I,K_{t+\delta})$. This completes the proof. It remains to prove the initial claim: to do so we may assume $t = 0\,,\lambda(0) = 1$ and we must prove that $\mathrm{diam}(K_s \to 0$ with s. Define

$$M_t = \max(\sqrt{(}t)\,, \sup_{s\leq t} |\lambda(s) - 1|$$

and consider $z \in \Delta$ such that $z - 1| \geq 10M_t$ and $\sigma = \inf\{s > 0\,;\, g_s(z) - 1| = M_t\}$. Then either $\sigma > t$ or $\sigma \leq t$: in the second case, if $M_t < 3/4$ we must have

$$M_t = |g_\sigma(z) - z| \leq \frac{\sigma}{M_t} \to \sigma \geq t.$$

In both cases $g_s(z)$ is defined at least up to time t, which means that $z \notin K_t$. It follows that $\mathrm{diam}(K_t) \leq 10M_t$.
Notice that the uniformity in the theorem is needed to ensure the continuity of the driving function. Indeed consider the following situation: let us consider the curve consisting in going from 1 to 3, then going around the circle of center $3 + i$ and radius 1 and then continuing up along the real axis. Denote by $a_t < b_t$ the two images by g_t of 3: then $\lambda(t)$ comes close to a_t while the point turns around the circle, but suddenly jumps after b_t when the curve crosses itself.

Assume now that the process (K_t) has a driving function. The preceeding theorem shows that we cannot hope to prove that $K_t = \gamma([0,t])$ for some simple curve γ. However we may hope that the process is generated by a curve in the following sense:

Définition 3.1.3. *We say that the Löwner process (Ω_t) is generated by the curve $\gamma : [0, +\infty[\to \overline{\Delta}$ if for all $t \geq 0$, Ω_t is the unbounded component of $\Delta\backslash\gamma([0,t])$.*

Unfortunately this is also false as the following counterexample shows: Let γ_1 be a curve that starts at 1 and spirals towards the circle of center 3 and radius 1 as t varies in $[0{,}1[$ and γ_2 a similar function from $]1,+\infty[$ such that $\gamma_2(t)\to\infty$ as $t\to+\infty$ and spirals towards the same circle as t decreases to 1 but without cutting γ_1. We then define $K_t=\gamma_1([0{,}t])$ if $t<1$, $K_1=\gamma_1([0{,}1[)\cup\overline{D(3{,}1)}$, $K_t=K_1\cup\gamma_2(]1{,}t])$ if $t>1$. A moment's reflection shows that the condition in the preceeding theorem is satisfied; this process has thus a driving function but is not generated by a curve. The point is here that K_t is not locally connected. In fact we have the following theorem:

Théorème 3.1.4. *Let (Ω_t) be a Löwner process with driving function U. Then this process is generated by a curve if and only if K_t is locally connected for all $t\geq 0$.*

Proof : We call a point $t-$accessible if $z\in K_t\setminus\cup_{s<t}K_s$ and if there exists an arc $\eta\,:\,[0{,}1[\to\mathbb{C}$ with $\eta(0)=z\,,\eta(]0{,}1])\subset\Omega_t$.

Lemme 3.1.4. *For all t there is at most one $t-$accessible point.*

Proof : Let z be a $t-$accessible point and η the corresponding curve. If $\varepsilon>0$ then $D(z{,}\varepsilon)\cap\cup_{s<t}K_s\neq\emptyset$ so that this set contains $z_\varepsilon\in K_{s_\varepsilon}$. We then consider the segment joining z and z_ε that one cuts as soon as one meets K_{s_ε}. Let η_ε be the curve starting from this point joining z with a straight line and continuing with η until we reach distance ε from z. Then diamη_ε is smaller than 2ε, implying that $\mathrm{diam}(g_{s_\varepsilon}(\eta_\varepsilon))\leq C\sqrt{\varepsilon}$. One deduces from this study that

$$\lim_{s\to t_-}g_s(z)=\lim_{\alpha\to 0}g_t(\eta(\alpha))$$

first exists and secondly must be equal tu U_t by the theory of ODE's. By the discussion in Caratheodory continuity theorem, there are no other $t-$accessible point.

to be continued.

3.2 Chordal Löwner equation

Let us start with the following simple situation: $\gamma\,:\,[0{,}\infty)\to\overline{\mathbb{H}}$ is a simple curve with $\gamma((0{,}\infty))\subset\mathbb{H}$, $\gamma(0)=0$ and $\lim_{t\to\infty}\gamma(t)=\infty$. For $t\geq 0$ we write $\mathbb{H}_t=\mathbb{H}\setminus\gamma([0{,}t])$ so that the sequence $\mathbb{H}_t$ is a decreasing sequence of simply connected subdomains of $\mathbb{H}$. By the preceeding chapter there exists a unique holomorphic and bijective mapping $g_t\,:\,\mathbb{H}_t\to\mathbb{H}$ such that $g_t(z)-z\to 0$ as $z\to\infty$. We want to prove that $t\mapsto g_t(z)$ is the flow of a remarkable differential equation, the Löwner Differential Equation.

First of all

$$g_t(z)=z+\frac{b(t)}{z}+O(\frac{1}{|z|^2})\tag{3.4}$$

where $b(t)=\mathrm{hcap}(\gamma([0{,}t]))$. Notice that $t\mapsto b(t)$ is an increasing function.

Proposition 3.2.1. *The function b is continuous and increasing on $\mathbb{R}_+$.*

Proof : We already know that b is increasing. The continuity will be an immediate consequence of the following lemma:

Lemme 3.2.1. *If $A_1 \subset A_2$ are two hulls such that $\forall z \in \partial A_2, d(z,A_1) < \varepsilon$, then*

$$\mathrm{hcap}(A_2) - \mathrm{hcap}(A_1) \leq C\varepsilon^{1/3}\mathrm{diam}(A_2)$$

for some constant C.

Proof : An immediate consequence of Beurling lemma is that, for $M > 1, z \in \partial A_2$,

$$\omega(z,B(z,Md(z,\partial(A_1))) \cap \mathbb{H}\backslash A_1; \partial B(z,Md(z,\partial(A_1)))) \leq CM^{-1/2}.$$

On the other hand, if we put

$$v_j(z) = \Im(z) - g_{A_j}(z), j = 1,2$$

we can write

$$v_2(z) - v_1(z) = \int_{\partial A_2} \left(\int_{\partial A_1} (\Im(u) - \Im(v))\, \omega(u,\mathbb{H}\backslash A_1; dv) \right) \omega(z,\mathbb{H}\backslash A_2; du).$$

If we split the integral on ∂A_1 into two parts whether $|v-u|$ is smaller or greater than $Md(u,\partial A_1)$, the just mentionned consequence of Beurling theorem allows us to bound this integral from above by $M\varepsilon + CM^{-1/2}$. We now choose $M = \varepsilon^{-2/3}$. A trivial estimate involving harmonic measure for the half-plane then implies that

$$\Im(z)(v_2(z) - v_1(z)) \leq C\varepsilon^{1/3}\mathrm{diam}(A_2).$$

We now come to the more difficult derivation of the differential equation. We first introduce some notation. We put $f_t = g_t^{-1}$; by theorem(2.3.2), $\lim_{t\to s+0} g_s(\gamma(t)) = U_s$ exists and by Caratheodory theorem, since f_t can be continuously extended to $\overline{\mathbb{H}}$,$f_t(U_t) = \gamma(t)$. Let $0 \leq s \leq t$: then $g_t = g_{t,s} \circ g_s$ where $g_{s,t} = g_{g_s(\gamma([s,t]))}$. Furthermore we have $b(t) = b(s) + \mathrm{hcap}(g_s(\gamma([s,t]))$.

Proposition 3.2.2. *: There is a constant $c(\gamma,t_0) > 0$ such that if $0 \leq s \leq t \leq t_0 \leq +\infty$ then*

$$\mathrm{diam}(g_s(\gamma([s,t])) \leq C\sqrt{\mathrm{diam}(\gamma([0,t_0]))\omega_\gamma(t-s)} \tag{3.5}$$

where ω_γ is the modulus of continuity defined as

$$\omega_\gamma(\delta) = \sup_{0\leq s\leq t\leq t_0, t-s\leq\delta} (|\gamma(t) - \gamma(s)|)$$

Proof : It will follow from Beurling thorem (2.5.1). let D denote the diameter of $\gamma([0,t_0])$ and $w = \gamma(t_0) + 10iD$. By corollary (2.7.1),$13D \geq \Im(g_s(w) \geq 7D$. Morover, we can write, by conformal invariance of harmonic measure,

$$\omega_1 = \omega(w,\mathbb{H}_t; \gamma([s,t])) = \omega_2 = \omega(g_s(w),g_s(\mathbb{H}_t); g_s(\gamma([s,t]))).$$

By Beurling theorem we have

$$\omega_1 \leq C\sqrt{\frac{\omega_{t-s}}{D}}.$$

A slightly more tricky application of the same Beurling theorem shows that

$$\omega_2 \geq c\frac{\mathrm{diam}(g_s(\gamma([s,t]))}{D}.$$

The proposition follows.

Corollaire 3.2.1. *: The function $t \mapsto U_t$ is continuous on $\mathbb{R}_+$.*

Proof : $g_t(z) - g_s(z) = g_{s,t}(\zeta) - \zeta, \zeta = g_s(z)$. By corollary(2.7.1, 3.5) it follows that g_s converges uniformly on $\mathbb{R}$ towards g_t as $s \to t$, a fact that implies the corollary.

The fact that b is increasing and continuous implies the useful result that we can always perform a time change to ensure that b is actually C^1. Assuming this we can prove the following theorem, due to Löwner, which shows that (g_t) is a remarkable flow:

Théorème 3.2.1. *: Suppose γ is a simple curve as above and that b is C^1,$b(t) \to +\infty$ as $t \to +\infty$. Then $t \mapsto g_t(z)$ is the solution of the differential equation*

$$y' = \frac{b'(t)}{y - U_t}, y(0) = z. \tag{3.6}$$

We can precise the theorem by adding that if $z = \gamma(t_0)$ for some $t_0 \in \mathbb{R}_+$ then the solution is defined up to time t_0 while it is defined in $\mathbb{R}_+$ if $z \notin \gamma((0, +\infty))$. In this last case $\Im(g_t(z))$ decreases to 0 as $t \to \infty$.

Proof : We write

$$g_{s+\varepsilon}(z) - g_s(z) = g_{s,s+\varepsilon}(\zeta + U_s) - U_s - \zeta$$

where $\zeta = g_s(z) - U_s$. We then observe that $g_{s+\varepsilon}(\zeta + U_s) - U_s = g_A(\zeta), \quad A = g_s(\gamma([s,s+\varepsilon])) - U_s$. By (2.19) we can then write

$$g_{s+\varepsilon}(z) - g_s(z) = \frac{b(s+\varepsilon) - b(s)}{g_s(z) - U_s} + \mathrm{diam}(\gamma([s,s+\varepsilon]))[b(s+\varepsilon) - b(s)]O(\frac{1}{|g_s(z) - U_s|^2}), \tag{3.7}$$

from which the result follows.

Let us come to the remark that follows the statement of the theorem. First of all, if $x \in \mathbb{R}, \quad x > U_0$ then $g_t(x)$ is defined for all $t \geq 0$. It follows that $\forall t \geq 0, g_t(x) > 0$ and thus that $t \mapsto g_t(x)$ is increasing: classical theory of differential equations then implies that it must go to $+\infty$ as t grows. If now $z \in \mathbb{H} \backslash \gamma([0, +\infty))$ then a simple computation shows that $t \mapsto \Im g_t(z)$ is decreasing. Moreover the limit as $t \to \infty$ must be 0. To see this we first change time so that $b(t) = 2t$: then if

$\Im g_t$ converges to $c > 0$ we would have that the derivative of the function stays $\leq -c' < 0$, a fact which obviously leads to a contradiction.

