NEW DEVELOPMENTS IN GEOMETRIC MECHANICS

Janusz Grabowski

(Polish Academy of Sciences)

GEOMETRY OF JETS AND FIELDS

Będlewo, 10-16 May, 2015

Contents

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Vector bundles as graded bundles

- A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n ,$$

$A(x) \in \mathrm{GL}(n, \mathbb{R})$.

- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0 and 'linear coordinates' y have degree 1. Linearity in $y's$ is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

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Graded bundles

- Canonical examples and constructions: $\mathsf{T}M$, T^*M, $E\otimes_M F$, $\wedge^k E$, etc.
- A straightforward generalization is the concept of a graded bundle $\tau: F\to M$ with a local trivialization by $U\times\mathbb{R}^n$ as before, and with the difference that the local coordinates $(y^1,\dots,y^n)$ in the fibres have now associated positive integer weights $w_1,\dots,w_n$, that are preserved by changes of local trivializations:
$$U\cap V\times\mathbb{R}^n\ni(x,y)\mapsto(x,A(x,y))\in U\cap V\times\mathbb{R}^n\,,$$
- One can show that in this case $A(x,y)$ must be polynomial in fiber coordinates, i.e. any graded bundle is a polynomial bundle.
- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. For instance, if $(y,z)\in\mathbb{R}^2$ are coordinates of degrees $1,2$, respectively, then the map $(y,z)\mapsto(y,z+y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.
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$$U\cap V\times\mathbb{R}^n\ni(x,y)\mapsto(x,A(x,y))\in U\cap V\times\mathbb{R}^n\,,$$

- One can show that in this case $A(x,y)$ must be polynomial in fiber coordinates, i.e. any graded bundle is a polynomial bundle.
- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. For instance, if $(y,z)\in\mathbb{R}^2$ are coordinates of degrees $1,2$, respectively, then the map $(y,z)\mapsto(y,z+y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.
- If all $w_i\le r$, we say that the graded bundle is of degree r.

Graded bundles

- In the above terminology, vector bundles are just graded bundles of degree 1.
- Graded bundles F_k of degree k admit, like many jet bundles, a tower of affine fibrations by their subbundles of lower degrees

$$F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M\,.$$

- Canonical examples: $\mathrm{T}^k M$, with canonical coordinates $(x, \dot{x}, \ddot{x}, \dddot{x}, \dots)$ of degrees, respectively, $0, 1, 2, 3,$ etc.
- Another example. If $\tau : E \to M$ is a vector bundle, then $\wedge^r \mathrm{T} E$ is canonically a graded bundle of degree r with respect to the projection

$$\wedge^r \mathrm{T}\tau : \wedge^r \mathrm{T}E \to \wedge^r \mathrm{T}M\,.$$

- Note that similar objects has been used in supergeometry by Kosmann-Schwarzbach, Voronov, Mackenzie, Roytenberg et al. under the name N-manifolds. However, we will work with classical, purely even manifolds during this talk.

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Graded Bundles

- With the use of coordinates (x^α, y^a) with degrees 0 for basic coordinates x^α, and degrees $w_a > 0$ for the fibre coordinates y^a, we can define on the graded bundle F a globally defined weight vector field (Euler vector field)
$$\nabla_F = \sum_a w_a y^a \partial_{y^a} .$$
- The flow of the weight vector field extends to a smooth action $\mathbb{R} \ni t \mapsto h_t$ of multiplicative reals on F, $h_t(x^\mu, y^a) = (x^\mu, t^{w_a} y^a)$. Such an action $h : \mathbb{R} \times F \to F$, $h_t \circ h_s = h_{ts}$, we will call a homogeneity structure.
- A function $f : F \to \mathbb{R}$ is called homogeneous of degree (weight) k if $f(h_t(x)) = t^k f(x)$; similarly for the homogeneity of tensor fields.
- Morphisms of two homogeneity structures (F_i, h^i), $i = 1, 2$, are defined as smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h^1_t = h^2_t \circ \Phi$. Consequently, a homogeneity substructure is a smooth submanifold S invariant with respect to h, $h_t(S) \subset S$.

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Double Graded Bundles

The fundamental fact (c.f. Grabowski-Rotkiewicz) says that graded bundles and homogeneity structures are in fact equivalent concepts.

