Revisiting Lie integrability by quadratures from a geometric perspective

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Abstract

The classical result of Lie on integrability by quadratures will be reviewed and some generalizations will be proposed. After a short review of the classical Lie theorem, a finite dimensional Lie algebra of vector fields is considered and the most general conditions under which the integral curves of one of the fields can be obtained by quadratures in a prescribed way will be discussed, determining also the number of quadratures needed to integrate the system. The theory will be illustrated with examples and an extension of the theorem where the Lie algebras are replaced by some distributions will also be presented.

This is a report on a recent collaboration with:

F. Falceto, J. Grabowski and M.F. Rañada

Outline

Motivation

I know Janusz for more than 20 years: summer of 1993 at El Escorial (Spain) during the meeting

“Advanced Topics in Classical and Quantum Systems".

After some years of meetings in different countries we started our collaboration

We have had a nice and fruitful collaboration both in Poland and in Spain, with other colleagues, mainly focused on:

A) Geometrical properties of differential equations (Lie systems and generalizations)

B) Deformation of algebraic structures and its physical applications

C) Integrability

Nowadays we are not only scientific collaborators but also very good friends

Lie–Scheffers systems: A geometric approach

Bibliopolis, Napoli, ISBN 88-7088-378-7 (2000).

Authors: José F. Cariñena, Janusz Grabowski and Giuseppe Marmo

Some physical applications of systems of differential equations admitting a superposition rule

Rep. Math. Phys. **48**, 47–58 (2001)

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Reduction of time–dependent systems admitting a superposition principle

Acta Applicandae Mathematicae 66, 67–87 (2001)

Autores: José F. Cariñena, Janusz Grabowski and Arturo Ramos

Superposition rules, Lie theorem and partial differential equations

Rep. Math. Phys. **60**, 237–258 (2007)

Authors: José F. Cariñena, Janusz Grabowski and Giuseppe Marmo

Quasi–Lie schemes: theory and applications

J. Phys. A: Math. Theor. **42**, 335206 (20 p.) (2009)

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Lie families: theory and applications

J. Phys. A: Math. Theor. **43**, 305201 (2010)

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Superposition rules for higher-order systems and their applications

J. Phys. A: Math. Theor. **45**, 185202 (26pp) (2012)

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Dirac–Lie systems and Schwarzian equations

J. Diff. Eqns. **257**, 2303–2340 (2014)

Autores: José F. Cariñena, Janusz Grabowski, Javier de Lucas and Cristina Sardón

Quantum Bihamiltonian Systems

Int. J. Mod. Phys. $\mathbf{A}$ $\mathbf{15}$, 4797–4810 (2000)

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Contractions: Nijenhuis and Saletan tensors for general algebraic structures

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Authors: José F. Cariñena, Janusz Grabowski and Giuseppe Marmo

Geometry of Lie integrability by quadratures

J. Phys. A: Math. Theor. **48**, 215206 (18pp) (2015)

Autores: José F. Cariñena, Fernando Falceto, Janusz Grabowski and Manuel F. Rañada

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Geometry of Lie integrability by quadratures

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Abstract
In this paper, we extend the Lie theory of integration by quadratures of systems of ordinary differential equations in two different ways. First, we consider a finite-dimensional Lie algebra of vector fields and discuss the most general conditions under which the integral curves of one of the fields can be obtained by quadratures in a prescribed way. It turns out that the conditions can be expressed in a purely algebraic way. In the second step, we generalize the construction to the case in which we substitute the Lie algebra of vector fields by a module (generalized distribution). We obtain a much larger class of explicitly integrable systems, replacing standard concepts of solvable (or nilpotent) Lie algebra with distributional solvability (nilpotency).

Keywords: integrable dynamical systems, integration by quadratures, solvable and nilpotent Lie algebras

1. Introduction

The integrability of a given system of differential equations is a recurrent subject of significant interest that has been an active field of research in recent years. The meaning of integrability, however, is not well defined and has a different definition within each theory; it is only rigorously defined in each specific field. Of course, integrability means that you can find the general solution in an algorithmic way, but for instance, you can restrict yourself to search for solutions of a previously selected class of functions, polynomial functions, rational functions, etc. The existence of additional structures such as compatible symplectic structures

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The meaning of Integrability

An autonomous system of first-order differential equations,

$$\dot{x}^i = f^i(x^1, \ldots, x^N)\ , \qquad i = 1, \ldots, N,$$

is geometrically interpreted in terms of a vector field Γ in a N-dimensional manifold M with a local coordinate expression

$$\Gamma = f^i(x^1, \ldots, x^N)\frac{\partial}{\partial x^i}\ .$$

The integral curves of Γ are the solutions of the given system.

Integrate the system means to determine the general solution of the system.

More specifically, integrability by quadratures means that you can determine the solutions (i.e. the flow of Γ) by means of a finite number of algebraic operations and quadratures of some functions.