In what we have just seen we have produced a continuous function from a curve inside the upper-half plane. For later purposes we prefer to see this function $t \mapsto U_t$ as a function from $\mathbb{R}_+$ in the space of positive measures on $\mathbb{R}$, namely $t \mapsto \delta_{U_t}$, the Dirac mass at point U_t. Using this language, we have the following converse to the last theorem:

Théorème 3.2.2. *: Suppose* $(\mu_t),\quad t \geq 0$ *is a one parameter family of positive finite Borel measures on* $\mathbb{R}$ *such that* $t \mapsto \mu_t$ *is continuous in the weak topology and such that* $\forall t \geq 0 \exists M_t \geq 0$ *such that* $s \leq t \Rightarrow \mu_s(\mathbb{R}) \leq M_t$ *and* $supp(\mu_s) \subset [-M_t, M_t]$*. For each* $z \in \mathbb{H}$ *let* $t \mapsto g_t(z)$ *be the unique solution of the differential equation*

$$y' = \int_{\mathbb{R}} \frac{d\mu_t(u)}{y-u} \tag{3.8}$$

with initial data $y(0) = z$*. For* $z \in \mathbb{H}$ *let* T_z *be the life-time of the solution and let* $\mathbb{H}_t = \{z \in \mathbb{H}; T_z > t\}$*. Then* g_t *is the unique conformal mapping from* $\mathbb{H}_t$ *into* $\mathbb{H}$ *such that* $g_t(z) - z \to 0, z \to \infty$ *and moreover*

$$g_t(z) = z + \frac{\int_0^t \mu_s(\mathbb{R}) ds}{z} + O(\frac{1}{|z|^2}), \quad z \to \infty. \tag{3.9}$$

Proof : As before we see that $\Im(g_t(z))$ decreases with t and that $T_z = \sup\{t; \Im(g_t(z)) \geq 0\}$. It follows that g_t maps in $\mathbb{H}$. Cauchy-Lipschitz theorm implies that g_t is injective and the analyticity (in z) of the equation proves that g_t is holomorphic. To finish the proof one just needs to prove that $g_t(\mathbb{H}_t) = \mathbb{H}$. To this end we introduce the "backward flow" defined as the solution of the equation

$$\dot{h_s}(w) = -\int_{\mathbb{R}} \frac{\mathrm{d}\mu_{t-s} u}{h_s(w) - u}, h_0(w) = w. \tag{3.10}$$

Since $\Im(h_s)$ is now increasing this function is defined for all $w \in \mathbb{H}$ and $0 \leq s \leq t$. Now if h_s is such that $h_0(w) = w$ then $\tilde{g}_s := h_{t-s}$ satisfies (3.6) and $\tilde{g}_0(w) = h_t(w)$. This implies $\tilde{g}_s(w) = g_s(h_t(w))$. In particular $\tilde{g}_t(w) = g_t(h_t(w)) = w$. The rest of the proof is easy and left to the reader.

As we have already pointed out the case $\mu_t = 2\delta_{U_t}$ leads to the important case

$$\dot{g}_t = \frac{2}{g_t - U_t}, g_0(z) = z. \tag{3.11}$$

A flow (g_t) satisfying such an equation is called a Löwner chain. A generalized Löwner chain will denote a flow satisfying the more general equation

$$\dot{g}_t = \frac{\dot{b}(t)}{g_t - U_t}, g_0(z) = z \tag{3.12}$$

where b is an increasing function from $\mathbb{R}_+$ onto itself. This equation corresponds to the case $\mu_t = \dot{b}(t)\delta_{U_t}$. We pass from the last to the former with just a time change as we have already seen.

We have just shown that if γ is a simple closed curve with hcap$(\gamma([0,t]) = 2t$ then the corresponding conformal maps with the right hydrodynamical normalization form a Löwner chain. But the converse is not true: if (g_t) is a Löwner chain then the corresponding domains do not generally come from a simple curve. The following theorem actually characterizes Löwner chains:

Définition 3.2.1. *An increasing family of hulls K_t in $\mathcal{Q}$ is called right-continuous at t if $\cap_{\delta>0}\overline{g_{t+\delta}(K_{t+\delta}\backslash K_t)}$ is a single point which we denote by U_t.*

If (K_t) is right continuous and if b is right differentiable at t then the above proof goes through to show that

$$\lim_{\delta\to 0}\frac{g_{t+\delta}(z)-g_t(z)}{\delta} = \frac{\dot{b}(t)}{g_t(z)-U_t}\ \forall z\in\mathbb{H}\backslash K_t. \tag{3.13}$$

Théorème 3.2.3. *: Let $(K_t)_{t\geq 0}$ be an increasing sequence of compact $\mathbb{H}-$hulls and (g_t) the corresponding normalized conformal mappings. Then (g_t) is a generalized Löwner chain if and only if (K_t) is right continuous everywhere, b is C^1 and U is continuous.*

Proof : The if part is proved the same way as (3.2.2). For the only if part we start with a flow of type (3.11) and we only need to prove right continuity of the compact hulls. But this follows immediately from the following

Lemme 3.2.2. *: Suppose $U_0 = 0$ and let, for $t\geq 0, R_t = \max\{\sqrt{t}, \sup_{s\leq t}(|U_s|)\}$ .Then, for $t\geq 0, K_t\subset B(0,4R_t)$.*

Proof : Suppose $|z| > 4R_t$ and let σ be the first time that $|g_s(z)-z|\geq R_t$. Then $|\dot{g}_s(z)|\leq 1/R_t, s\leq\min(t,\sigma)\Rightarrow |z-g_s(z)|\leq s/R_t, s\leq\min(t,\sigma)$. Hence either $\sigma > t$ or $\sigma > R_t^2\geq t$, so $\sigma\geq t$.

3.3 Chains generated by curves

In this section we continue to consider curves γ with $\gamma(0)\in\mathbb{R}$ but we generalize the situation by allowing γ to come back on $\mathbb{R}$ or to self intersect. So $\gamma:\mathbb{R}_+\to\overline{\mathbb{H}}$ is a continuous function with $\gamma(0)\in\mathbb{R}$. For $t\geq 0$ we define $\mathbb{H}_t$ as the unbounded component of $\mathbb{H}\backslash\gamma([0,t])$ and $K_t = \mathbb{H}\backslash\mathbb{H}_t$. The new phenomenon is that K_t may be much larger than $\cup_{s<t}K_s$. This happens at times t when $\gamma(t)\in\mathbb{R}$ or when the curve self-intersects. Consider for example $\gamma(t) = e^{2i\pi t}, 0\leq t\leq 1$: if $t<1$, K_t is a piece of a circle while K_1 is a half-disk. However, if $\partial_t = \partial\mathbb{H}_t\cap\mathbb{H}$ then

Proposition 3.3.1. *: $\partial_t\cap\mathbb{H}\subset\overline{\cup_{s<t}K_s}$*

Proof : It is clear that $\overline{\cup_{s<t}K_s} \supset \gamma([0,t])$: so, if $z \in \mathbb{H}\backslash\overline{\cup_{s<t}K_s}$ there exists $\varepsilon > 0$ such that $B(z,\varepsilon)$ is included in a component of $\mathbb{H}\backslash\gamma([0,t])$ and thus cannot intersect ∂_t.

Définition 3.3.1. *A Löwner chain* (3.11) *is said to be generated by the curve γ if for every $t > 0$ the domain of g_t is the unbounded component of $\mathbb{H}\backslash\gamma([0,t])$.*

As we have seen, simple curves generate Löwner chains. But not all curves do so. Even if hcap is continuous it may be that it stays constant on some interval (if $\gamma(t) \in K_s$ for $t \in [a,b]$ with $a > s$ for instance. This implies that we cannot reparametrize the curve so that $\text{hcap}(\gamma[0,t]) = 2t$. This gives a first example of a curve not giving rise to a Löwner chain. Another example is a curve with a double point: the function U_t would be discontinuous.

On the other hand there are non-simple curves that generate Löwner chains. Our next task will be precisely to characterize those Löwner chains that are generated by curves.
We consider a Löwner chain g_t with driving function U_t. As before we set $\mathbb{H}_t$ as to be the set of points $z \in \mathbb{H}$ such that Löwner flow started from z lives at least up to time t. We also recall that $K_t = \mathbb{H}\backslash\mathbb{H}_t$ and that $\partial_t = \partial\mathbb{H}_t \cap \mathbb{H}$. With the above notations let us define

$$J_t = \overline{K_t}\backslash \cup_{s<t} \overline{K_s}. \tag{3.14}$$

We say that z is $t-$accessible if $z \in J_t$ and if there exists a curve $\eta : [0,1] \to \mathbb{C}$ such that $\eta(0) = z, \eta((0,1]) \subset \mathbb{H}_t$.

Lemme 3.3.1. *if $t > 0$ and z is a $t-$accessible point then there exists a strictly increasing sequence $s_j \to t$ and a sequence (z_j) of s_j-accessible points converging to z.*

Proof : Consider z $t-$accessible. Since $z \in \partial\mathbb{H}_t$, $\quad z \in \cup_{s<t}\overline{K_s}$. For every $\varepsilon > 0$ there thus exists $s_\varepsilon = s, \zeta \in \overline{K}_s \cap B(z,\varepsilon)$. Drawing the line segment from z to ζ we then find an $s-$ accessible point in the same ball. We get the lemma by letting $\varepsilon \to 0$ since for every $s < t$ $z \in \mathbb{H}_s$.

Proposition 3.3.2. *: For each $t > 0$ there is at most one $t-$accessible point and ∂_t is included in the closure of the set of $s-$accessible points for $s \leq t$.*

Proof : The key of the proof is theorem(2.3.2). Let z be $t-$accessible: then there exists a access to z inside $\mathbb{H}_t$ and this access can be extended by last lemma to an access to a $s-$accessible point very close to z, s being also very close to t. Altogether this path can be made as small as we wish: using then (3.5) it follows that the limit of $g_t(\zeta)$ along the access in $\mathbb{H}_t$ must be equal to $\lim_{s\to t-} g_s(z) = U_t$. By the theorem we have just proven, z must be unique. The rest of the proposition is easy and left to the reader.

Proposition 3.3.3. *: Let $V(y,t) = g_t^{-1}(iy + U_t)$. If $\gamma(t) = \lim_{y\to 0} V(y,t)$ exists* *for $t \geq 0$ and this function is continuous on $[0, +\infty)$ then g_t is the Löwner chain generated by the curve γ.*

Proof : By proposition (3.3.2) the point $\gamma(t)$ is the only possible $t-$accessible point. Therefore the set of $s-$accessible points for $s \leq t$ is contained in $\gamma([0,t])$. By the second part of proposition (3.3.2) the Löwner chain (g_t) is generated by γ.

We can now state the characterization:

Théorème 3.3.1. *: The Löwner chain (g_t) is generated by a curve if and only if for each $t \geq 0, \overline{K_t}$ is locally connected.*

Proof : The $\Rightarrow$ part is obvious. Suppose conversely that $t \geq 0, \overline{K_t}$ is locally connected. Then, using Caratheodory theorem, g_t^{-1} extends contnuously to the closed half-plane and the function γ is well-defined. It remains to prove that γ is continuous. But $\gamma(t)$,$\gamma(t+\delta) \in \overline{K_{t+\delta} \backslash K_t}$ and this set has small diameter since first $K_{t,t+\delta}$ has small diameter because (g_t) is a Löwner chain and secondly g_t^{-1} is continuous up to the boundary.

We give an example of a Löwner chain which is not generated by a curve. We start with the logarithmic spiral

$$\lambda(t) = (t-1)e^{i \ln |t-1|}, 0 \leq t \leq 2, \tag{3.15}$$

and we define

$$\gamma(t) = F(\lambda(t)), F(z) = i[\frac{|z|+1}{|z|} z + 2] \tag{3.16}$$

for $t \in [0,2] \backslash \{1\}, \gamma(t) = (t-2)i, t \geq 2$. We then define the hulls $K_t = \gamma((0,t]), t < 1, K_1 = \gamma((0,1)) \cup |z| = 1, K_t = K_1 \cup \gamma((1,t]), t > 1$. It is easy to see that if $t < s$ we can find a cross-cut $c(t,s)$ separating $K_s \backslash K_t$ from ∞ in $\mathbb{H}_t$ and whose diameter tend to 0 as $t - s \to 0$. By Beurling lemma, this imples that we have a Löwner chain. It is not generated by a curve since K_1 is not locally connected.

In fact it can be proven in this example that the driving function is Hölder with exponent 1/2.

We end this section with the

Proposition 3.3.4. *Suppose (g_t) is a Löwner chain generated by a curve γ such that the driving function satisfies*

$$\exists r < \sqrt{2} \quad \forall s < t \quad |U_t - U_s| \leq r\sqrt{t-s}, \tag{3.17}$$

then the curve γ is simple.

Proof : It starts with a very simple but very useful characterization of simple paths. Let g_t be a Löwner chain generated by γ. For $s > 0$ the Löwner chain $gt^{(s)} = g_{s+t} \circ g_s^{-1}$ is generated by the path $\gamma^s(t) = g_s(\gamma(s+t))$ and has driving function $U_t^{(s)} = U_{s+t}$. If $\gamma(s) = \gamma(s+t)$ for some $s,t > 0$ then $\gamma^{(s)}(t) = g_s(\gamma(s)) = U_s \in \mathbb{R}$. Hence γ is a simple path if and only if for all $s > 0$,$\gamma^{(s)}((0,+\infty)) \cap \mathbb{R} = \emptyset$.

