Higher order mechanics on graded bundles: some mathematical background

Andrew James Bruce

Institute of Mathematics of the Polish Academy of Sciences

andrew@.impan.pl

14/05/2015

Joint work with K. Grabowska & J. Grabowski

`arXiv:1502.06092` and `arXiv:1409.0439`

In science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before. But in the case of poetry, it's the exact opposite!

P.A.M. Dirac

Overview and Motivation

Overview and Motivation

1. Graded bundles

Overview and Motivation

1. Graded bundles
2. Weighted groupoids and algebroids

Overview and Motivation

1. Graded bundles
2. Weighted groupoids and algebroids
3. The Lie functor and integration

Overview and Motivation

1. Graded bundles
2. Weighted groupoids and algebroids
3. The Lie functor and integration

Why the interest?

Overview and Motivation

1. Graded bundles
2. Weighted groupoids and algebroids
3. The Lie functor and integration

Why the interest?

'Categorified' objects in the category of Lie groupoids and Lie algebroids

Overview and Motivation

1. Graded bundles
2. Weighted groupoids and algebroids
3. The Lie functor and integration

Why the interest?

'Categorified' objects in the category of Lie groupoids and Lie algebroids

- Mackenzie's 'double structures', for example double Lie groupoids and double Lie algebroids *etc*, all related to Poisson geometry.

Overview and Motivation

1. Graded bundles
2. Weighted groupoids and algebroids
3. The Lie functor and integration

Why the interest?

'Categorified' objects in the category of Lie groupoids and Lie algebroids

- Mackenzie's 'double structures', for example double Lie groupoids and double Lie algebroids *etc*, all related to Poisson geometry.
- $\mathcal{VB}$-groupoids generalise linear representations of Lie groupoids. (see for example Gracia-Saz & Mehta 2010)

Graded Bundles

Graded Bundles

Manifold F, homogeneous coordinates (y^a_w), where $w = 0, 1, \cdots, k$

Graded Bundles

Manifold F, homogeneous coordinates (y_w^a), where $w = 0, 1, \cdots, k$
Associated with a smooth action

$$\mathsf{h} : \mathbb{R}_{\geq 0} \times F \to F,$$

of the multiplicative monoid $(\mathbb{R}_{\geq 0}, \cdot)$

A function is homogeneous of order q if

$$\mathsf{h}_t(f) = t^q f,$$

for all $t > 0$.

A function is homogeneous of order q if

$$\mathsf{h}_t(f) = t^q f,$$

for all $t > 0$.

Only non-negative integer weights are allowed.

The action reduced to $\mathbb{R}_{>0}$ is the one-parameter group of diffeomorphisms integrating the weight vector field

The action reduced to $\mathbb{R}_{>0}$ is the one-parameter group of diffeomorphisms integrating the weight vector field

Weight vector field is h-complete.

The action reduced to $\mathbb{R}_{>0}$ is the one-parameter group of diffeomorphisms integrating the weight vector field

Weight vector field is h-complete.

The action can be canonically extended to $\mathrm{h} : \mathbb{R} \times F \to F$ and we shall call this extended action a *homogeneity structure*.

Fundamental result: Any smooth action of $(\mathbb{R}, \cdot)$ leads to a graded bundle.

Fundamental result: Any smooth action of $(\mathbb{R}, \cdot)$ leads to a graded bundle.

For $t \neq 0$, h_t is a diffeomorphism and when $t = 0$ it is a smooth surjection $\tau = \mathrm{h}_0$ onto $F_0 = M$, the fibres being $\mathbb{R}^N$.

Fundamental result: Any smooth action of $(\mathbb{R}, \cdot)$ leads to a graded bundle.

For $t \neq 0$, h_t is a diffeomorphism and when $t = 0$ it is a smooth surjection $\tau = \mathrm{h}_0$ onto $F_0 = M$, the fibres being $\mathbb{R}^N$.

Get a series of 'affine fibrations'

$$F = F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 = M$$

Fundamental result: Any smooth action of $(\mathbb{R}, \cdot)$ leads to a graded bundle.

For $t \neq 0$, h_t is a diffeomorphism and when $t = 0$ it is a smooth surjection $\tau = \mathsf{h}_0$ onto $F_0 = M$, the fibres being $\mathbb{R}^N$.

Get a series of 'affine fibrations'

$$F = F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 = M$$

A point of F_l is a class of points in F described by coordinates of weight $\leq l$.

Fundamental result: Any smooth action of $(\mathbb{R}, \cdot)$ leads to a graded bundle.