The two main techniques in the process of solving the system:

- Determination of constants of motion

  Constants of motion provide us foliations such that Γ is tangent to the leaves, and reducing in this way the problem to a family of lower dimensional problems, one on each leaf

- Search for symmetries of the vector field

  The knowledge of infinitesimal groups of symmetries of the vector field (i.e. of the system of differential equations), suggests us to use adapted local coordinates, the system decoupling then into lower dimensional subsystems.

More specifically, the knowledge of r functionally independent (i.e. such that $dF_1 \wedge \cdots \wedge dF_r \neq 0$) constants of motion $F_1, \ldots, F_r$, allows us to reduce the problem to that of a family of vector fields $\widetilde{\Gamma}_c$ defined in the $N - r$ dimensional submanifolds M_c given by the level sets of the vector function of rank r, $(F_1, \ldots, F_r) : M \to \mathbb{R}^r$.

Of course the best situation is when $r = N - 1$: the leaves are one-dimensional, giving us the solutions to the problem, up to a reparametrization.

There is another way of reducing the problem. Given an infinitesimal symmetry (i.e. a vector field X such that $[X, \Gamma] = 0$), in a neighbourhood of a point where X is different form zero we can choose adapted coordinates, $(y^1, \dots, y^N)$, for which X is written (Straightening out Theorem)

$$X = \frac{\partial}{\partial y^N} \ .$$

Then $[X, \Gamma] = 0$ implies that Γ has the form

$$\Gamma = \bar{f}^1(y^1, \dots, y^{N-1}) \frac{\partial}{\partial y^1} + \dots + \bar{f}^{N-1}(y^1, \dots, y^{N-1}) \frac{\partial}{\partial y^{N-1}} + \bar{f}^N(y^1, \dots, y^{N-1}) \frac{\partial}{\partial y^N} \ ,$$

and its integral curves are obtained by solving the system

$$\left\{ \begin{array}{rcll} \dfrac{dy^i}{dt} & = & \bar{f}^i(y^1, \dots, y^{N-1}) \ , & \qquad i = 1, \dots, N-1 \\ \dfrac{dy^N}{dt} & = & \bar{f}^N(y^1, \dots, y^{N-1}). & \end{array} \right.$$

We have reduced the problem to a subsystem involving only the first $N-1$ equations, and and once this has been solved, the last equation is used to obtain the function $y^N(t)$ by means of one quadrature.

Note that the new coordinates $y^1, \ldots, y^{N-1}$, are constants of the motion for X and therefore we cannot easily find such coordinates in a general case.

Moreover, the information provided by two different symmetry vector fields cannot be used simultaneously in the general case, because it is not possible to find local coordinates $(y^1, \ldots, y^N)$ such that

$$X_1 = \frac{\partial}{\partial y^{N-1}} \ , \qquad X_2 = \frac{\partial}{\partial y^N} \ ,$$

unless that $[X_1, X_2] = 0$.

In terms of adapted coordinates for Γ the integration is immediate, the solution being

$$y^k(t) = y_0^k, \quad k = 1, \ldots, N-1, \qquad y^N(t) = y^N(0) + t.$$

This proves that the concept of integrability by quadratures depends on the choice of initial coordinates.

However, it will be proved that when Γ is part of a family of vector fields satisfying appropriate conditions, then it is integrable by quadratures for any choice of initial coordinates

Both, constants of motion and infinitesimal symmetries, can be used simultaneously if some compatibility conditions are satisfied.

We can say that a system admitting $r < N - 1$ functionally independent constants of motion, $F_1, \ldots, F_r$, is integrable when we know furthermore s commuting infinitesimal symmetries $X_1, \ldots, X_s$, with $r + s = N$ such that

$$[X_a, X_b] = 0,\ a, b = 1, \ldots, s, \qquad \text{and} \qquad X_a F_\alpha = 0, \quad \forall a = 1, \ldots, s, \alpha = 1, \ldots r.$$

The constants of motion determine a s-dimensional foliation (with $s = N - r$) and the former condition means that the restriction of the s vector fields X_a to the leaves are tangent to such leaves.

Sometimes we have additional geometric structures that are compatible with the dynamics. For instance, a symplectic structure ω on a $2n$-dimensional manifold M.

Such 2-form relates, by contraction, in a one-to-one way vector fields and 1-forms.

Vector fields X_F associated with exact 1-forms dF are said to be Hamiltonian vector fields.

Compatible means that the dynamical vector field itself is a Hamiltonian v.f. X_H.

Particularly interesting is Arnold–Liouville definition of (Abelian) complete integrability ($r = s = n$). The vector fields are $X_a = X_{F_a}$ and, for instance, $F_1 = H$.

The regular Poisson bracket defined by ω (i.e. $\{F_1, F_2\} = X_{F_2} F_1$), allows us to express the above tangency conditions as

$$X_{F_b} F_a = \{F_a, F_b\} = 0,$$

– i.e. the n functions are constants of motion in involution and the corresponding Hamiltonian vector fields commute.