Theorem

For any homogeneity structure h on a manifold F, there is a smooth submanifold $M = h_0(F) \subset F$, a non-negative integer $k \in \mathbb{N}$, and an $\mathbb{R}$-equivariant map $\Phi_h^k : F \to \mathrm{T}^k F_{|M}$ which identifies F with a graded submanifold of the graded bundle $\mathrm{T}^k F$. In particular, there is an atlas on F consisting of local homogeneous functions.

As two graded bundle structure on the same manifold are just two homogeneity structures, the obvious concept of compatibility leads to the following: A double graded bundle is a manifold equipped with two homogeneity structures h^1, h^2 which are compatible in the sense that

$$h^1_t \circ h^2_s = h^2_s \circ h^1_t \quad \text{for all } s, t \in \mathbb{R}\,.$$

This covers of course the concept of a double vector bundle of Pradines and Mackenzie, and extends to n-tuple graded bundles in the obvious way.

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Double graded bundles - examples

- Lifts. If $\tau : F \to M$ is a graded bundle of degree k, then $\mathrm{T}F$ and T^*F carry canonical double graded bundle structure: one is the obvious vector bundle, the other is of degree k. A double graded bundle whose one structure is linear we will call a $\mathcal{GL}$-bundle. There are also lifts of graded structures on F to $\mathrm{T}^r F$.
- In particular, if $\tau : E \to M$ is a vector bundle, then $\mathrm{T}E$ and T^*E are double vector bundles. The latter is isomorphic with T^*E^*. As a linear Poisson structure on E^* yields a map $\mathrm{T}^*E^* \to \mathrm{T}E^*$, a Lie algebroid structure on E can be encoded as a morphism of double vector bundles, $\varepsilon : \mathrm{T}^*E \to \mathrm{T}E^*$ (!)
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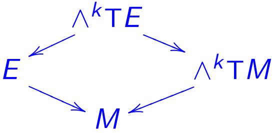

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T^*M - phase space

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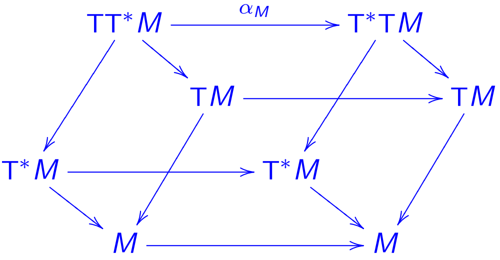

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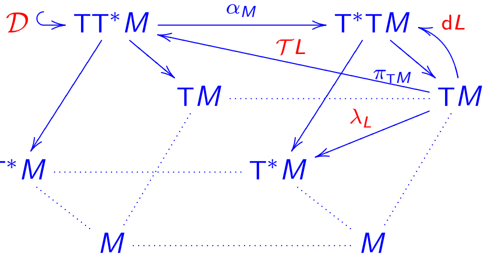

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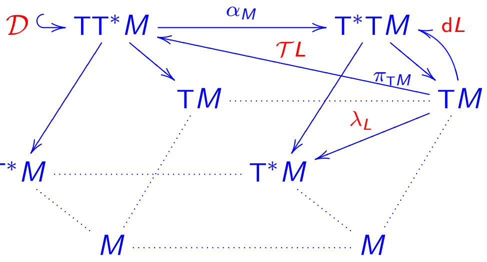

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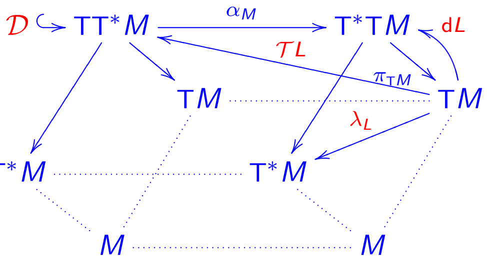

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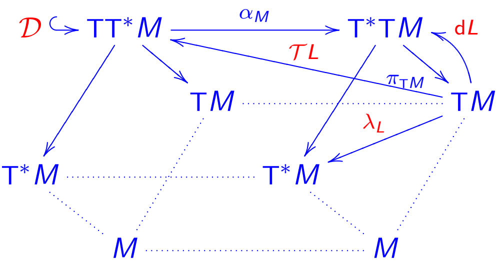

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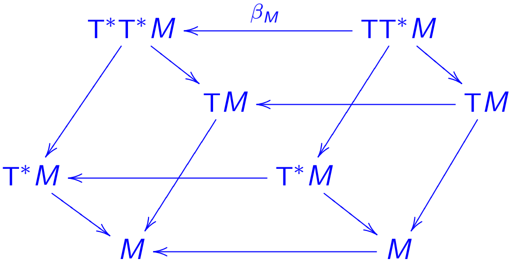