For $t \neq 0$, h_t is a diffeomorphism and when $t = 0$ it is a smooth surjection $\tau = \mathsf{h}_0$ onto $F_0 = M$, the fibres being $\mathbb{R}^N$.

Get a series of 'affine fibrations'

$$F = F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 = M$$

A point of F_l is a class of points in F described by coordinates of weight $\leq l$.

Grabowski & Rotkiewicz 2012

Weighted Lie algebroids

Recall: Lie algebroid $(E \to M, [,], \rho) \leftrightsquigarrow$ Q-manifold $(\Pi E, Q)$ where Q is of weight one.

Weighted Lie algebroids

Recall: Lie algebroid $(E \to M, [,], \rho) \leftrightsquigarrow$ Q-manifold $(\Pi E, Q)$ where Q is of weight one.

Definition

A weighted Lie algebroid of degree k is a Lie algebroid $(\Pi E, Q)$ equipped with a homogeneity structure of degree $k-1$ such that

$$\Pi\widehat{\mathsf{h}}_t : \Pi E \to \Pi E$$

is a Lie algebroid morphism for all $t \in \mathbb{R}$. That is

$$Q \circ (\Pi\widehat{\mathsf{h}}_t)^* = (\Pi\widehat{\mathsf{h}}_t)^* \circ Q.$$

Examples

Examples

- $\mathcal{VB}$-algebroids are weighted Lie algebroids of degree 1.
  Bursztyn + Cabrera + de Hoyo (2014)

Examples

- $\mathcal{VB}$-algebroids are weighted Lie algebroids of degree 1.
  Bursztyn + Cabrera + de Hoyo (2014)
- The tangent bundle of a graded bundle.

Examples

- $\mathcal{VB}$-algebroids are weighted Lie algebroids of degree 1.
  Bursztyn + Cabrera + de Hoyo (2014)
- The tangent bundle of a graded bundle.
- Higher order tangent bundles of a Lie algebroid.

Question: What are the global objects that 'integrate' weighted Lie algebroids?

Question: What are the global objects that ‘integrate’ weighted Lie algebroids?

On to Lie groupoids...

Weighted Lie groupoids

Weighted Lie groupoids

Definition

A *weighted Lie groupoid* of degree k is a Lie groupoid $\Gamma_k \rightrightarrows B_k$, together with a homogeneity structure $\underline{\mathrm{h}} : \mathbb{R} \times \Gamma_k \to \Gamma_k$ of degree k, such that $\underline{\mathrm{h}}_t$ is a Lie groupoid morphism for all $t \in \mathbb{R}$.

Unravel:

- B_k is a graded bundle of degree k.

Unravel:

- B_k is a graded bundle of degree k.
- Let $\underline{\mathrm{h}}_t = (\mathrm{h}_t, \mathrm{g}_t)$

<u>Unravel</u>:

- B_k is a graded bundle of degree k.
- Let $\underline{\mathsf{h}}_t = (\mathsf{h}_t, \mathsf{g}_t)$

$$
\begin{array}{ccc}
\Gamma_k & \xrightarrow{\mathsf{h}_t} & \Gamma_k \\
\underline{s}\downarrow\;\downarrow\underline{t} & & \underline{s}\downarrow\;\downarrow\underline{t} \\
B_k & \xrightarrow[\mathsf{g}_t]{} & B_k
\end{array}
$$

Unravel:

- B_k is a graded bundle of degree k.
- Let $\underline{\mathrm{h}}_t = (\mathrm{h}_t, \mathrm{g}_t)$

$$\begin{array}{ccc} \Gamma_k & \xrightarrow{\mathrm{h}_t} & \Gamma_k \\ {\scriptstyle \underline{s}}\downarrow\;\downarrow{\scriptstyle \underline{t}} & & {\scriptstyle \underline{s}}\downarrow\;\downarrow{\scriptstyle \underline{t}} \\ B_k & \xrightarrow[\mathrm{g}_t]{} & B_k \end{array}$$

$\underline{s} \circ \mathrm{h}_t = \mathrm{g}_t \circ \underline{s}$, and $\underline{t} \circ \mathrm{h}_t = \mathrm{g}_t \circ \underline{t}$

Unravel:

- B_k is a graded bundle of degree k.
- Let $\underline{\mathrm{h}}_t = (\mathrm{h}_t, \mathrm{g}_t)$

$$\begin{array}{ccc} \Gamma_k & \xrightarrow{\mathrm{h}_t} & \Gamma_k \\ \underline{s}\downarrow\ \downarrow\underline{t} & & \underline{s}\downarrow\ \downarrow\underline{t} \\ B_k & \xrightarrow[\mathrm{g}_t]{} & B_k \end{array}$$