Our aim is to study integrability in absence of additional compatible structures, the main tool being properties of Lie algebras containing the given vector field, very much in the approach started by Lie.

The problem of integrability by quadratures depends on the determination by quadratures of the necessary first-integrals and on finding adapted coordinates, or, in another words, in finding a sufficient number of invariant tensors.

The set $\mathfrak{X}_\Gamma(M)$ of strict infinitesimal symmetries of $\Gamma \in \mathfrak{X}(M)$ is a linear space:

$$\mathfrak{X}_\Gamma(M) = \{X \in \mathfrak{X}(M) \mid [X, \Gamma] = 0\} \ .$$

The flow of a vector field $X \in \mathfrak{X}_\Gamma(M)$ preserves the set of integral curves of Γ.

The set of vector fields generating flows preserving the set of integral curves of Γ up to a reparametrization is a real linear space containing $\mathfrak{X}_\Gamma(M)$ and will be denoted

$$\mathfrak{X}^\Gamma(M) = \{X \in \mathfrak{X}(M) \mid [X, \Gamma] = f_X\, \Gamma\} \ , \quad f_X \in C^\infty(M).$$

Vector fields in $\mathfrak{X}^\Gamma(M)$ preserve the one-dimensional distribution generated by Γ.

One chan check that:

- $\mathfrak{X}^\Gamma(M)$ is a real Lie algebra
- $\mathfrak{X}_\Gamma(M)$ is a subalgebra of $\mathfrak{X}^\Gamma(M)$.

However $\mathfrak{X}_\Gamma(M)$ is not an ideal in $\mathfrak{X}^\Gamma(M)$.

Lie theorem of integrability by quadratures

The first important result is due to Lie who established the following theorem:

Theorem: *If n vector fields $X_1,\ldots,X_n$, which are linearly independent in each point of an open set $U \subset \mathbb{R}^n$, generate a solvable Lie algebra and are such that $[X_1, X_i] = \lambda_i X_1$ with $\lambda_i \in \mathbb{R}$, then the differential equation $\dot{x} = X_1(x)$ is solvable by quadratures in U.*

Consider first the simplest case $n = 2$.

The differential equation can be integrated if we are able to find a first integral F (i.e. $X_1 F = 0$), such that $dF \neq 0$ in U. The straightening out theorem says that such a function F locally exists.

Such a function F implicitly defines one variable, for instance x_2, in terms of the other one by $F(x_1, \phi(x_1)) = k$.

If X_1 and X_2 are such that $[X_1, X_2] = \lambda_2 X_1$, and α_0 is a 1-form, defined up to multiplication by a function, such that $i(X_1)\alpha_0 = 0$, as X_2 is linear independent of X_1 at each point, $i(X_2)\alpha_0 \neq 0$, and we can see that the 1-form $\alpha = (i(X_2)\alpha_0)^{-1}\alpha_0$ is such that $i(X_1)\alpha = 0$ and satisfies the condition $i(X_2)\alpha = 1$. Such 1-form α is closed, because X_1 and X_2 generate $\mathfrak{X}(\mathbb{R}^2)$ and

$$d\alpha(X_1, X_2) = X_1\alpha(X_2) - X_2\alpha(X_1) + \alpha([X_1, X_2]) = \alpha([X_1, X_2]) = \lambda_2\,\alpha(X_1) = 0.$$

Therefore, there exists, at least locally, a function F such that $\alpha = dF$, and it is given by

$$F(x_1, x_2) = \int_\gamma \alpha,$$

where γ is any curve with end in the point (x_1, x_2), is the function we were looking for, because $dF = \alpha$ and then

$$i(X_1)\alpha = 0 \Longleftrightarrow X_1 F = 0, \qquad i(X_2)\alpha = 1 \Longleftrightarrow X_2 F = 1.$$

Recalling some basic concepts of cohomology

Let be $\mathfrak{g}$ a Lie algebra and $\mathfrak{a}$ a $\mathfrak{g}$-module: $\mathfrak{a}$ is a linear space that is the carrier space for a linear representation Ψ of $\mathfrak{g}$, $\Psi : \mathfrak{g} \to \mathrm{End}\, \mathfrak{a}$ – i.e. satisfying

$$\Psi(a)\Psi(b) - \Psi(b)\Psi(a) = \Psi([a, b]), \quad \forall a, b \in \mathfrak{g}.$$

By a k-cochain we mean a k-linear alternating map $\alpha : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathfrak{a}$.

$C^k(\mathfrak{g}, \mathfrak{a})$ denotes the linear space of k-cochains.