To prove the proposition it suffices thus to show that (3.17) implies $\forall x > U_0, \forall t > 0, g_t(x) \neq U_t$.
Since $r < \sqrt{2}$ we can choose $\rho \in (0,1)$ such that $2\rho^3 r^{-2} - \rho > \rho^{-1} - 1$. Let t_1 be the first time that $|U_t| \geq \rho x$ and t_2 the first time that $g_t - U_t > x/\rho$. Since $t \mapsto g_t(x)$ is increasing it suffices to show that $t_2 \leq t_1$. By (3.17), $t_1 \geq (\frac{\rho x}{r})^2$. On the other hand, Löwner equation gives $\dot{g}_t(x) \geq 2\rho/x, t \leq t_2$. If then $\min t_1, t_2 \geq (\frac{\rho x}{r})^2$

$$g_{(\frac{\rho x}{r})^2}(x) \geq x + (2\rho^3/r^2)x \geq x[1 + 2\rho^3/r^2 - \rho] + U_{(\frac{\rho x}{r})^2} > x/\rho + U_{(\frac{\rho x}{r})^2} \quad (3.18)$$

which implies $\min(t_1, t_2) = t_2$, what we wanted.

Chapitre 4

Stochastic Processes and Brownian Motion

4.1 Construction of Brownian Motion

We consider a probability space $(\Omega,\mathcal{F},P)$, a measurable set $E,\mathcal{E}$ and a set T.

Définition 4.1.1. *: A stochastic process indexed by T and with values in E is a family (X_t) of measurable functions $\Omega \to E$.*

The space E is the state space while T represents time. Most of the time $T = \mathbb{N}$ or $\mathbb{Z}$ (discrete case) or $\mathbb{R}_+, \mathbb{R}$ (continuous case).
We now proceed to construct the most important stochastic process, i.e. Brownian Motion (BM). To this end we start with the

Proposition 4.1.1. *:Let H be a separable Hilbert space. There exists a probability space $(\Omega,\mathcal{F},P)$ and a family $(X_h), h \in H$ of real random variables such that*

- *(i) $h \mapsto X_h$ is linear,*
- *(ii) X_h is for $h \in H$ a centered Gaussian variable with*

$$E(X_h)^2 = \|h\|^2.$$

Proof : Consider an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. We know that there exists a probability space and a sequence of independent reduced Gaussian variables. It then suffices to define $X_h = \sum_{n\geq 0} < h,e_n > g_n$.

Définition 4.1.2. *:When $H = L^2(A,\mathcal{A},\mu)$ then the mapping $h \mapsto X_h$ is a Gaussian measure with intensity μ.*

The reason for this definition is that we can define for $F \in \mathcal{A}\,;\, \mu(F) < \infty, X(F) = X_{1_F}$. Since in a Gaussian space L^2 convergence and almost sure convergence are equivalent it is true that if $\mu(F) < \infty, F = \cup F_n$ then

$$X(F) = \sum_{n=0}^{\infty} X(F_n)$$

a.s. It is not true though that for almost all $\omega, F \mapsto X(F)(\omega)$ is a measure. Let us also notice that if $F,G \in \mathcal{A}, \mu(F), \mu(G) < \infty$, then

$$E(X(F)X(G)) = \mu(F \cap G).$$

Let us start the construction of BM. We put $(A,\mathcal{A},\mu) = (\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+),\text{Lebesgue measure})$ and for each $t \geq 0$ we choose an element B_t in the class $X([0,t])$. Let us study the properties of this stochastic process:
1) By the last remark above this process has independent increments, i.e. if $t_0 < t_1 < ... < t_n$ the variables $B_{t_{i+1}} - B_{t_i}$ are independent for $i = 0,...,n-1$.
2) With the same notations the (vectorial) variable $(B_{t_0},B_{t_1},...,B_{t_n})$ is Gaussian.
3) For each $t, E(B_t^2) = t$.
To have a good definition of Brownian motion we need further the paths $t \mapsto X_t(\omega)$ to be a.s. continuous. But in order for this statement to be meaningful we need that the set of $\omega' s$ for which the path is continuous to be measurable, and there is no reason for that. To overcome this difficulty we will use the following notions:
Définition 4.1.3. *: Two processes X,X' (not necessarilly defined on the same probability space but with the same state space) are said to be a version of each other if for every sequence of times $t_1,..,t_n$, the variables $(X_{t_1},..,X_{t_n})$ and $(X'_{t_1},..,X'_{t_n})$ have the same law.*
Définition 4.1.4. *: Two processes X,X' defined on the same probability space and with the same state space are said to be a modification of each other if for every t, a.s. $X_t = X'_t$. They are called indistinguishable if a.s. $\forall t,X_t(\omega) = X'_t(\omega)$.*

If two processes are modifications of each other then they are versions of each other. Also, since a continuous function on $\mathbb{R}$ is determined by its values on $\mathbb{Q}$ two processes that are a.s. continuous and that are modifications of each other are indistinguishable.
Théorème 4.1.1. *(Kolmogorov's criterium) A real-valued process for which there exists $\alpha \geq 1, \beta, C > 0$ such that for every t,h*

$$E[|X_{t+h} - X_t|^\alpha] \leq Ch^{1+\beta}$$

has a modification which is almost-surely continuous.
Proof : We put, for $j \in \mathbb{N}, K_j = \sup\{|X_t - X_s|,\ t,s \text{ dyadic of order } j,\ |t-s| = 2^{-j}\}$.
Then $E(K_j^\alpha) \leq \sum_{\text{allpossible s,t}} E[|X_t - X_s|^\alpha] \leq 2^j c 2^{-j(1+\beta)} = c2^{-j\beta}$. Let now s,t be two dyadic number in [0,1] such that $|s-t| \in [2^{-m-1},2^{-m}]$. Let s_j,t_j be the biggest dyadic numbers of order j which are $\leq s,t$. Then

$$X_s - X_t = \sum_m^\infty (X_{s_{j+1}} - X_{s_j}) + (X_{s_m} - X_{t_m}) + \sum_m^\infty (X_{t_{j+1}} - X_{t_j})$$

from which it follows that

$$|X_t - X_s| \leq 2 \sum_m^{\infty} K_j.$$

Let us then define

$$M_\gamma = \sup\{\frac{|X_t - X_s|}{|t-s|^\gamma}, s \neq t \,\text{dyadic}\}.$$

Then

$$M_\gamma \leq C \sup_{m \in \mathbb{N}} (2^{m\gamma} \sum_m^{\infty} K_j) \leq C \sum_0^{\infty} 2^{j\gamma} K_j.$$

Now

$$\left(E(M_\gamma^\alpha)^{1/\alpha}\right) \leq C \sum 2^{j\gamma} \left(E(K_j^\alpha)^{1/\alpha}\right) \leq C \sum 2^{j(\gamma\alpha - \beta)} < \infty$$

if $\gamma < \beta/\alpha$. It follows that a.s. $t \mapsto X_t(\omega)$ is uniformly continuous on the dyadics and thus has a unique extension $\tilde{X}_t$ continuous on $\mathbb{R}$. By Fatou's lemma $\tilde{X}_t$ is the desired modification. The theorem applies in our situation since $B_{t+h} - B_t$ is Gaussian centered with variance h because then

$$E(B_{t+h} - B_t)^{2p} = C_p h^p.$$

We more precisely get that a.s. $t \mapsto B_t$ is $\gamma-$Hölder $\forall \gamma < 1/2$.

4.2 Canonical processes

If X is a stochastic process then for each ω we may view $t \mapsto X_t(\omega)$ as a map from T in E, i.e. an element of $\mathcal{F}(T,E) = E^T$. thus if $w \in E^T$ we think of $w(t), t \in T$ as the coordinates of w that we denote $Y_t(Y_t(w) = w(t))$. Now we can endow E^T with the product σ-algebra (E^T), i.e. the smallest σ-algebra making all the coordiante mappings Y_t measurable. It can also be described as the σ-algebra generated by the products $\prod A_t$ where $A_t = E$ for all $t \in T$ except a finite number for which $A_t \in \mathcal{E}$. We now return to our process X and define a map from Ω in E^T by

$$\Phi(\omega)(t) = X_t(\omega).$$

This mapping is measurable by definition of (E^T). Let us call P_X the image of P by Φ; the processes X_t, P and Y_t, P_X are then versions of each other.

Définition 4.2.1. *We call Y the canonical version of X and P_X the law of X.*

If the process X has continuous paths with $T = \mathbb{R}_+$ we can proceed as before on the space $C(\mathbb{R}_+,E)$. Doing this with BM we then get

Théorème 4.2.1. *There exists a unique probability measure W on $C(\mathbb{R}_+,\mathbb{R})$ for which the coordinate process is a Brownian motion. It si called the Wiener measure on the Wiener space $C(\mathbb{R}_+,\mathbb{R})$.*

4.3 Filtrations and stopping times

Définition 4.3.1. *: A filtration on a measurable space* $(\Omega,\mathcal{F})$ *is an increasing family* $(\mathcal{F}_t)_{t\geq 0}$ *of sub-*σ*-algebras of* $\mathcal{F}$*. A measurable space endowed with a filtration is called a filtered space.*

Définition 4.3.2. *: A process* (X_t) *on a filtered space is called adapted to the filtration if* $\forall t \geq 0$*,* X_t *is* $\mathcal{F}_t$ *measurable.*

Any process is adapted to its natural filtration $\mathcal{F}_t^0 = \sigma(X_s,s \leq t)$ which is the smallest filtration to which X is adapted. We define for any filtration

$$\mathcal{F}_t^- = \bigvee_{s<t} \mathcal{F}_s, \mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s, \mathcal{F}_\infty = \bigvee_{s\geq 0} \mathcal{F}_s.$$

Définition 4.3.3. *A stopping time relative to a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *is a map* $T : \Omega \rightarrow [0,+\infty]$ *such that for every* $t \geq 0,\{T \leq t\} \in \mathcal{F}_t$*.*

If T is a stopping time we define $\mathcal{F}_T$ as the σ-algebra of sets A such that $A \cap \{T \leq t\} \subset \mathcal{F}_t,t \geq 0$.

Proposition 4.3.1. *: If* E *is a metric space and if* X *is the coordinate process on* $C(\mathbb{R}_+,E)$ *then if* $A \subset E$ *is closed then*

$$D_A(\omega) = \inf\{t \geq 0; X_t(\omega) \in A\}$$

is a stopping time for its natural filtration.

Proof : $\{D_A \leq t\} = \{\omega; \inf\{d(X_s(\omega),A),s \in \mathbb{Q},s \leq t\} = 0\}$.

4.4 Martingales

In what follows we always have a probability space $(\Omega,\mathcal{F},P)$, an interval T of $\mathbb{N}$ or $\mathbb{R}_+$ and a filtration $(\mathcal{F}_t)_{t\in T}$ of sub $\sigma - algebras$ of $\mathcal{F}$.

Définition 4.4.1. *A real-valued process* $(X_t), t \in T$ *such that* $\forall t \in T, E(|X_t|) < +\infty$ *which is* $\mathcal{F}_t-$*adapted is called a sub-martingale (resp. a super-martingale, resp. a martingale) if*
$\forall s < t \in T, X_s \leq E[X_t|\mathcal{F}_s]$ *(resp.* $X_s \geq E[X_t|\mathcal{F}_s]$*, resp.* $X_s = E[X_t|\mathcal{F}_s]$*).*

The two following propositions are versions, valid for essentially finite martingales, of the very general optional stopping theorem to be stated below.

Proposition 4.4.1. *If* (X_n) *is a martingale and* (H_n) *is a positive bouded process such that for* $n \geq 1,H_n$ *is* $\mathcal{F}_{n-1}-$*measurable. Then the process*

$$Y_0 = X_0, Y_n = Y_{n-1} + H_n(X_n - X_{n-1})$$

is a martingale.

Proof : Obvious.
We denote by $H \cdot X$ the process defined in this proposition. As will become clear later, this is a discrete version of Ito's stochastic integral.

Corollaire 4.4.1. *With the same notations, if T is a stopping time, then the stopped process $X^T = X_{T\wedge t}$ is a martingale.*
Proof : it suffices to apply the preceeding proposition to $H_n = 1_{T\geq n}$.
We come now to a first version of the optional stopping theorem:
Théorème 4.4.1. *If $S \leq T$ are two bounded stopping times and (X_n) is a martingale then $X_S = E(X_T|\mathcal{F}_S)$.*
Proof : If $M \in \mathbb{R}$ is such that $S \leq T \leq M$ then, putting $H_n = 1_{T\geq n} - 1_{S\geq n}$, we have

$$(H \cdot X)_n - X_0 = X_T - X_S$$

if $n > M$ and it follows that $E(X_S) = E(X_T)$.If we apply this equality to the stopping times $\tilde{S} = S1_B + M1_{^cB}, \tilde{T} = T1_B + M1_{^cB}$ with $B \in \mathcal{F}_S$ we get that

$$E[X_T 1_B] = E[X_S 1_B]$$

i.e. the desired result.