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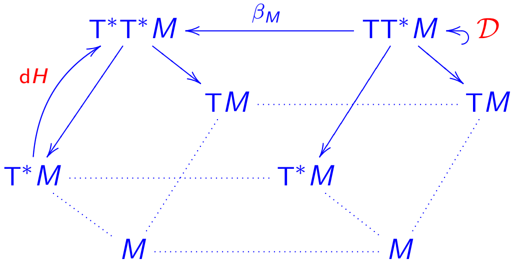

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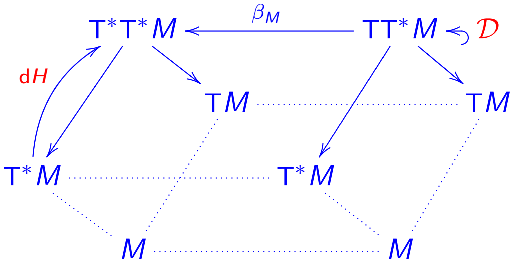

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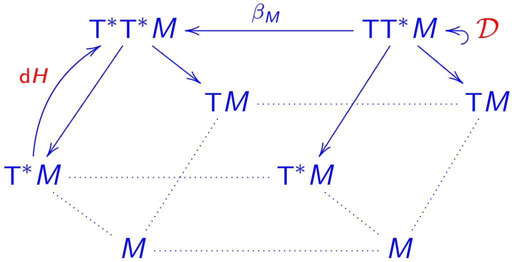

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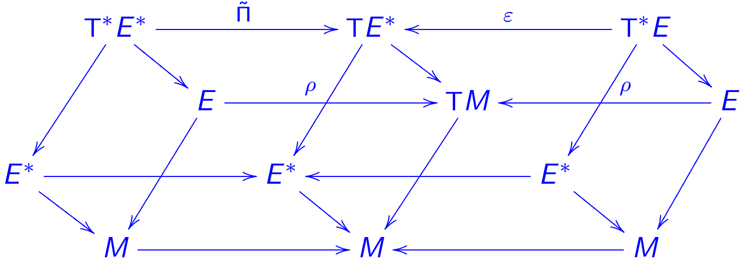
Algebroid setting



$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \mathcal{T}L(E)$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathsf{T}^*E^*$ $\qquad$ $\mathcal{D} = \tilde{\Pi}(\mathrm{d}H(E^*))$ $\qquad$ $\mathcal{D}_L \subset \mathsf{T}^*E$

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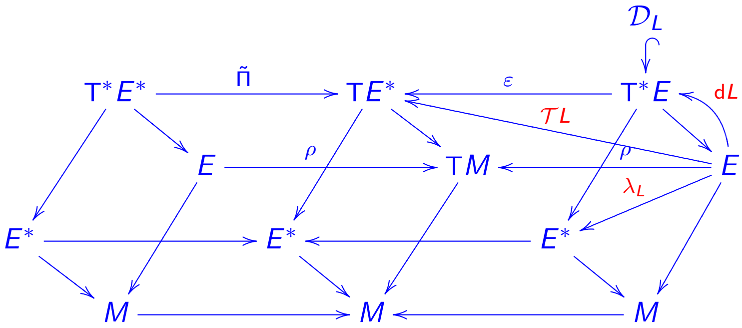

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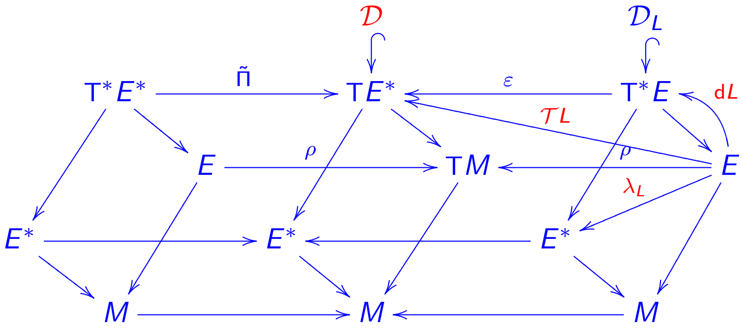