For every $k \in \mathbb{N}$ we define $\delta_k : C^k(\mathfrak{g}, \mathfrak{a}) \to C^{k+1}(\mathfrak{g}, \mathfrak{a})$ by

$$\begin{aligned}(\delta_k \alpha)(a_1, \ldots, a_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \Psi(a_i) \alpha(a_1, \ldots, \widehat{a}_i, \ldots, a_{k+1}) + \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([a_i, a_j], a_1, \ldots, \widehat{a}_i, \ldots, \widehat{a}_j, \ldots, a_{k+1}),\end{aligned}$$

where $\widehat{a}_i$ denotes, as usual, that the element a_i is omitted.

The linear maps δ_k can be shown to satisfy $\delta_{k+1} \circ \delta_k = 0$.

The linear operator δ on $C(\mathfrak{g},\mathfrak{a})=\bigoplus_{n=0}^{\infty} C^k(\mathfrak{g},\mathfrak{a})$ whose restriction to each $C^k(\mathfrak{g},\mathfrak{a})$ is δ_k, satisfies $\delta^2=0$. We will then denote

$$\begin{aligned} B^k(\mathfrak{g},\mathfrak{a}) &= \{\alpha\in C^k(\mathfrak{g},\mathfrak{a}) \mid \exists\beta\in C^{k-1}(\mathfrak{g},\mathfrak{a}) \text{ such that } \alpha=\delta\beta\} = \mathrm{Image}\,\delta_{k-1}, \\ Z^k(\mathfrak{g},\mathfrak{a}) &= \{\alpha\in C^k(\mathfrak{g},\mathfrak{a}) :\mid \delta\alpha=0\} = \ker\delta_k . \end{aligned}$$

The elements of $Z^k(\mathfrak{g},\mathfrak{a})$ are called k-cocycles, and those of $B^k(\mathfrak{g},\mathfrak{a})$ are called k-coboundaries.

Since $\delta^2=0$, we see $B^k(\mathfrak{g},\mathfrak{a})\subset Z^k(\mathfrak{g},\mathfrak{a})$. The k-th cohomology group $H^k(\mathfrak{g},\mathfrak{a})$ is

$$H^k(\mathfrak{g},\mathfrak{a}) := \frac{Z^k(\mathfrak{g},\mathfrak{a})}{B^k(\mathfrak{g},\mathfrak{a})} ,$$

and we will define $B^0(\mathfrak{g},\mathfrak{a})=0$, by convention.

An interesting example: $\mathfrak{g}$ = a finite-dimensional Lie subalgebra of $\mathfrak{X}(M)$, $\mathfrak{a}=\bigwedge^p(M)$, and the action given by $\Psi(X)\zeta=\mathcal{L}_X\zeta$.

The case $p=0$, has been used, for instance, in the study of weakly invariant differential equations as shown in a paper with M.A. del Olmo and P. Winternitz, Lett. Math. Phys. **29**, 151 (1993). The cases $p=1,2$, are also interesting in mechanics.

Coming back to the particular case $p = 0$, $\mathfrak{a} = \bigwedge^0(M) = C^\infty(M)$, $\mathfrak{g} = \mathfrak{X}(M)$, the elements of $Z^1(\mathfrak{g}, \bigwedge^0(M))$ are linear maps $h : \mathfrak{g} \to C^\infty(M)$ satisfying

$$(\delta_1 h)(X, Y) = \mathcal{L}_X h(Y) - \mathcal{L}_Y h(X) - h([X, Y]) = 0 \ , \qquad X, Y \in \mathfrak{X}(M),$$

and those of $B^1(\mathfrak{g}, C^\infty(M))$ are those h for which $\exists g \in C^\infty(M)$ with

$$h(X) = \mathcal{L}_X g \ .$$

Lemma *Let $\{X_1, \dots, X_n\}$ be a set of n vector fields whose values are linearly independent at each point of an n-dimensional manifold M. Then:*

1) The necessary and sufficient condition for the system of equations for $f \in C^\infty(M)$

$$X_i f = h_i, \qquad h_i \in C^\infty(M), \quad i = 1, \dots, n,$$

to have a solution is that the 1-form $\alpha \in \bigwedge^1(M)$ such that $\alpha(X_i) = h_i$ be an exact 1-form.

2) If the previous n vector fields generate a n-dimensional real Lie algebra $\mathfrak{g}$ (i.e. there exist real numbers $c_{ij}{}^k$ such that $[X_i, X_j] = c_{ij}{}^k\, X_k$), then the necessary condition for the system of equations to have a solution is that the $\mathbb{R}$-linear function $h : \mathfrak{g} \to C^\infty(M)$ defined by $h(X_i) = h_i$ is a cochain that is a cocycle.