4.4.1 Maximal inequalities

Théorème 4.4.2. *Let X be a (sub-)martingale indexed by $T = \{1,...,N\}$ then for every $p \geq 1, \lambda > 0$,*

$$\lambda P(\{\sup_{t\in T} |X_t| \geq \lambda\}) \leq \int_{\sup_n(|X_n|)\geq\lambda} |X_N| dP).$$

Proof : The process $(|X_n|)$ is a submartingale: λ being fixed we intoduce the stopping time $T = \inf\{n; X_n \geq \lambda\}$ if this set is not empty, and $T = N$ otherwise. By the previous results

$$E(|X_N|) \geq E(|X_T|) = \int_{\sup_n(|X_n|)\geq\lambda} \sup_n(|X_n|) dP + \int_{\sup_n(|X_n|)<\lambda} |X_N| dP$$

$$\geq \lambda P(\sup(|X_n|) > \lambda) + \int_{\sup_n(|X_n|)<\lambda} |X_N| dP.$$

Substracting $\int_{\{} \sup_n(|X_n|) < \lambda\} |X_N| dP$ from the first and last term we get what we want.
Corollaire 4.4.2. *With the hypothesises of the preceeding theorem, denoting $X^* = \sup_t |X_t|$, we have, for $p > 1$,*

$$E(X^{*p}) \leq (\frac{p}{p-1}) \sup_t E(|X_t|^p).$$

Proof : Let μ be the law of X^*; then $E(X^{*p}) = \int_0^\infty \lambda^p d\mu$ and by an integration by parts we get by theorem (4.4.2), $E(X^{*p}) = \int_0^\infty p\lambda^{p-1}P(X^* \geq \lambda)d\lambda \leq \int_0^\infty p\lambda^{p-1}\frac{1}{\lambda}(\int_{|X_N|\geq\lambda}|X_N|dP)d\lambda$. To estimate the last integral we interchange the order of integration to get

$$E(X^{*p}) \leq pE(|X_N|\int_0^{|X_N|}\lambda^{p-2}d\lambda) \leq \ (\frac{p}{p-1})E(|X_N|^p).$$

4.4.2 Law of the iterated logarithm

Théorème 4.4.3. *Let B denote the standard real Brownian motion. Then, a.s.,*

$$\overline{\lim}_{t\to 0}\frac{B_t}{\sqrt{2t\ln(\ln\frac{1}{t})}} = 1 \tag{4.1}$$

Proof : It starts with the
Lemme 4.4.1. *The process $Y_{t,\alpha} = \exp(\alpha B_t - \alpha^2 t/2)$ is a martingale.*
Proof : $E[Y_{t,\alpha}|\mathcal{F}_s] = E[Y_{s,\alpha}\exp(\alpha(B_t - B_s) - \alpha^2(t-s)/2|\mathcal{F}_t] = Y_{s,\alpha)}E[Z_{t,s}|\mathcal{F}_s]$ and the result follows from the fact that Z is independent of $\mathcal{F}_s$ and that $E[Z]=1$.
We define now $S_t = \sup\{B_s, s\leq t\}$:
Lemme 4.4.2. *For $a>0, P[S_t > at] \leq \exp(-a^2t/2)$.*
Proof : We have $\exp(\alpha S_t - \alpha^2 t/2) = \sup_{s\leq t} Y_{s,\alpha}$ hence

$$P[S_t \geq at] \leq P[\sup_{s\leq t} Y_{s,\alpha} \geq \exp(\alpha at - \alpha^2 t/2)] \leq \exp(-\alpha at + \alpha^2 t/2)E[Y_{t,\alpha}]$$

by the maximal inequality. But $E[Y_{t,\alpha}] = E[Y_{0,\alpha}] = 1$ and $\inf_{\alpha>0}(-\alpha at + \alpha^2 t/2) = -a^2t/2$ and the result follows.
We now come to the proof of the theorem: let $h(t) = \sqrt{2t\ln(\ln\frac{1}{t})}$ and $\theta, \delta \in (0,1)$. We define

$$\alpha_n = (1+\delta)\theta^{-n}h(\theta^n) \quad \beta_n = h(\theta^n)/2.$$

Using the same reasonning as in the previous lemmas, we get

$$P[\sup_{s\leq 1}(B_s - \alpha_n s/2) \geq \beta_n] \leq e^{-\alpha_n\beta_n} = Kn^{-(1+\delta)}$$

for some constant K. By Borel-Cantelli lemma, for almost every ω there exists $n_0(\omega)$ such that for $n \geq n_0(\omega)$,$s \in [\theta^n,\theta^{n-1})$,

$$B_s(\omega) \leq \frac{\alpha_n\theta^{n-1}}{2} + \beta_n = [\frac{1+\delta}{2\theta} + \frac{1}{2}]h(\theta^n) \leq [\frac{1+\delta}{2\theta} + \frac{1}{2}]h(s).$$

As a result

$$\overline{\lim}_{s\to 0}\frac{B_s}{h(s)} \leq \frac{1+\delta}{2\theta} + \frac{1}{2}a.s.$$

and we get the $\leq$ inequality in the theorem by letting $\theta \to 1, \delta \to 0$.
For the proof of the opposite inequality we consider the events

$$A_n = \{B_{\theta^n} - B_{\theta^{n+1}} \geq (1 - \sqrt{\theta})h(\theta^n)\}.$$

These events are independent and a striaightforward computation shows that

$$P(A_n) \geq \frac{a}{1+a^2} e^{-a^2/2}$$

with $a = (1-\sqrt{\theta})\sqrt{\frac{2 \ln \ln \theta^{-n}}{1-\theta}}$ which makes $P(A_n)$ greater than $n^{-\gamma}$,$\gamma = (1-2\sqrt{\theta}+\theta)/(1-\theta) < 1$. By Borel-Cantelli lemma again we have that a.s.

$$B_{\theta^n} > (1 - \sqrt{\theta})h(\theta^n) + B_{\theta^{n+1}}.$$

Since $-B$ is also a Brownian motion we know that $-B_{\theta^{n+1}}(\omega) < 2h(\theta^{n+1})$ from $n_0(\omega)$ on. it follows that $B_{\theta^n} > h(\theta^n)(1 - 5\sqrt{\theta})$ infinitely often, and the theorem is proven.

4.4.3 Optional Stopping Theorem

We recall that a family $(X_t)_{t \in T}$ of random variables is said to be uniformly integrable if

$$\forall \varepsilon > 0 \exists \delta > 0; \forall t \in T \forall E \in \mathcal{F}_t, P(E) < \delta \Rightarrow \int_E |X_t| dP < \varepsilon.$$

An important example of uniformly integrable family is that of a bounded family in L^p for some $p > 1$.

Théorème 4.4.4. *For a martingale $(X_t)_{t \in \mathbb{R}_+}$ the following three conditions are equivalent:*
1) (X_t) converges in L^1.
2) There exists a random variable $X_\infty \in L^1$ such that $\forall t \geq 0$,$X_t = E(X_\infty|\mathcal{F}_t$,
3) The family X_t is uniformly integrable.

Proof : The fact that 2) $\Rightarrow$ 3) is obvious. If 3) holds then in particular $\sup_t E(|X_t|) < +\infty$. Let us then show that a martingale satisfying this last property is converging a.s. Let f be a function $T \to \overline{\mathbb{R}}_+, t_1 < t_2 < .. < t_d$ a finite subset F of T: if $a < b$ are two reals we define $s_1 = \inf\{t_i;\ f(t_i) > b\}, s_2 = \inf\{t_i > s_1;\ f(t_i) < a\}$ and inductively $s_{2k+1} = \inf\{t_i > s_{2k};\ f(t_i) > b\}, s_{2k+2} = \inf\{t_i > s_{2k+1};\ f(t_i) < a\}(\inf(\emptyset) = t_d)$. We then put

$$D(f,F,[a,b]) = \sup\{n;\ s_{2n} < t_d\}$$

and define the downcrossing of $[a,b]$ by f as

$$D(f,[a,b]) = \sup_{F \subset T, \text{ finite}} D(f,F,[a,b]).$$

Lemme 4.4.3. *If X is a martingale then*

$$\forall a < b, (b-a)E(D(X,[a,b]) \leq \sup_{t\in T} E\left[(X_t-b)^+\right].$$

Proof : We may assume that $T = F$ is finite. The $s_k's$ are now stopping times and $A_k = \{s_k < t_d\} \in \mathcal{F}_{s_k}$. Moreover $A_k \supset A_{k+1}, X_{s_{2n-1}} > b \,\text{on}\, A_{2n-1}, X_{s_{2n}} < a \,\text{on}\, A_{2n}$. Therefore, by corollary (4.4.1)

$$0 \leq \int_{A_{2n-1}} (X_{s_{2n-1}} - b)dP \leq \int_{A_{2n-1}} (X_{s_{2n}} - b)dP \leq (a-b)P(A_{2n}) + \int_{A_{2n-1}\setminus A_{s_{2n}}} (X_{2n}-b)dP.$$

Consequently, since $s_{2n} = t_d$ on the complement of A_{2n},

$$(b-a)P(A_{2n}) \leq \int_{A_{2n-1}\setminus A_{2n}} (X_{t_d} - b)^+ dP.$$

But $A_{2n} = \{D(X,T,[a,b]) > n\}$ and the sets $A_{2n-1}\setminus A_{2n}$ are disjoint: the result then follows by adding these inequalities.
Recall that we want to prove that if $\sup_t E(|X_t|) < +\infty$ then X_t is a.s. converging as $t \to \infty$. If this were not the case then there would exist $a < b$ such that $\underline{\lim}_{t\to\infty}(X_t) < a < b < \overline{\lim}_{t\to\infty}(X_t)$ on a set of positive probability. But this would imply that $D(X,[a,b]) = +\infty$ on this set, which is impossible by the preceeding lemma. With the use of classical measure theory, the implication 3)$\Rightarrow$ 1) is thus proven. The fact that 1)$\Rightarrow$ 2) follows by passing to the limit as $s \to \infty$ in the equality

$$X_t = E(X_{t+s}|\mathcal{F}_t).$$

Théorème 4.4.5. *(Optional stopping theorem) If X is a martingale and if S,T are two bounded stopping times with $S \leq T$ then*

$$X_S = E[X_T|\mathcal{F}_S]. \tag{4.2}$$

If X is uniformly integrable, the family (X_S) where S runs through the set of all stopping times is uniformly integrable and if $S \leq T$,

$$X_S = E[X_T|\mathcal{F}_S] = E[X_\infty|\mathcal{F}_S]. \tag{4.3}$$

Proof : It suffices to prove (4.3) because a matingale defined on a closed interval is uniformly integrable. It is true if S,T are bounded by (4.4.1) and the result follows by approximation.

The preceeding theorem is false if the martingale is not assumed to be uniformly integrable. To see this, consider a positive martingale going to 0, (for example $X_t = \exp(B_t - t/2)$ where B_t is a usual Brownian, $X_0 = 1$): if $T = \inf\{t \geq 0;\, X_t \leq \alpha\}$ then $E[X_T] = \alpha \neq E[X_0] = 1$.

4.5 Stochastic Integration.

4.5.1 Quadratic Variations.

Définition 4.5.1. *A process A is called of finite variation if it is adapted and if the paths $t \mapsto A_t(\omega)$ are right-continuous and of bounded variation.*

If X is a progressively measurable process (i.e. if for every t the map $(s,\omega) \mapsto X_s(\omega)$ is measurable on $[0,t] \times \Omega$) and bounded on every interval $[0,t]$ then one can define

$$(X \cdot A)_t = \int_0^t X_s(\omega) dA_s(\omega).$$

We aim to define a similar integral for martingales A. This cannot be defined as before because of the

Proposition 4.5.1. *If M is a continuous martingale of bounded variation then M is constant.*

Proof : Let $t_1,...,t_n$ be a subdivision of $[0,t]$. Then if we assume that $M_0 = 0$ we have

$$E[M_t^2] \leq E\left[\sum_{i=0}^{n-1}(M_{t_{i+1}}^2 - M_{t_i}^2)\right] = E\left[\sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2\right] \leq V \sup_i |M_{t_{i+1}} - M_{t_i}| \to 0$$

as the mesh goes to 0. This means that one cannot proceed to a path by path integration. Instead we are going to use a more global method and the notion of quadratic variation.

If $\Delta = \{t_0 < ... < t_k < ..\}$ is a subdivision of $\mathbb{R}_+$ we define its modulus as $\sup\{t_{k+1} - t_k, k \geq 0\}$ and, if M is a process, we define, for $t \geq 0$,

$$T_t^\Delta = \sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2 + (M_t - M_{t_n})^2$$

where n is such that $t_n \leq t < t_{n+1}$.