$\underline{s} \circ \mathrm{h}_t = \mathrm{g}_t \circ \underline{s}$, and $\underline{t} \circ \mathrm{h}_t = \mathrm{g}_t \circ \underline{t}$

$\mathrm{h}_t(g \circ h) = \mathrm{h}_t(g) \circ \mathrm{h}_t(h)$

Examples

- If $\mathcal{G} \rightrightarrows M$ is Lie groupoid the $\mathsf{T}^k\mathcal{G} \rightrightarrows \mathsf{T}^k M$ is a weighted Lie groupoid of degree k

Examples

- If $\mathcal{G} \rightrightarrows M$ is Lie groupoid the $\mathsf{T}^k\mathcal{G} \rightrightarrows \mathsf{T}^k M$ is a weighted Lie groupoid of degree k
- $\mathcal{VB}$-groupoids = degree 1 weighted Lie groupoids
  Bursztyn + Cabrera + de Hoyo (2014)

Theorem

If $\Gamma_k \rightrightarrows B_k$ is a weighted Lie groupoid of degree k, then we have the following tower of weighted groupoid structures of lower order:

Theorem

If $\Gamma_k \rightrightarrows B_k$ is a weighted Lie groupoid of degree k, then we have the following tower of weighted groupoid structures of lower order:

$$
\begin{array}{ccccccccc}
\Gamma_k & \xrightarrow{\tau^k} & \Gamma_{k-1} & \xrightarrow{\tau^{k-1}} & \cdots & \xrightarrow{\tau^2} & \Gamma_1 & \xrightarrow{\tau^1} & \mathcal{G} \\
\underline{s}_k \downarrow \downarrow \underline{t}_k & & \underline{s}_{k-1} \downarrow \downarrow \underline{t}_{k-1} & & & & \underline{s}_1 \downarrow \downarrow \underline{t}_1 & & \underline{\sigma} \downarrow \downarrow \underline{\tau} \\
B_k & \xrightarrow[\pi^k]{} & B_{k-1} & \xrightarrow[\pi^{k-1}]{} & \cdots & \xrightarrow[\pi^2]{} & B_1 & \xrightarrow[\pi^1]{} & M
\end{array}
$$

In particular, $\Gamma_1 \rightrightarrows B_1$ is a $\mathcal{VB}$-groupoid.

Weighted Lie theory

Theorem

If $\Gamma_k \rightrightarrows B_k$ *is a weighted Lie groupoid of degree* k *with respect to a homogeneity structure* $\underline{\mathsf{h}}$ *on* Γ_k, *then* $\mathsf{A}(\Gamma_k) \to B_k$ *is a weighted Lie algebroid of degree* $k+1$ *with respect to the homogeneity structure* $\widehat{\mathsf{h}}$ *defined by*

$$\widehat{\mathsf{h}}_t = (\underline{\mathsf{h}}_t)' = \mathsf{Lie}(\underline{\mathsf{h}}_t)$$

*

Weighted Lie theory

Theorem

If $\Gamma_k \rightrightarrows B_k$ is a weighted Lie groupoid of degree k with respect to a homogeneity structure $\underline{\mathsf{h}}$ on Γ_k, then $\mathsf{A}(\Gamma_k) \to B_k$ is a weighted Lie algebroid of degree $k+1$ with respect to the homogeneity structure $\widehat{\mathsf{h}}$ defined by

$$\widehat{\mathsf{h}}_t = (\underline{\mathsf{h}}_t)' = \mathsf{Lie}(\underline{\mathsf{h}}_t) \qquad *$$

Theorem

Let $E_{k+1} \to B_k$ be a weighted Lie algebroid of degree $k+1$ with respect to a homogeneity structure $\widehat{\mathsf{h}}$ and Γ_k its (source simply-connected) integration groupoid. Then Γ_k is a weighted Lie groupoid of degree k with respect to the homogeneity structure $\underline{\mathsf{h}}$ uniquely determined by $*$.

Closing remarks

Closing remarks

- Compatible grading $\rightarrow$ morphisms in the appropriate category

Closing remarks

- Compatible grading $\rightarrow$ morphisms in the appropriate category
- Weighted Poisson–Lie groupoids, weighted Lie bi-algebroids and weighted Courant algebroids

Closing remarks

- Compatible grading $\rightarrow$ morphisms in the appropriate category
- Weighted Poisson–Lie groupoids, weighted Lie bi-algebroids and weighted Courant algebroids
- Expect further links with physics