Proof.- 1) If i, j, if $X_i f = h_i$ and $X_j f = h_j$, then, as $\exists f_{ij}{}^k \in C^\infty(M)$ such that $[X_i, X_j] = f_{ij}{}^k X_k$,

$$X_i(X_j f) - X_j(X_i f) = [X_i, X_j]f = f_{ij}{}^k X_k f \implies X_i(h_j) - X_j(h_i) - f_{ij}{}^k h_k = 0,$$

and as $\alpha(X_i) = h_i$, we obtain that

$$d\alpha(X_i, X_j) = X_i \alpha(X_j) - X_j \alpha(X_i) - \alpha([X_i, X_j]) = X_i(h_j) - X_j(h_i) - f_{ij}{}^k h_k.$$

Consequently, a necessary condition for the existence of the solution of the system is that α be closed.

2) Consider $\mathfrak{a} = C^\infty(M)$ and the cochain determined by the linear map h. Now the necessary condition for the existence of the solution is written as:

$$X_i(h_j) - X_j(h_i) - c_{ij}{}^k h_k = (\delta_1 h)(X_i, X_j) = 0.$$

The is just the 1-cocycle condition.

Most properties of differential equations are of a local character: closed forms are locally exact and we can restrict ourselves to appropriate open subsets U of M, i.e. open submanifolds, where the closed 1-form is exact, .

Then if α is closed, it is locally exact, $\alpha = df$ in a certain open U, $f \in C^\infty(U)$, and the solution of the system can be found by one quadrature: the solution function f is given by the quadrature

$$f(x) = \int_{\gamma_x} \alpha,$$

where γ_x is any path joining some reference point $x_0 \in U$ with $x \in U$.

We also remark that α is exact, $\alpha = df$, if and only if $\alpha(X_i) = df(X_i) = X_i f = h_i$, i.e. h is a coboundary, $h = \delta f$.

In the particular case of the appearing functions h_i being constant the condition for the existence of local solution reduces to $\alpha([X,Y]) = 0$, for each pair of elements, X and Y in $\mathfrak{g}$, i.e. α vanishes on the derived Lie algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. In particular when $\mathfrak{g}$ is Abelian there is not any condition

A generalization of Lie theory of integration

Consider a family of n vector fields, $X_1, \ldots, X_n$, defined on a n-dimensional manifold M and assume that they close a Lie algebra L over the real numbers

$$[X_i, X_j] = c_{ij}{}^k\, X_k\,, \quad i, j, k = 1, \ldots, n,$$

and that in addition they span a basis of T_xM at every point $x \in M$. We pick up an element in the family, X_1, the dynamical vector field.

To emphasize its special rôle we will often denote it by $\Gamma \equiv X_1$.

Our goal, is to obtain the integral curves $\Phi_t : M \to M$ of Γ

$$(\Gamma f)(\Phi_t(x)) = \frac{d}{dt} f(\Phi_t(x)), \quad \forall f \in C^\infty(x),$$

using quadratures (operations of integration, elimination and partial differentiation).

The number of quadratures is given by the number of integrals of known functions depending on a finite number of parameters, that are performed.

Γ plays a distinguished and important rôle since it represents the dynamics to be integrated.

Our approach is concerned with the construction of a sequence of nested Lie subalgebras $L_{\Gamma,k}$ of the Lie algebra L, and it will be essential that Γ belongs to all the subalgebras. This construction will be carried out in several steps.

The first one will be to reduce, by one quadrature, the original problem to a similar one but with a Lie subalgebra $L_{\Gamma,1}$ of the Lie algebra L (with $\Gamma \in L_{\Gamma,1}$) whose elements span at every point the tangent space of the leaves of a certain foliation.

If iterating the procedure we end up with an Abelian Lie algebra we can, with one more quadrature, obtain the flow of the dynamical vector field.

We determine the foliation through a family of functions that are constant on the leaves. We first consider the ideal

$$L_{\Gamma,1} = \langle \Gamma \rangle + [L, L]\,, \qquad \dim L_{\Gamma,1} = n_1,$$

that, in order to make the notation simpler, we will assume to be generated by the first n_1 vector fields of the family (i.e. $L_{\Gamma,1} = \langle \Gamma, X_2, \ldots, X_{n_1} \rangle$). This can be always achieved by choosing appropriately the basis of L.

Now take $\zeta_1 \in L^0_{\Gamma,1} = \text{annihilator of } L_{\Gamma,1} = \{\text{elements in } L^* \text{ killing vectors of } L_{\Gamma,1}\}$ and define the 1-form α_{ζ_1} on M by its action on the vector fields in L:

$$\alpha_{\zeta_1}(X) = \zeta_1(X), \quad \text{for } X \in L.$$

As $\alpha_{\zeta_1}(X)$ is a constant function on M, for any vector field in L, we have

$$d\alpha_{\zeta_1}(X, Y) = \alpha_{\zeta_1}([X, Y]) = \zeta_1([X, Y]) = 0, \quad \text{for } X, Y \in L, \zeta_1 \in L^0_{\Gamma,1}.$$

Therefore the 1-form α_{ζ_1} is closed and by application of the result of the lemma the system of partial differential equations

$$X_i Q_{\zeta_1} = \alpha_{\zeta_1}(X_i), \quad i = 1, \ldots, n, \quad Q_{\zeta_1} \in C^\infty(M),$$

has a unique (up to the addition of a constant) local solution which can be obtained by one quadrature.