Définition 4.5.2. *We say that a process M is of finite quadratic variation if there exists a process denoted by $< M,M >$ such that T_t^Δ converges in probability towards $< M,M >$ as the modulus of Δ goes to 0.*

Théorème 4.5.1. *A continuous and bounded martingale M is of finite quadratic variation. Moreover, $< M,M >$ is the unique continuous increasing adapted process vanishing at 0 such that $M^2 - < M,M >$ is a martingale.*

Proof : We only outline it. We first easily see that if Δ is a subdivision then $M^2 - T^\Delta$ is a continuous martingale. It thus remains only to show that if Δ_n is a sequence of subdivisions of the interval $[0,a]$ whose modulus converges to 0 then $T_a^{\Delta_n}$ converges in L^2. We have thus to show that if $|\Delta| + |\Delta'| \to 0$ then

$E[|T_a^\Delta - T_a^{\Delta'}|^2] \to 0$. We complete the proof in the case Δ' is Δ completed by a point s_i in each interval $[t_i,t_{i+1}]$: then

$$|T_a^\Delta - T_a^{\Delta'}| = 2(M_{t_i} - M_{s_i})(M_{t_{i+1}} - M_{s_i})$$

and thus $E[|T_a^\Delta - T_a^{\Delta'}|^2] \le 4E[\sup |M_{t_{i+1}} - M_{s_i}|^4]^{1/2} E[(T_a^{\Delta'})^2]^{1/2}$ and it is sufficient to prove that $E[(T_a^{\Delta'})^2]$ remains bounded as the modulus goes to 0.
In order to prove this we write

$$(T_a^\Delta)^2 = (\sum_{i=0}^{n-1} (M_{t_{i+1}} - M_{t_i})^2)^2$$

$$= 2\sum_{k=0}^{n-1} (T_a^\Delta - T_{t_k}^\Delta)(T_{t_{k+1}}^\Delta - T_{t_k}^\Delta) + \sum_{k=0}^{n-1} (M_{t_{i+1}} - M_{t_i})^4.$$

But

$$E[T_a^\Delta - T_{t_k}^\Delta | \mathcal{F}_k] = E[(M_a - M_{t_k})^2 | \mathcal{F}_k]$$

and thus

$$E[(T_a^\Delta)^2] = 2\sum_{k=0}^{n-1} E[(M_a - M_{t_k})^2 (T_{t_{k+1}}^\Delta - T_{t_k}^\Delta) + \sum_{k=0}^{n-1} E[(M_{t_{i+1}} - M_{t_i})^4]$$

$$\le 12C^2 E[T_a^\Delta] \le 48C^4$$

where C is a bound for the martingale M.

This theorem is very interesting but its hypothesises are very strong. It does not cover for instance the case of the Brownian motion (a non-uniformly integrable martingale) though Brownian motion has a quadratic variation, namely $B_t^2 - t$ is a martingale. In order to cover this case we need the notion of local martingale.

Définition 4.5.3. *An adapted right continuous process X is called a local martingale if there exists stopping times $T_n, n \ge 0$ increasing to $+\infty$ a.s. such that for every n the process $X^{T_n} 1_{[T_n > 0]}$ is a uniformly integrable martingale.*

In this statement we have used the notation $X^T = X_{T \wedge t}$. If the process X is continuous we can further use the stopping time $S_n = \inf\{t > 0; |X_t| = n\}$ and replace T_n by $T_n \wedge S_n$, meanning that we can assume that the martingale $X^{T_n} 1_{[T_n > 0]}$ is bounded.
We may now state the general

Théorème 4.5.2. *If M is a continuous local martingale there exists a unique continuous increasing process $< M,M >$ such that $M^2 - < M,M >$ is a continuous local martingale.*

To prove this thorem we use a sequence T_n of stopping times increasing to ∞ such that for all n, $X_n = X^{T_n} 1_{[T_n>0]}$ is a bounded martingale. By the theorem for bounded martingales there exists an increasing process A_n such that $X_n^2 - A_n$ is a bounded martingale. It is easy to see that $A_{n+1}^{T_n} = A_n$ on $[T_n > 0]$ and we can thus define unambiguously $< M,M >$ by setting it to be equal to A_n on $[T_n > 0]$. This process is the one we were looking for.
The next theorem generalizes the preceeding in the sense that it polarizes it:
Théorème 4.5.3. *If M,N are two continuous local martingales there exists a unique process $< M,N >$ with bounded variation, vanishing at 0, such that $MN- < M,N >$ is a local martingale.*
Proof : $< M,N >= \frac{1}{4}[< M + N,M + N > - < M - N,M - N >]$.

Théorème 4.5.4. *If M,N are two local martingales and H,K are two measurable processes then, a.s. for all $t \leq \infty$,*

$$\int_0^t |H_s||K_s||d < M,N >_s | \leq (\int_0^t |H_s|^2 |d < M,M >_s |)^{1/2} (\int_0^t |K_s|^2 |d < N,N >_s |)^{1/2} \tag{4.4}$$

Proof : It suffices to prove the theorem for processes of the form

$$K = K_0 1_0 + K_1 1_{]0,t_1]} + ... + K_n 1_{]t_{n-1},t_n]}.$$

We now define $< M,N >_s^t =< M,N >_t - < M,N >_s$. Since almost surely for every $r \in \mathbb{Q}$ we have

$$< M,M >_s^t +2r < M,N >_s^t +r^2 < N,N >_s^t =< M + rN,M + rN >_s^t \geq 0,$$

we must have

$$| < M,N >_s^t | \leq (< M,M >_s^t)^{1/2} (< N,N >_s^t)^{1/2} a.s.$$

As a result,

$$| \int_0^t H_s K_s d < M,N >_s | \leq \sum_i |H_i K_i|| < M,N >_{t_i}^{t_{i+1}} | \leq \sum_i |H_i K_i| (< M,M >_{t_i}^{t_{i+1}})^{1/2} (< N,N >_{t_i}^{t_{i+1}})^{1}$$

and the result follows by application of Cauchy-Schwarz inequality.
Corollaire 4.5.1. *(Kunita Watanabe inequality) If $1/p + 1/q = 1, p \geq 1$, then*

$$E[\int_0^\infty |H_s||K_s||d < M,N >_s] \leq \|(\int_0^\infty |H_s|^2 |d < M,M >_s |)^{1/2}\|_p \|(\int_0^\infty |K_s|^2 |d < N,N >_s |)^{1/2}\|_q \tag{4.5}$$

We now introduce the important (Hardy) space H^2, the space of L^2 martingales. We have already seen that this space is in a natural one to one correspondance with L^2. Thus H^2 is a Hilbert space for the norm

$$\|M\|_{\mathbb{H}^2} = E[M_\infty^2]^{1/2}.$$

The subspace H_0^2 consists of those martingales in H^2 such that $M_0 = 0$.

Théorème 4.5.5. *A continuous local martingale M is in H^2 if and only if $M_0 \in L^2$ and $E[< M,M >_\infty] < \infty$.*

Proof : Let T_n be a sequence of stopping times such that $M^{T_n} 1_{[T_n>0]}$ is bounded. We can write

$$E[M_{T_n\wedge t}^2 1_{[T_n>0]}] - E[< M,M >_{T_n\wedge t} 1_{[T_n>0]}] = E[M_0^2 1_{[T_n>0]}]$$

and the result follows by passing to the limit as $n \to \infty$.

4.6 Stochastic Integration

For reasons that will appear clearly later we need a notion of integration along brownian paths. But this cannot be done naively since Brownian motion is not of bounded variation: Riemann sums do not converge pathwise but will be shown to converge in probability. Before we come to this point we define integration with respect to the elements of H^2.

Définition 4.6.1. *if $M \in H^2$ we define $\mathcal{L}^2(M)$ the space of progressively measurable processes K such that*

$$\|K\|_M^2 = E[\int_0^\infty K_s^2 d < M,M >_s] < +\infty.$$

We can define a bounded measure on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ by putting

$$P_M(\Gamma) = E[\int_0^\infty 1_\Gamma(s,\omega) d < M,M >_s (\omega)$$

and $\mathcal{L}^2(M)$ appears as the space of P_M-square integrable, progressively measurable functions and we can then define as usual the Hilbert space $L^2(M)$.

Théorème 4.6.1. *Let $M \in H^2$: for each $K \in H^2$ there exists a unique element of H_0^2, denoted by $K \cdot M$ such that for every $N \in H^2$*

$$< K \cdot M,N >= K\cdot < M,N >$$

(notice that the two $\cdot$ have a different meanning). Moreover the map $K \mapsto K \cdot M$ is an isometry between $L^2(M)$ and H_0^2.

Proof : Uniqueness is obvious. To prove existence we observe, by Kunita-Watanabe inequality, that for every $N \in H_0^2$ we have

$$|E[\int_0^\infty K_s d < M,N >_s]| \leq \|N\|_{H^2}\|K\|_M$$

which implies that the map $N \mapsto E[(K\cdot < M,N >)_\infty]$ is a continuous linear form on the Hilbert space H_0^2 . There is thus an element $K \cdot M \in H_0^2$ such that

$$\forall N \in H_0^2, E[(K \cdot M)_\infty N_\infty] = E[(K\cdot < M,N >)_\infty].$$

Let T be a stopping time; me may write

$$E[(K \cdot M)_T N_T] = E[E[(K \cdot M)_\infty|\mathcal{F}_T]N_T] = E[(K \cdot M)_\infty N_T]$$

$$= E[(K \cdot M)_\infty N_\infty^T] = E[(K\cdot < M,N^T >)_\infty]$$

$$= E[(K\cdot < M,N >^T)_\infty] = E[(K\cdot < M,N >)_T]$$

which proves that $(K \cdot M)N - K\cdot < M,N >$ is a martingale and thus the first result. The fact that $K \mapsto K \cdot M$ is an isometry is obvious. Finally in the general case $M \in H^2$ we simply set $K \cdot M = K \cdot (M - M_0)$ and all the properties are easily checked.

Définition 4.6.2. *The martingale $K \cdot M$ is the Ito integral or stochastic integral of K wrt M and is also denoted by*

$$(K \cdot M)_t = \int_0^t K_s dM_s.$$

Let $\mathcal{E}$ be the space of elementary processes, i.e. processes of the form

$$K = K_{-1}1_0 + \sum_i K_i 1_{]t_i,t_{i+1}]}$$

where (t_i) is a sequence increasing to $+\infty$. In this case it is not hard to see that

$$(K \cdot M)_t = \sum_{i=0}^{n-1} K_i(M_{t_{i+1}} - M_{t_i}) + K_n(M_t - M_{t_n})$$

whenever $t \in [t_n,t_{n+1}[$. the following theorem is left as an exercise to the reader:

Théorème 4.6.2. *If $K \in L^2(M)$,$H \in L^2(K \cdot M)$,then $HK \in L^2(M)$ and*

$$(HK) \cdot M = H \cdot (K \cdot M).$$

Now we want to define a stochastic integral wrt general local martingales, the main purpose being integration wrt the Brownian. For this purpose we introduce the

Définition 4.6.3. *If M is a continuous local martingale we call $L^2_{loc}(M)$ the space of progressively measurable processes K for which there exists a sequence of stopping times T_n increasing to ∞ such that*

$$E[\int_0^{T_n} K_s^2 d < M,M >_s] < +\infty.$$

Contrarily as it may seem at a first glance, this notion is very general. It englobes for instance all locally bounded processes and thus, in particular, all continuous processes.

Théorème 4.6.3. *For any $K \in L^2_{loc}(M)$ there exists a unique continuous local martingale denoted $K \cdot M$ such that for any continuous local martingale N,*

$$< K \cdot M,N >= K \cdot < M,N > .$$

Proof : One can choose a sequence of stopping times T^n such that $M^{T_n} \in H^2$ and $K^{T_n} \in L^2(M^{T_n})$ and thus define $X^{(n)} = K^{T_n} \cdot M^{T_n}$.

Lemme 4.6.1. *If T is a stopping time,*

$$K \cdot M^T = K 1_{[0,T]} \cdot M = (K \cdot M)^T.$$

The proof is left to the reader.
This lemma implies that $X^{(n+1)} = X^{(n)}$ on $[0,T_n]$. This defines unambiguously a process $K \cdot M$ and all the properties are easily derived.

4.7 Itô's formula

From now on we will call semimartingale any process that can be expressed as a sum of a local martingale and a process of finite variation. If X is a continuous semimartingale, for which functions F of a real variable is it true that $F(X)$ is still a semimartingale? Itô's formula will in particular give an answer to this question. We start with the special case $F(x) = x^2$.

Proposition 4.7.1. *If X,Y are continuous semimartingales then*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + < X,Y >_t .$$

Proof : The case $X = Y$ follows almost immediately from the obvious formula:

$$\sum_i (X_{t_{i+1}} - X_{t_i})^2 = X_t^2 - X_0^2 - 2 \sum_i X_{t_i}(X_{t_{i+1}} - X_{t_i}).$$

The general case is obtained by the usual polarization.

