

Higher order mechanics on graded bundles: some mathematical background

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In science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before. But in the case of poetry, it's the exact opposite!

P.A.M. Dirac





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- $2. \ Weighted \ groupoids \ and \ algebroids$

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Why the interest?

'Categorified' objects in the category of Lie groupoids and Lie algebroids

- Mackenzie's 'double structures', for example double Lie groupoids and double Lie algebroids *etc*, all related to Poisson geometry.
- ► VB-groupoids generalise linear representations of Lie groupoids. (see for example Gracia-Saz & Mehta 2010)

Graded Bundles





Manifold F, homogeneous coordinates (y_w^a) , where $w = 0, 1, \cdots, k$



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$$h: \mathbb{R}_{\geq 0} \times F \to F,$$

of the multiplicative monoid $(\mathbb{R}_{\geq 0}, \cdot)$



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$$\mathsf{h}_t(f) = t^q f,$$

for all t > 0.



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Only non-negative integer weights are allowed.



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The action can be canonically extended to $h : \mathbb{R} \times F \to F$ and we shall call this extended action a *homogeneity structure*.





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$$F = F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 = M$$



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Grabowski & Rotkiewicz 2012

Weighted Lie algebroids



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Definition

A weighted Lie algebroid of degree k is a Lie algebroid $(\Pi E, Q)$ equipped with a homogeneity structure of degree k - 1 such that

$$\Pi \widehat{\mathsf{h}}_t : \Pi E \to \Pi E$$

is a Lie algebroid morphism for all $t \in \mathbb{R}$. That is

$$Q \circ (\Pi \widehat{\mathsf{h}}_t)^* = (\Pi \widehat{\mathsf{h}}_t)^* \circ Q.$$





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- The tangent bundle of a graded bundle.
- Higher order tangent bundles of a Lie algebroid.



Question: What are the global objects that 'integrate' weighted Lie algebroids?



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On to Lie groupoids...

Weighted Lie groupoids





Definition

A weighted Lie groupoid of degree k is a Lie groupoid $\Gamma_k \rightrightarrows B_k$, together with a homogeneity structure $\underline{h} : \mathbb{R} \times \Gamma_k \to \Gamma_k$ of degree k, such that \underline{h}_t is a Lie groupoid morphism for all $t \in \mathbb{R}$.



Unravel:

• B_k is a graded bundle of degree k.



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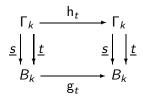
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$$\begin{array}{c} \Gamma_k & \xrightarrow{h_t} & \Gamma_k \\ \underline{s} & \downarrow & \underline{t} & \underline{s} & \downarrow & \underline{t} \\ B_k & \xrightarrow{g_t} & B_k \end{array} \\ \underline{s} \circ h_t = g_t \circ \underline{s}, \quad \text{and} \quad \underline{t} \circ h_t = g_t \circ \underline{t} \end{array}$$



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 $\mathsf{h}_t(g \circ h) = \mathsf{h}_t(g) \circ \mathsf{h}_t(h)$



Examples

If G ⇒ M is Lie groupoid the T^kG ⇒ T^kM is a weighted Lie groupoid of degree k



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- If G ⇒ M is Lie groupoid the T^kG ⇒ T^kM is a weighted Lie groupoid of degree k
- ► VB-groupoids = degree 1 weighted Lie groupoids Bursztyn + Cabrera + de Hoyo (2014)



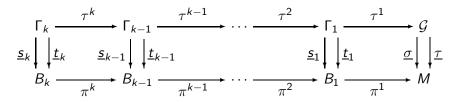
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In particular, $\Gamma_1 \rightrightarrows B_1$ is a \mathcal{VB} -groupoid.

Weighted Lie theory



Theorem

If $\Gamma_k \rightrightarrows B_k$ is a weighted Lie groupoid of degree k with respect to a homogeneity structure <u>h</u> on Γ_k , then $A(\Gamma_k) \rightarrow B_k$ is a weighted Lie algebroid of degree k + 1 with respect to the homogeneity structure \hat{h} defined by

$$\widehat{\mathsf{h}}_t = (\underline{\mathsf{h}}_t)' = \mathsf{Lie}(\underline{\mathsf{h}}_t) *$$

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Theorem

Let $E_{k+1} \rightarrow B_k$ be a weighted Lie algebroid of degree k + 1 with respect to a homogeneity structure \hat{h} and Γ_k its (source simply-connected) integration groupoid. Then Γ_k is a weighted Lie groupoid of degree k with respect to the homogeneity structure <u>h</u> uniquely determined by *.



Closing remarks





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- Weighted Poisson-Lie groupoids, weighted Lie bi-algebroids and weighted Courant algebroids



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- Weighted Poisson-Lie groupoids, weighted Lie bi-algebroids and weighted Courant algebroids
- Expect further links with physics