If we fixe the same reference point x_0 for any ζ_1, α_{ζ_1} depends linearly on ζ_1 and, if γ_x is independent of ζ_1, we have that the correspondence

$$L^0_{\Gamma,1} \ni \zeta_1 \mapsto Q_{\zeta_1} \in C^\infty(M),$$

defines an injective linear map.

The system expresses that the vector fields in $L_{\Gamma,1}$ (including Γ) are tangent to

$$N_1^{[Y_1]} = \{x \mid Q_{\zeta_1}(x) = \zeta_1(Y_1),\, \zeta_1 \in L^0_{\Gamma,1}\} \subset M$$

for any $[Y_1] \in L/L_{\Gamma,1}$. Locally, for an open neigbourhood U, the $N_1^{[Y_1]}$'s define a smooth foliation of n_1-dimensional leaves.

Now, we repeat the previous procedure by taking $L_{\Gamma,1}$ as the Lie algebra and any leaf $N_1^{[Y_1]}$ as the manifold. The new subalgebra $L_{\Gamma,2} \subset L_{\Gamma,1}$ is defined by

$$L_{\Gamma,2} = \langle \Gamma \rangle + [L_{\Gamma,1}, L_{\Gamma,1}]\,, \qquad \dim L_{\Gamma,2} = n_2\,,$$

and taking $\zeta_2 \in L^0_{\Gamma,2} \subset L^*_{\Gamma,1}$ (the annihilator of $L_{\Gamma,2}$), we arrive at a new system of partial differential equations

$$X_i Q^{[Y_1]}_{\zeta_2} = \zeta_2(X_i), \quad i = 1, \ldots, n_1, \quad Q^{[Y_1]}_{\zeta_2} \in C^\infty(N_1^{[Y_1]})\,,$$

that can be solved with one quadrature and such $Q^{[Y_1]}_{\zeta_2}$ depends linearly on ζ_2.

It will be useful to extend $Q_{\zeta_2}^{[Y_1]}$ to U. We first introduce the map

$$U \ni x \mapsto [\overset{x}{Y_1}] \in L_{\Gamma,0}/L_{\Gamma,1}$$

where x and $[\overset{x}{Y_1}]$ are related by the equation $Q_{\zeta_1}(x) = \zeta_1(\overset{x}{Y_1})$, that correctly determines the map. Now, we define $Q_{\zeta_2} \in C^\infty(U)$ by

$$Q_{\zeta_2}(x) = Q_{\zeta_2}^{[\overset{x}{Y_1}]}(x).$$

Note that by construction $x \in N_1^{[\overset{x}{Y_1}]}$ and, therefore the definition makes sense.

The resulting function $Q_{\zeta_2}(x)$ is smooth provided the reference point of the lemma changes smoothly from leave to leave.

The construction is then iterated by defining

$$N_2^{[Y_1][Y_2]} = \{x \mid Q_{\zeta_1}(x) = \zeta_1(Y_1), \quad Q_{\zeta_2}(x) = \zeta_2(Y_2), \text{ with } \zeta_1 \in L^0_{\Gamma,1}, \zeta_2 \in L^0_{\Gamma,2}\} \subset M,$$

for $[Y_1] \in L_{\Gamma,0}/L_{\Gamma,1}$ and $[Y_2] \in L_{\Gamma,1}/L_{\Gamma,2}$. Note that $L_{\Gamma,2}$ generates at every point the tangent space of $N_2^{[Y_1][Y_2]}$, therefore we can proceed as before.

The algorithm ends if after some steps, say k, the Lie algebra $L_{\Gamma,k} = \langle X_1, \ldots, X_{n_k} \rangle$, whose vector fields are tangent to the n_k-dimensional leaf $N_k^{[Y_1],\ldots,[Y_k]}$, is Abelian.

In this moment the system of equations

$$X_i Q_{\zeta_k}^{[Y_1],\dots,[Y_k]} = \zeta_k(X_i), \quad i=1,\dots,n_{k-1}, \quad Q_{\zeta_k}^{[Y_1],\dots,[Y_k]} \in C^\infty(N_k^{[Y_1],\dots,[Y_k]}),$$

can be solved locally by one more quadrature for any $\zeta_k \in L^*_{\Gamma,k}$.

Remark that, as the Lie algebra $L_{\Gamma,k}$ is Abelian, the integrability condition is always satisfied and we can take ζ_k in the whole of $L^*_{\Gamma,k}$ instead of $L^0_{\Gamma,k}$. then, as before, we extend the solutions to U and call them Q_{ζ_k}.