Notice that in the case where X is a local martingale, we already know that $X^2- < X,X >$ is a local martingale: Itô's formula gives a formula for this local martingale. In the case X is of finite variation, Itô's formula reduces to the ordinary integration by parts. In the case of Brownian motion, Itô's formula reads

$$B_t^2 - t = 2\int_0^t B_s dB_s.$$

We now come to the famous Itô formula. In order to state it in a sufficient generality we introduce the notion of d-dimensional vector local (continuous semi) martingale. It is a $\mathbb{R}^d$ valued process $X = (X_1,...,X_d)$ such that each of its components is a local (continuous semi) martingale.

Théorème 4.7.1. *(Itô's formula) Let $F : \mathbb{R}^d \to \mathbb{R}$ be a C^2 function and X a continuous vector semimartingale; then $F(X)$ is a continuous semimartingale and*

$$F(X_t) = F(X_0) + \sum_i \int_0^t \frac{\partial F}{\partial x_i}(X_s)dX_s + \frac{1}{2}\sum_{i,j}\int_0^t \frac{\partial^2 F}{\partial x_i x_j}(X_s)d < X_i,X_j >_s .$$

Proof : We outline it in the case $d = 1$.Suppose that Itô's formula is valid for the function F and let us consider the function $G = xF$. Then by (**??**) we have

$$G = G(X_0) + X \cdot F(X) + F(X) \cdot X + < X,F(X) > .$$

On the other hand, since F satisfies Itô's formula

$$F(X) = F(X_0) + F'(X) \cdot X + F(X) \cdot X + \frac{1}{2}F''(X) \cdot < X,X > .$$

If we replace $F(X)$ by this expression we obtain

$$X \cdot F(X) = X \cdot (F'(X) \cdot X) + \frac{1}{2}X \cdot (F''(X) \cdot < X,X >= (XF'(X)) \cdot X + \frac{1}{2}XF''(X) \cdot < X,X >$$

Similarly,

$$< X,F(X) > = < X,F'(X) \cdot X > + \frac{1}{2} < X,F''(X) \cdot < X,X >>$$

$$= F'(X) < X,X > + \frac{1}{2}F''(X) < X, < X,X >>= F'(X) < X,X > .$$

On the other hand

$$G(X_0) + G'(X) \cdot X + \frac{1}{2}G''(X) \cdot < X,X >$$

$$= G(X_0) + XF'(X) \cdot X + F'(X) \cdot < X,X > + \frac{1}{2} XF''(X) \cdot < X,X > ,$$

and we get that Itô's formula is valid for G. It follows that its is valid for all polynômials; an easy approximation argument then implies that it is valid for any C^2 function.
We state an first important consequence of this formula:
Théorème 4.7.2. *If f is a complex function defined on $\mathbb{R} \times \mathbb{R}_+$ of class C^2 and satisfying the heat equation*

$$\frac{\partial f}{\partial y} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = 0,$$

then for any continuous local martingale M the process $f(M, < M,M >)$ is a local martingale. In particular the process

$$\mathcal{E}^\lambda(M) = \exp\{\lambda M_t - \frac{\lambda^2}{2} < M,M >_t\}$$

is a local martingale. If $\lambda = 1$ we speak of this process as the exponential of M.
Proof : Itô's formula gives, writing $N =< M,M >$, that

$$f(M,N)_t = f(M,N)_0 + \int_0^t \frac{\partial f}{\partial x} dM_s + \int_0^t \frac{\partial f}{\partial y} dN_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2} d < M,M >_s = \int_0^t \frac{\partial f}{\partial x} dM_s$$

4.8 Martingales as Time-changed Brownian Motion

Théorème 4.8.1. *(Paul Lévy) For a continuous adapted d-dimensional process X vanishing at 0 the following three conditions are equivalent:*

- *(i) X is a Brownian Motion.*
- *(ii) X is a continuous local martingale and $< X^i,X^j >_t = \delta_{ij} t, 1 \leq i,j \leq d$.*
- *(iii) X is a continuous local martingale and for any d-uple $f_1,...,f_d$ of $L^2(\mathbb{R}_+)$ functions the process*

$$\mathcal{E}_t^{if} = \exp\left\{ i \sum_k \int_0^t f_k(s) dX_s^k + \frac{1}{2} \sum_k \int_0^t f_k(s)^2 ds \right\}$$

is a complex local martingale.

Proof : (in the case $d = 1$).(i)⇒(ii) is known already. The fact that (ii)⇒ (iii) follows from theorem (4.7.2) applied with $\lambda = i, dM = f dX$. Suppose finally that (iii) holds: we apply it with $f = \xi 1_{[0,T]}$ and it gives that the process

$$\mathcal{E}_t^{if} = \exp\left\{ i\xi X_{t\wedge T} + \frac{1}{2} \xi^2 t \wedge T \right\}$$

is a martingale. For $A \in \mathcal{F}_s, s < t < T$ we get

$$E[1_A \exp\{i\xi(X_t - X_s)\}] = P(A) \exp\left\{\left(-\frac{\xi^2}{2}(t-s)\right)\right\},$$

which implies that $X_t - X_s$ is independent of $\mathcal{F}_s$ and has a Gaussian distribution with variance $(t-s)$; hence (i) holds.
We now come to the fundamental characterization of martingales. For this purpose we need the notion of time-change. Consider a right continuous increasing adapted process A; we can associate to this process the stopping times $C_s = \inf\{t; A_s > t\}$. The reader is encouraged to check that (C_s) is a right-continuous process and that the filtration $\mathcal{F}_{C_s}$ is also right continuous. Moreover, for any t, the random variable A_t is a $(\mathcal{F}_{C_s})$-stopping time.
Définition 4.8.1. *A time-change is a family of stopping times* (C_s),$s \geq 0$ *such that a.s.* $s \mapsto C_s$ *is increasing and riht-continuous.*

If C is a time-change and X is a progressive process we define $\hat{X}_t = X_{C_t}$,$\hat{\mathcal{F}}_t = \mathcal{F}_{C_t}$. The process $\hat{X}$ is called the time-changed process of X.
We want to prove that the class of semimartingales is stable under this operation. We formally prove that $\hat{X}$ is a local martingale if X, from which the result follows. So let X be a local martingale and T a stopping time such that X^T is bounded. The time $\hat{T} = \inf\{t; C^t \geq T\}$ is a $\hat{\mathcal{F}}_t$-stopping time and $\hat{X}_t^{\hat{T}} = X_{C_t}^T$. By the optional stopping theorem $\hat{X}^{\hat{T}}$ is a martingale. Considering sequences of such stopping times we obtain that $\hat{X}$ is a local martingale.
Théorème 4.8.2. *(Dambis,Dubins-Schwarz). If* M *is a continuous local martingale vanishing at* 0 *and such that* $< M,M >_\infty = \infty$ *then, if we set*

$$T_t = \inf\{s :< M,M >_s > t\},$$

$B_t = M_{T_t}$ *is a* $\mathcal{F}_{T_t}$*-Brownian motion and* $M_t = B_{<M,M>_t}$.
Proof : By the result outlined before the theorem B is a continuous local $(\mathcal{F}_{T_t})$-martingale and $< B,B >_t = < M,M >_{T_t} = t$. it is thus a Brownian motion by Paul Lévy's characterization.

Chapitre 5

Stochastic Löwner Evolution

5.1 Bessel Processes

We start by considering standard Brownian motion in $\mathbb{R}^d$, i.e. $B = (B_1,...,Bd)$. We denote by R the process $R = \|B\| = \sqrt{B_1^2 + ... + B_d^2}$. If we apply Itô's formula we get

$$dR = \frac{\frac{d-1}{2}}{R}dt + \sum_{d}^{j=1} \frac{B_j}{R} dB_j$$

But $M = \sum_d^{j=1} \frac{B_j}{R} dB_j$ is a local martingale with $< M,M >_t = t$ so that it is a Brownian motion. This motivates the

Définition 5.1.1. *For $x > 0$ we define a Bessel $d-$process as a solution of the stochastic differential equation (SDE)*

$$dX_t^x = \frac{a}{X_t^x}dt + dB_t, X_0^x = x$$

where a=(d-1)/2.

If we solve the above SDE, it is understood that we take the same ω for different values of x. It follows that if $x < y$ then $X_t^x < X_t^y$ (by uniqueness of solution) for all values of t less than T_x, the life-time of X_t^x, that is

$$T_x = \sup\{t > 0\,;\, X_t^x > 0\}$$

This implies in particular that $T_x \leq T_y$.

It will be useful to notice the scaling law of Bessel processes $\frac{1}{x}X_{x^2t}^x \approx X_t^x$, $\approx$ meanning having the same law.

The following theorem shows the different phases of Bessel processes that will reflect in the different phases of SLE later on:

Théorème 5.1.1. *According to the value of a, we have:*

1. *If $a \geq 1/2$, then for all $x > 0, T_x = +\infty$ a.s. and $\overline{\lim}_{t\to\infty} X_t^x = +\infty a.s.$*

2. If $a = 1/2$ *then* $\inf_{t>0} X_t^x = 0$ *a.s.*
3. If $a > 1/2$ *then for all* $x > 0, X_t^x \longrightarrow \infty$ *a.s.*
4. If $a < 1/2$ *then for all* $x > 0, T_x < \infty$ *a.s.*
5. If $1/4 < a < 1/2, x < y$ *then* $P(T_x = T_y) > 0$.
6. If $a \leq 1/4, x < y$, *then* $T_x < T_y$ *a.s.*

Proof : Let $0 < x_1 < x_2$ be fixed numbers and consider $x \in [x_1,x_2]$. We define $\sigma = \inf\{t > 0\,;\, X_x^t \in \{x_1,x_2\}\}$ and $\Phi(x;x_1,x_2) = P(X_\sigma^x = x_2)$. It is obvious that

$$\Phi(X_{t\wedge\sigma}^x) = E\left[\Phi(X_\sigma^x)|\mathcal{F}_t\right]$$

and hence that $\Phi(X_{t\wedge\sigma}^x)$ is a martingale. It follows that the drift term in Itô formula must vanish and this reads

$$\frac{1}{2}\Phi''(x) + \frac{a}{x}\Phi'(x) = 0.$$

Knowing that $\Phi(x_1) = 0, \Phi(x_2) = 1$ we have the formulas

$$\Phi(x) = \frac{x^{1-2a} - x_1^{1-2a}}{x_2^{1-2a} - x_1^{1-2a}}, a \neq \frac{1}{2},$$

$$\Phi(x) = \frac{\ln(x) - \ln(x_1)}{\ln(x_2) - \ln(x_1)}, a = \frac{1}{2}.$$

We start with the properties of the case $a \geq 1/2$:
First of all

$$\lim_{x_1\to 1} \Phi(x;x_1,x_2) = 1$$

in this case. It follows immediately that for all $x_2 > 0, X_t^x$ will reach x_2 before 0. The second part of the fist point follows. To prove the first it suffices to see that X_t^x cannot reach ∞ in finite time. To see this last point consider T_n the first arrival at 2^n and S_n the greatest $t \leq T_{n+1}$ such that $X_t^x = 2^n$. Then it is easy to see that the expectation of $T_{n+1} - S_n$ is greater than $c4^n$ and an easy argument using Borel-Cantelli lemma allows to conclude.
The second point follows from the fact that

$$\lim_{x_2\to+\infty} \Phi(x;x_1,x_2) = 0$$

if $a = 1/2$, from which it follows that for every $x_1 > 0$ there exists $M > 0$ such that X_t^x will reach x_1 before M with probability 1. The second point follows.
We come to the third point: we already know that $\overline{\lim} X_t^x = +\infty$. Let T_n the first passage to 2^n. We have

$$\lim_{x_2\to+\infty} \Phi(x;x_1,x_2) = 1 - (\frac{x_1}{x})^{2a-1} = l.$$

More precisely

$$|\Phi(x;x_1,x_2)-l| \leq (x^{1-2a}-x_2^{1-2a})(\frac{x_1}{x_2})^{2a-1}$$

and we deduce from this inequality that the probability that between T_n and T_{n+1} the process reaches $2^n/M_n$ is less than C/M_n^{2a-1}. Taking

$$M_n = n^{\frac{2}{2a-1}}$$

we conclude with Borel-Cantelli.
For the rest of the proof we assume $a<1/2$: we have

$$\Phi(x;0,x_2) = (\frac{x}{x_2})^{1-2a} \to 0, x_1 \to 0,$$

and thus a.s. there exists $x_2>0$ such that X_t^x reaches 0 before x^2. This proves the forth point.
We come to the proof of the $5th$ point: we already know that $T_x \leq T_y < +\infty$. Put $q(x,y) = P(T_x = T_y)$: by scaling, it is obvious that $q(x,y) = q(1,y/x)$.
Lemme 5.1.1. *For all fixed $t>0, \lim_{r\longrightarrow\infty} P(T_r<t)=0$.*
Proof : A small computation using Itô shows that

$$X_t^r - r = (2a+1)t + \int_0^t 2X_s^r dB_s$$

so that

$$-r = (2a+1)T_r + \int_0^{T_r} 2X_s dB_s$$

and the result follows by Tchebychev inequality.
As a corollary, $\lim_{r\longrightarrow\infty} q(1,r) = 0$.
Lemme 5.1.2. *The event $\{T_1 = T_y\}$ is equal (up to a set of probability 0) to the set*

$$\left\{ \sup_{t<T_1} \frac{X_t^y - X_t^1}{X_t^1} < +\infty \right\}.$$

Proof : It is obvious that the last statement implies that $T_1 = T_y$. Conversely, by the strong Markov property,

$$P\left\{ T_y = T_1\,;\, \sup_{t>0} \left\{ \frac{X_t^y - X_t^1}{X_t^1} \right\} \geq r \right\} \leq q(1,1+r),$$

which goes to 0 as r goes to ∞.
Let $Z_t = \ln(\frac{X_t^y - X_t^1}{X_t^1})$. By Itô's formula,

$$dZ_t = \left[(\frac{1}{2}-2a)\frac{1}{X_t^2} + a\frac{X_t^y - X_t^x}{X_t^y X_t^{x2}} \right] dt - \frac{1}{X_t^1} dB_t.$$

Define a time-change $r(t)$ by $\int_0^{r(t)} \frac{ds}{X_s^{1\,2}} = t$:
Lemme 5.1.3. $I = r^{-1}(T_x) = +\infty$.