With all these ingredients we can find the flow of Γ by performing only algebraic operations. In fact, consider the formal direct sum

$$\Xi = L^0_{\Gamma,1} \oplus L^0_{\Gamma,2} \oplus \cdots \oplus L^0_{\Gamma,k} \oplus L^*_{\Gamma,k}$$

that, as one can check, has dimension n.

The linear maps $L^0_{\Gamma,i} \ni \zeta_i \mapsto Q_{\zeta_i} \in C^\infty(U)$ can be extended to Ξ so that to any $\xi \in \Xi$ we assign a $Q_\xi \in C^\infty(U)$. Now consider a basis

$$\{\xi_1,\dots,\xi_n\} \subset \Xi.$$

The associated functions $Q_{\xi_j}, j=1,\dots,n$ are independent and satisfy

$$\Gamma Q_{\xi_j}(x) = \xi_j(\Gamma)\,, \quad j=1,2,\dots,n,$$

where it should be noticed that as $\Gamma \in L_{\Gamma,l}$ for any $l = 0, \ldots, k$, the right hand side is well defined, and we see from here that in the coordinates given by the Q_{ξ_j}'s the vector field Γ has constant components and, then, it is trivially integrated

$$Q_{\xi_j}(\Phi_t(x)) = Q_{\xi_j}(x) + \xi_j(\Gamma)t.$$

Now, with algebraic operations, one can derive the flow $\Phi_t(x)$. Altogether we have performed $k+1$ quadratures.

Algebraic properties

The previous procedure works if it reaches an end point (i.e. if there is a smallest non negative integer k such that

$$L_{\Gamma,k} = \langle\Gamma\rangle + [L_{\Gamma,k-1}, L_{\Gamma,k-1}] \text{ for } k > 0\,, \qquad L_{\Gamma,0} = L,$$

is an Abelian algebra). In that case we will say that (M, L, Γ) is Lie integrable of order $k+1$.

The content of the previous section can, thus, be summarized in the following

Theorem: If (M, L, Γ) is Lie integrable of order r, then the integral curves of Γ can be obtained by r quadratures.

We will discuss below some necessary and sufficient conditions for the Lie integrability.

Proposition: If (M, L, Γ) is Lie integrable then L is solvable.

Proof.- Let $L_{(i)}$ be the elements of the derived series, $L_{(i+1)} = [L_{(i)}, L_{(i)}]$, $L_{(0)} = L$, (note that $L_{(i)} = L_{0,i}$). Then,

$$L_{(i)} \subset L_{\Gamma,i},$$

and if the system is Lie integrable (i.e. $L_{\Gamma,k}$ is Abelian for some k), then we have $L_{(k+1)} = 0$ and, therefore, L is solvable.

Proposition: If L is solvable and A is an Abelian ideal of L, then (M, L, Γ) is Lie integrable for any $\Gamma \in A$.

Proof.- Using that A is an ideal containing Γ, we can show that

$$A + L_{\Gamma,i} = A + L_{(i)}.$$

We proceed again by induction; if the previous holds, then

$$\begin{aligned} A + L_{\Gamma,i+1} &= A + [L_{\Gamma,i}, L_{\Gamma,i}] = A + [A + L_{\Gamma,i}, A + L_{\Gamma,i}] = \\ &= A + [A + L_{(i)}, A + L_{(i)}] = A + L_{(i+1)}. \end{aligned}$$

Now L is solvable if some $L_{(k)} = 0$ and therefore $L_{\Gamma,k} \subset A$, i.e. it is Abelian and henceforth the system is Lie integrable.

Note that the particular case in which $A = \langle\Gamma\rangle$ corresponds to the standard Lie theorem.

Nilpotent algebras of vector fields also play an interesting role in the integrability of vector fields.

Proposition: If L is nilpotent, (M, L, Γ) is Lie integrable for any $\Gamma \in L$.

Proof.- Let us consider now the central series $L^{(i+1)} = [L, L^{(i)}]$ with $L^{(0)} = L$.

L nilpotent means that there is a k such that $L^{(k)} = 0$. Now, by induction, it is easy to see that $L_{\Gamma,i} \subset \langle\Gamma\rangle + L^{(i)}$ and therefore $L_{\Gamma,k} = \langle\Gamma\rangle$. Then, $L_{\Gamma,k}$ is Abelian and the system is Lie integrable.

From the previous propositions, we can derive the following,

Corollary 1 *Let (M, L, Γ) be Lie integrable of order r, then:*

(a) If r_s is the minimum positive integer such that $L_{(r_s)} = 0$, then $r \geq r_s$.

(b) If L is nilpotent r_n is the smallest natural number such that $L^{(r_n)} = 0$, $r \leq r_n$.

An interesting example

We now analyze the particular case of a recently studied superintegrable system.

The system is Hamiltonian, that is, the dynamical vector field $\Gamma = X_H$ is obtained from a Hamitonian function H by making use of a sympletic structure ω_0 defined in a cotangent bundle T^*Q.

We are now interested in considering this system just as a dynamical system (without mentioning the existence of a sympletic structure) and focusing our attention on the Lie structure of the symmetries.

The dynamics is given by the following vector field X_1 defined in $M = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates (x, y, p_x, p_y)

$$X_1 = p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} - \frac{k_2}{y^{2/3}} \frac{\partial}{\partial p_x} + \frac{2}{3} \frac{k_2\, x + k_3}{y^{5/3}} \frac{\partial}{\partial p_y}\,,$$

where k_2 and k_3 are arbitrary constants.

Now we denote by X_i, $i=2,3,4$ the following vector fields

$$\begin{aligned}
X_2 &= \left(6\,p_x^2+3\,p_y^2+k_2\frac{6x}{y^{2/3}}+k_3\frac{6}{y^{2/3}}\right)\frac{\partial}{\partial x}+(6\,p_xp_y+9\,k_2y^{1/3})\frac{\partial}{\partial y}\\
&- k_2\frac{6}{y^{2/3}}\,p_x\frac{\partial}{\partial p_x}+\left(4k_2\frac{x}{y^{5/3}}-3\frac{1}{y^{2/3}}\,p_y\right)\frac{\partial}{\partial p_y}\,,
\end{aligned}$$

$$\begin{aligned}
X_3 &= \left(4\,p_x^3+4\,p_xp_y^2+\frac{8(k_2x+k_3)}{y^{2/3}}p_x+12k_2\,y^{1/3}\,p_y\right)\frac{\partial}{\partial x}\\
&+ \left(4p_x^2\,p_y+12k_2\,y^{1/3}\,p_x\right)\frac{\partial}{\partial y}-4k_2\frac{1}{y^{2/3}}p_x^2\,\frac{\partial}{\partial p_x}\\
&+ \left(\frac{8}{3}\frac{k_2x+k_3}{y^{5/3}}p_x^2-4k_2\frac{1}{y^{2/3}}p_xp_y-12\,k_2^2\frac{1}{y^{1/3}}\right)\frac{\partial}{\partial p_y}\,,
\end{aligned}$$

and

$$\begin{aligned}
X_4 &= \left(6p_x^5+12\,p_x^3p_y^2+24\frac{k_3+k_2x}{y^{2/3}}p_x^3+108\,k_2y^{1/3}p_x^2p_y+324\,k_2^2y^{2/3}p_x\right)\frac{\partial}{\partial x}\\
&+ \left(6\,p_x^4p_y+36\,k_2y^{1/3}p_x^3\right)\frac{\partial}{\partial y}-6\,\left(\frac{k_2}{y^{2/3}}p_x^4-972k_2^3\right)\frac{\partial}{\partial p_x}\\
&+ \left(4\,\frac{k_3+k_2x}{y^{5/3}}p_x^4-12\frac{k_2}{y^{2/3}}-108\,k_2^2\frac{1}{y^{1/3}}p_x^2\right)\frac{\partial}{\partial p_y}\,.
\end{aligned}$$

Then we have

(i) The three vector field X_i Lie commute with $X_1 = \Gamma$

$$[X_1, X_i] = 0\,, \quad i = 2, 3, 4.$$

(ii) The Lie brackets of the X_i between themselves are given by

$$[X_2, X_3] = 0\,, \qquad [X_2, X_4] = 1944\, k_2^3\, \Gamma\,, \qquad [X_3, X_4] = 432\, k_2^3\, X_2\,.$$

Therefore, we have the following situation: .

First, Γ and the three vector fields X_2, X_3, X_4 generate a four-dimensional real Lie algebra L.

Second, the derived algebra $L_{(1)} \subset L$ is two-dimensional and it is generated by X_1 and X_2, i.e. $L_{(1)}$ is Abelian.

Finally, the second derived algebra $L_{(2)}$ reduces to the trivial algebra, that is, $L_{(2)} = [L_{(1)}, L_{(1)}] = \{0\}$.

Therefore the Lie algebra L is solvable with $r_s = 2$. However, $L^{(2)} = [L, L_{(1)}]$ is not trivial but $L_{(1)}$ is the one-dimensional ideal ideal in L generated by X_1, and this implies that the Lie algebra is nilpotent with $r_n = 3$.

(M, L, Γ) is Lie integrable for any $\Gamma \in L$, but the order of the system depends on the choice of the dynamical field:

a) (M, L, Γ) is Lie integrable of order 2 (the minimum possible value) for $\Gamma = X_i, i = 1, 2, 3$ or any combination of them

b) (M, L, Γ) is Lie integrable of order 3 (the maximum according to the corollary for $\Gamma = X_4$ (or any combination in which the coefficient of X_4 does not vanish).

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THANKS FOR YOUR ATTENTION !!!

CONGRATULATIONS, JANUSZ!!!

and

THANKS FOR ALL