Proof : It suffices to show that

$$\int_0^{T_x} = +\infty.$$

to do so we assume that $x = 1$ and denote by T_j the first arrival at 2^{-j}. We also put

$$Y_j = \int_{T_{j-1}}^{T_j} \frac{ds}{X_s^2} :$$

Then $I = \sum Y_j = +\infty$ a.s. because the variables Y_j are independent with the same distribution (by scaling) and with positive expectation.
Let $\tilde{Z}(t) = Z_{r(t)}$. Then $\tilde{Z}_t$ satisfies

$$d\tilde{Z}_t = \left[(\frac{1}{2} - 2a) + a\frac{X^y_{r(t)} - X^1_{r(t)}}{X^1_{r(t)}} \right] dt + d\tilde{B}_t,$$

where $\tilde{B}_t = -\int_0^{r(t)} X_s^{1-1} dB_s$ is a standard Brownian motion. After integration, we obtain

$$\tilde{Z}_t = \tilde{Z}_0 \tilde{B}_t + (\frac{1}{2} - 2a)t + a \int_0^t \frac{X^y_{r(s)} - X^1_{r(s)}}{X^1_{r(s)}} ds.$$

If $a \leq 1/4$ then $\tilde{Z}_t$ takes arbitrarily large values; by the preceeding discussion, we get point 5).
Suppose finally that $1/4 < a < 1/2$: choose $b \in (1/4,a)$ and let $\varepsilon = 2(a-b)/a$. Suppose $x = 1,y = 1 + \varepsilon/2$ and let σ be the first time that $X^y_{r(s)} - X^1_{r(s)} = \varepsilon X^1_{r(s)}$. For $0 \leq T_1 \wedge \sigma, \tilde{Z}_t \leq \tilde{Z}^*_t$ where

$$d\tilde{Z}^*_t = (\frac{1}{2} - 2b)dt + d\tilde{B}_t$$

Since $1/2 - 2b < 0$ there is a positive probability that $\tilde{Z}^*_t$ never reaches $\ln \varepsilon$ starting at $\ln \frac{\varepsilon}{2}$. On this event the same occurs for $\tilde{Z}_t$, which implies (5.1.2). We have thus shown that $q(1,1 + \varepsilon/2) > 0$; since $X^y_{r(s)} - X^1_{r(s)}$ decreases with t, it follows easily that $q(x,y) > 0$ for all $0 < x < y$.

5.2 Definitions for SLE

We want to define a Löwner process having certain properties. This process will be define by a driving function $U_t, t \geq 0$ which is a continuous real random process. We recall that this means that we consider the differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, g_0(z) = z$$

and the growing family of sets K_t is then defined as the set of initial values having a life-time $\leq t$.The mapping g_t can then be seen as the Riemann mapping from $\mathcal{H}\backslash K_t$ with the hydrodynamic normalization $g_t(z) = z + ..$ at ∞. If $s < t$ we define $g_{s,t} = g_t \circ g_s^{-1}$ and $\overline{g}_{s,t}(z) = g_{s,t}(z + U_s) - U_s$. The choice of the driving function will be done in order that:

1. the distribution of $\overline{g}_{s,t}$ depends only on $t - s$,
2. Markovian property:$\overline{g}_{s,t}$ is independent of $g_r, r \leq s$.
3. the distribution of K_t is symmetric wrt the imaginary axis.

It is an exercise to see that the only possibility for the driving function is $U_t = \lambda B_t$ for some positive constant λ, B_t being a standard $1D$ Brownian motion. For reasons that will become clear later we set $\lambda = \sqrt{\kappa}$ and set the

Définition 5.2.1. *The chordal stochastic Löwner evolution with parameter* $\kappa \geq 0(SLE_\kappa)$ *is the random collection of conformal maps* g_t *solving the ODE*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, g_0(z) = z.$$

An easy but important property of SLE is its scaling:

Proposition 5.2.1. *If* g_t *is a* SLE_κ *then it is the same for* $\tilde{g}(z) = r^{-1}g_{r^2t}(rz)$ *and if* γ *is a* $SLE-$*path, the same is true for* $\tilde{\gamma}(t) = r^{-1}\gamma(r^2t$.

We recall that, since U_t is continuous, the corresponding increasing family of sets K_t is continuously growing. However the sole continuity does not warranty that this family is generated by a curve, i.e. that there exists a path γ such that for $t \geq 0$,K_t is the unbounded component of $\mathcal{H}\backslash\gamma([0,t])$. However, in the case of SLE:

Théorème 5.2.1. *For every* $\kappa \geq 0$ SLE_κ *is generated by a curve.*

The proof of this theorem is difficult and we postpone it to the end of this series of lectures but we will allow us to use it before.

Définition 5.2.2. *A chordal* SLE *path is a random curve* γ *that generates chordal* SLE_κ*.*

In particular, if γ is a SLE path then

$$g_t(\gamma(t)) = \sqrt{\kappa}B_t$$

This value of g_t has of course to be interpreted as a proper limit.
An important remark is that (5.2.1) remains valid if $z \in \mathbb{R}$ and that the solution is then real (and may stop to exists after time T_z as for all starting points). The importance of this remark will be clear after we reinterpretate the following calculation in the case $z \in \mathbb{R}$. Put

$$\hat{g}_t(z) = \frac{g_t(z) - \sqrt{\kappa}B_t}{\sqrt{\kappa}} :$$

then $\hat{g}_t(\gamma(t)) = 0$ and $\hat{g}_t(z)$ satisfies the following SDE:

$$dX_t = \frac{2/\kappa}{X_t}dt - dB_t$$

which is Bessel equation. This remark is the key to the following theorem which gives a description of the phase transitions of the family of SLE's:

Théorème 5.2.2. *According to the different values of κ we have the following phases:*

1. *If $0 \leq \kappa \leq 4$ γ is a single curve such that $\gamma(0,+\infty) \subset \mathbb{H}$ and $\lim_{t\to\infty} \gamma(t) = \infty$.*
2. *If $4 < \kappa < 8$ then with probability 1, $\bigcup_{t>0} \overline{K}_t = \overline{\mathbb{H}}$ but $\gamma([0,\infty[) \cap \mathbb{H} \neq \mathbb{H}$. Also, $\lim_{t\to\infty} \gamma(t) = \infty$.*
3. *If $\kappa \geq 8$ then γ is a space filling curve, i.e. $\gamma[0,+\infty) = \overline{\mathbb{H}}$.*

Proof : It starts with the

Lemme 5.2.1. *For $x > 0$ we recall that T_x stands for the life-time of $g_t(x)$ which is the same as the first time $\hat{g}_t(z) = 0$. We then have*

1. *If $\kappa \leq 4$ then a.s.$T_x = \infty, \forall x > 0$.*
2. *If $\kappa > 4$ then a.s. $T_x < \infty, \forall x > 0$.*
3. *If $\kappa \geq 8$ then a.s. $T_x < T_y, \forall 0 < x < y$.*
4. *If $4 < \kappa < 8$ then a.s. $P(\{T_x = T_y\}) > 0, \forall 0 < x < y$.*

Proof : It is just a rephrasement of theorem(5.1.1) with $a = 2/\kappa$.

We now come back to the proof of the theorem. We will need the following notation: if $s \geq 0, \gamma^s(t) = g_s(\gamma(t+s)) - \sqrt{\kappa}B_s$ which has the same distribution as γ. To prove 1) we notice that if $\gamma(t) \in (0,+\infty)$ then $T_{\gamma(t)} < \infty$. Also if $\exists t_1 < t_2$ with $\gamma(t_1) = \gamma(t_2)$ then for every $q \in [t_1,t_2[, \gamma^q(0,\infty) \cap \mathbb{R} \neq \emptyset$,which contradicts the first part of the proof.

Let us prove now that $\underline{\lim}_{t\to\infty}|\gamma(t)| = +\infty$ if $\kappa \leq 4$. Let $\delta \in (0,1/4), x > 1$ and let $t_\delta = \inf\{t > 0; d(\gamma(t),[1,x]) \leq \delta\}$. Now, obviously,

$$g_{t_\delta}(1/2) - \sqrt{\kappa}B_{t_\delta} = \lim_{y\to\infty} \omega(iy,\mathbb{H};[\sqrt{\kappa}B_{t_\delta}, g_{t_\delta}(1/2)])$$

$$= \lim_{y\to\infty} \omega(iy,\mathbb{H}_{t_\delta}; \text{the part of } \partial\mathbb{H}_{t_\delta} \text{ between } 1/2 \text{ and } g_{t_\delta}(t_\delta)) \leq C\delta.$$

Now assume first that $\kappa < 4$. Then we know that

$$\lim_{t\to\infty}(g_t(1/2) - \sqrt{\kappa}B_t) = \infty,$$

from which it follows first that $d(\gamma([0,\infty[),[1,x]) > 0$ and then ,by scaling, that

$$\forall 0 < x_1 < x_2, d(\gamma([0,\infty[),[x_1,x_2]) > 0.$$

To finish the proof we now consider τ, the hitting time of the unit circle for γ (by scaling if necessary, we may assume it is finite). For all $\varepsilon > 0$ there exists

$0 < x_1 < x_2$ such that with probability $\geq 1-\varepsilon$ the two images of 0 under g_τ are in $[\sqrt{\kappa}B_\tau - x_2,\sqrt{\kappa}B_\tau - x_1] \cup [\sqrt{\kappa}B_\tau + x_1,\sqrt{\kappa}B_\tau + x_2]$. It follows from what we have just seen and the strong Markov property that with probability at least $1-\varepsilon$,

$$d(g_\tau(\gamma([\tau,+\infty[) - \sqrt{\kappa}B_\tau,[-x_2,-x_1] \cup [x_1,x_2]) > 0,$$

and finally that $d(0,\gamma([\tau,+\infty[) > 0$. By scaling, the property follows.
Case $\kappa = 4$: To be done....
Case $4 < \kappa < 8$: We will say that a point $z \in \mathbb{H}$ is swallowed if $T_z < \infty$ but $z \notin \bigcup_{t<T_z} \overline{K}_t$. Swallowed points form an open set and lifetime is constant in each connected component. By lemma(5.2.1) there is a positive probability that for some $x > 1,T_x = T_1$. In fact, by an easy scaling argument, this probability is equal to 1 and $\gamma(T_1)$ is the largest real x with $T_x = T_1$. Let $\varepsilon = d(1,\gamma([0,T_1]))$.Then all points in $\mathbb{H} \cap B(1,\varepsilon)$ are swallowed:this shows that the curve γ does not fill the half-plane. Also let T be the first time that both $1,-1$ are swallowed; then there exists a disk centered at 0 whose intersection with the half-plane is included in K_T. Thus for every u there exists $\varepsilon,t = t(\varepsilon,u)$ such that with probability $\geq 1-u$, $B(0,\varepsilon) \cap \mathbb{H} \subset K_t$. y scaling this must hold for all ε. This implies that $d(0,\mathbb{H}\backslash K_t) \to \infty$ and in particular that $\gamma(t) \to \infty$ as $t \to \infty$.
Case $\kappa \geq 8$. Notice first that lemma(5.2.1) shows that every real point belongs to the curve γ. Let us now prove the same for every point of the half-plane. First of all there cannot be any swallowed point in this case since there cannot be any real swallowed point from the fact that $T_x < T_y$ if $x < y$. It follows that $K_t = \gamma([0,t])$. The result will then follow if we can prove that the random variable

$$\Delta(x) = d(x+i,\gamma([0,+\infty[)$$

is identically equal to 0. To this end we change a little the notations. Writing $a = 2/\kappa$, $h_t(z) = g_t(\sqrt{\kappa}z)/\sqrt{\kappa}$, then h_t satisfies the Löwner equation

$$\dot{h}_t(z) = \frac{a}{h_t(z) + B_t}$$

and $Z_t = h_t + B_t$ satisfies the Bessel type equation

$$dZ_t = \frac{a}{Z_t}dt + dB_t.$$

We write $Z_t = X_t + iY_t$ and we consider the time-change defined by

$$t = \int_0^{\sigma(t)} \frac{ds}{x_s^2 + Y_2^s},$$

which really means that time becomes a function of Y. If then A_t is any process linked with the problem we put $\tilde{A}_t = A_{\sigma(t)}$. Suppose now that the curve does not fill the half-plane. Then by scaling we may assume that there exists $x \in$

$\mathbb{R}$,$\Delta(x) \neq 0$ and $T(x+i) = +\infty$ by the above discussion. By Koebe theorem, $\Delta(x)$ is comparable to $\exp^{-D(x)}$ where

$$D(x) = \lim_{t\to\infty} \ln \frac{|h_t'(x+i)|}{\Im(h_t(x+i))}.$$

Put

$$D_t(z) = \ln \frac{h_t'(z)}{\Im(h_t(z)}.$$

An easy computation shows that

$$\partial_t(\ln |h_t'\rangle| = a\frac{Y_t^2 - X_t^2}{(X_t^2+Y_t^2)^2}$$

while

$$\partial_t(\ln \Im(h_t)) = -a\frac{1}{X_t^2+Y_t^2}.$$

Finally

$$\partial_t(D_t) = \frac{2aY_t^2}{(X_t^2+Y_t^2)^2}$$

and thus

$$D(x) = 2a\int_0^{+\infty} \frac{Y_t^2}{(X_t^2+Y_t^2)^2}dt.$$

Let $D_t(x)$ be the integral from 0 to t and putting $K_t = \ln(X_t/Y_t)$, we see with the help of Itô's formula, that

$$d\tilde{C}_t = [2a - \frac{1}{2} - \frac{1}{2}e^{-2\tilde{C}_t}]dt + \sqrt{1+e^{-2\tilde{C}_t}}d\tilde{B}_t$$

and

$$\dot{\tilde{D}}_t = \frac{2a}{1+e^{-2\tilde{C}_t}} \Rightarrow D(x) = \int_0^\infty \frac{2a}{1+e^{-2\tilde{C}_t}}dt.$$

The fact that $\kappa \geq 8$ corresponds to the fact that $a \leq 1/4$, in which case the drift term is negative. This last fact implies that whatever large is $T > 0$ there exists $t > T$ such that $\tilde{C}_s \leq 0$,$s \in [t,t+1]$. But this implies that $D(x) = +\infty$ and the proof is complete.

5.3 dimension of SLE paths

In this section we will try to convince the reader that the box-dimension of $SLE_{kappa}-$paths is $1+\frac{\kappa}{8}$ if $\kappa < 8$. This will follow from the following theorem, in which the notations are those of the last paragraph:

Théorème 5.3.1. *$P(\Delta(x) \leq \varepsilon) \sim \varepsilon^{1-\frac{\kappa}{8}}$ if $\kappa < 8$.*

Proof : (Sketch)
To estimate this probability one computes explicitely the characteristic function

$$E\left[e^{ibD(x)}\right].$$

To do so we put $K_t = X_t/Y_t$: we performed a change of variable in the last section, leading to $\tilde{K}_t$. One here perform a second one, namely $\hat{\sigma}'(t) = (\tilde{K}^2_{\hat{\sigma}(t)} + 1)^{-1}$, leading to

$$d\hat{K}_t = \frac{2a\hat{K}_t}{1+\hat{K}_t^2}dt + d\hat{B}_t, dD_t(x) = \frac{2a\hat{K}_t}{(1+\hat{K}_t^2)^2}.$$

We now seek for a function ψ such that $\psi(\hat{K}_t)e^{ibD_t(x)}$ is a local martingale. An application of Itô's formula shows that the function ψ must be solution of the ODE

$$\frac{1}{2}y'' + \frac{2ax}{1+x^2}y' + \frac{ib}{(1+x^2)^2}y = 0.$$

There exists a solution to this equation which is bounded and tending to 1 as $t \to +\infty$. Applying optional stopping theorem we then get

$$\psi(x) = E\left[\psi(\hat{K}_t e^{ibD_t})\right] = E\left[e^{ibD_\infty}\right]$$

.

To be continued.

5.4 Locality for SLE_6

In this section we consider K_t,$t \geq 0$ a chordal SLE_κ.For convenience we will write $W_t = \sqrt{\kappa}B_t$. We also consider a hull A which is at positive distance from 0. Let Φ be the normalized conformal mapping from $\mathbb{H}\backslash A$ onto $\mathbb{H}$. Let T be the first time that K_t intersects A. For $t \leq T$ we can define $\tilde{K}_t = \Phi(K_t)$. The goal of this section is to compare the growth of K_t and $\tilde{K}_t$.
Let Φ_t be the normalized Riemmann mapping from $\mathbb{H}\backslash g_t(A)$ onto $\mathbb{H}$ where g_t is the Löwner process describing K_t (notice that $\Phi = \Phi_0$). Then, if $\tilde{g}_t$ is the Löwner process describing $\tilde{K}_t$, we have

$$\Phi_t \circ g_t = \tilde{g}_t \circ \Phi_0.$$

Write $\tilde{W}_t = \Phi_t(W_t)$ so that the differential equation satisfied by $\tilde{g}_t$ reads

$$\partial_t \tilde{g}_t(z) = \frac{2\partial_t(\text{hcap}(\tilde{K}_t))}{\tilde{g}_t(z) - \tilde{W}_t}.$$

It remains to understand the evolution of $\text{hcap}(\tilde{K}_t))$ and $\tilde{W}_t$.
For the first quantity we write, for $0 < s < t$, $g_t = g_{s,t} \circ g_s$ and parallely $\tilde{g}_t = \tilde{g}_{s,t} \circ \tilde{g}_s$. Then we can write $\text{hcap}(\tilde{K}_t) = \text{hcap}(\tilde{K}_s) + \text{hcap}(\tilde{K}_{s,t})$ and $\lim_{t \to s} \frac{\text{hcap}(\tilde{K}_{s,t})}{t-s} = \Phi_s'^2(W^s)$ because of the scaling of hcap.
In order to evaluate the second quantity we start with the identity

$$\Phi_t = \tilde{g}_t \circ \Phi \circ g_t^{-1}$$

that we differentiate wrt t. Using the inverse Löwner equation

$$\partial_t (g_t^{-1}(z)) = -2 \frac{(g_t^{-1})'(z)}{z - W_t}$$

from which it is easy to deduce that

$$\partial_t \Phi_t(z) = \frac{2\Phi_t'(W_t)^2}{\Phi_t(z) - \tilde{W}_t} - \frac{2\Phi_t'(z)}{z - W_t}.$$

By Schwarz reflection the time derivative of $\Phi_t(z)$ exists for $z = W_t$ and we must have

$$(\partial_t \Phi_t)(W_t) = \lim_{z \to W_t} [\frac{2\Phi_t'(W_t)^2}{\Phi_t(z) - \tilde{W}_t} - \frac{2\Phi_t'(z)}{z - W_t}] = -3\Phi_t''(W_t).$$

We finally make use of Itô's formula which gives:

$$d\tilde{W}_t = (\partial_t \Phi_t)(W_t)dt + \Phi_t'(W_t)dW_t + \frac{\kappa}{2}\Phi_t''(W_t)dt$$

hence,

$$d\tilde{W}_t = \Phi_t'(W_t)dW_t + [\frac{\kappa}{2} - 3]\Phi_t''(W_t)dt.$$

We can now state the main result of this section:
Théorème 5.4.1. *If $\kappa = 6$ then, modulo time-change, the process $\tilde{K}_t - \Phi(0), t < T$ has the same law as K_t.*
Proof : The time-change is of course $\text{hcap}(K_t) = \int_0^t \Phi_s'(W_s)^2 ds =< \tilde{W} >_t$. Hence if we define $\tilde{W}_t = \hat{W}_{\text{hcap}(K_t)}$ then $\hat{W} - \hat{W}_0$ and W have the same law. Moreover, if we define $\hat{g}$ by $\tilde{g}_t = \hat{g}_{\text{hcap}(K_t)}$ we have

$$\partial_t \hat{g}_t(z) = \frac{2}{\hat{g}_t(z) - \hat{W}_t}.$$

5.5 Restriction Property for $SLE_{8/3}$

In this section we keep the same notations as in the preceeding one; we would like to understand the evolution of $\Phi_t'(W_t)$. To this end we differentiate the equation (5.4):

$$\partial_t \Phi_t'(z) = -\frac{2\Phi_t'(W_t)^2 \Phi_t'(z)}{(\Phi_t(z) - \tilde{W}_t)^2} + \frac{2\Phi_t'(z)}{(z - W_t)^2} - \frac{2\Phi_t''(z)}{z - W_t}.$$

Taking the limit as $z \to W_t$ we obtain:

$$\partial_t \Phi_t'(W_t) = -\frac{\Phi_t''(W_t)^2}{2\Phi_t'(W_t)} - \frac{4}{3}\Phi_t'''(W_t).$$

If we then apply Itô's formula, we get

$$d[\Phi_t'(W_t)] = \Phi_t''(W_t)dW_t + \left[\frac{\Phi_t''(W_t)^2}{2\Phi_t'(W_t)} + (\kappa/2 - 4/3)\Phi_t'''(W_t)\right] dt.$$

From now on in this section we specialize $\kappa = 8/3$. Put $X_t = \Phi_t'(W_t)$. We look for an index α such that X_t^α is a local martingale (in fact a bounded martingale in this case since $X_t \leq 1$). Applying Itô's formula we see that $\alpha = 5/8$ does the job and that

$$d\left[\Phi_t'(W_t)^{5/8}\right] = \frac{5\Phi_t''(W_t)}{8\Phi_t'(W_t)^{3/8}} dW_t.$$

We can now state

Proposition 5.5.1. *For chordal $SLE_{8/3}$ and any hull A not containing* 0*,*

$$P(\forall t \geq 0, K_t \cap A = \emptyset) = \Phi_A'(0)^{5/8}.$$

Proof : Let us denote by M_t the local martingale $\Phi_t'(W_t)^{5/8}, t \leq T$. First of all notice that this is actually a martingale bounded by 1. Indeed, if we denote by u, v the real and imaginary parts of Φ_t', then by parity $\partial u/\partial y$ is equal to 0 on the real line while $\partial v/\partial y \in [0,1]$ also on the real line since one easily sees by maximum principle that $v(z) \leq y$ on $\mathcal{H}$. It is not difficult to see that if $T = \infty$ then, if τ_R stands for the hitting time of the circle centered at 0 with radius R,

$$\lim_{R\to\infty} \Phi_{\tau_R}'(W_{\tau_R}) = 1.$$

On the other hand, if $T < +\infty$ then $lim_{t \to T}\Phi_t'(W_t) = 0$. It follows that

$$P(T = \infty) = E[M_T] = E[M_0] = \Phi_A'(0)^{5/8}.$$

We can now state the theorem about the restriction property:

Théorème 5.5.1. *Suppose that A_0 is a hull; tyhen the conditionnal law of $K_\infty = \cup_{t>0} K_t$ given $K_\infty \cap A_0 = \emptyset$ is identical to the law of $\Psi_{A_0}(K_\infty)$.*

Proof : The law of K_∞ is characterized by the knowledge of $P(K_\infty \cap A = \emptyset)$ for all hulls A not containing 0. Let A be such a hull:

$$P(\Psi_{A_0}(K_\infty) \cap A = \emptyset | K_\infty) \cap A_0 = \emptyset)$$

$$= \frac{P(K_\infty \cap (\mathbb{H} \backslash \Psi_{A_0}^{-1} \circ \Psi_A^{-1}(\mathbb{H}) = \emptyset}{P(K_\infty \cap A_0 = \emptyset)}$$

$$= \left(\frac{\Psi'_{A_0}(0)\Psi'_A(0)}{\Psi'_{A_0}(0)}\right)^{5/8} = P(K_\infty \cap A = \emptyset).$$