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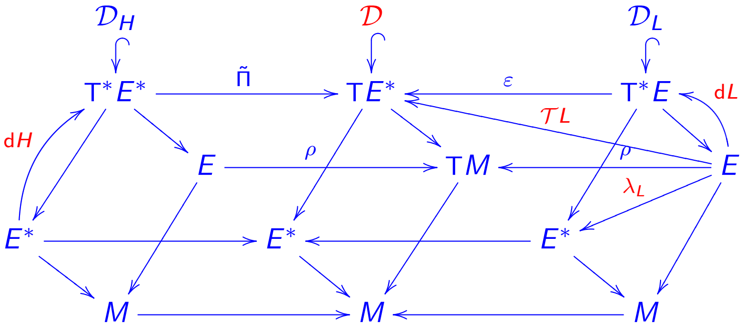

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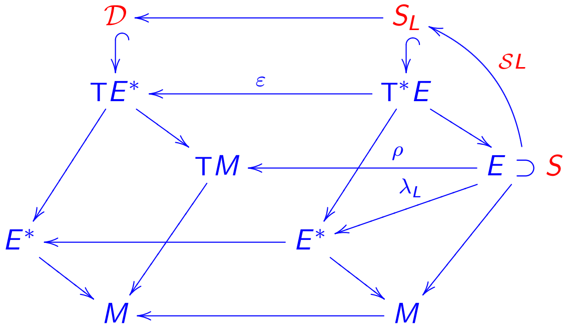
Algebroid setting with vakonomic constraints



where S_L is the lagrangian submanifold in T^*E induced by the Lagrangian on the constraint S, and $\mathcal{S}L : S \to \mathsf{T}^*E$ is the corresponding relation,

$$S_L = \{\alpha_e \in \mathsf{T}^*_e E : e \in S \text{ and } \langle \alpha_e, v_e \rangle = \mathrm{d}L(v_e) \text{ for every } v_e \in \mathsf{T}_e S\}.$$

The vakonomically constrained phase dynamics is just $\mathcal{D} = \varepsilon(S_L) \subset \mathsf{T}E^*$.

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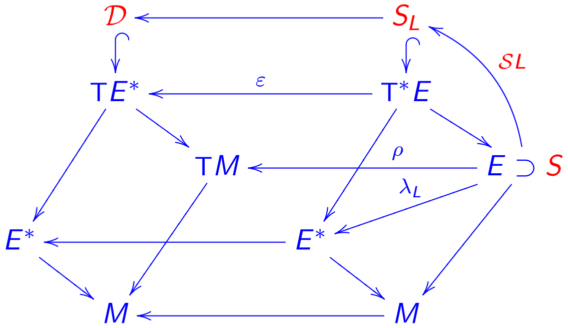

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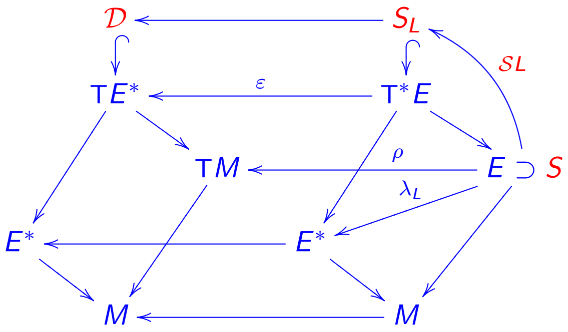

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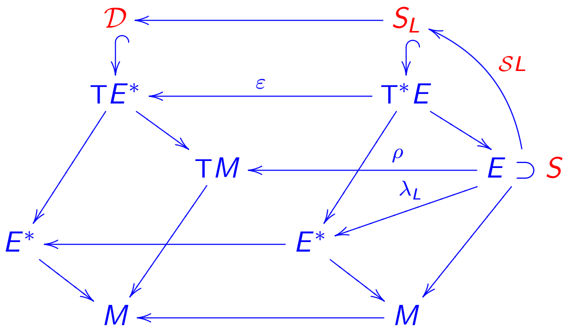

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Higher order Lagrangians

The mechanics with a higher order Lagrangian $L : \mathsf{T}^k Q \to \mathbb{R}$ is traditionally constructed as a vakonomic mechanics, thanks to the canonical embedding of of the higher tangent bundle $\mathsf{T}^k Q$ into the tangent bundle $\mathsf{T}\mathsf{T}^{k-1} Q$ as an affine subbundle of holonomic vectors.

Thus we work with the standard Tulczyjew triple for $\mathsf{T}M$, where $M = \mathsf{T}^{k-1} Q$, with the presence of vakonomic constraint $\mathsf{T}^k Q \subset \mathsf{T}\mathsf{T}^{k-1} Q$:

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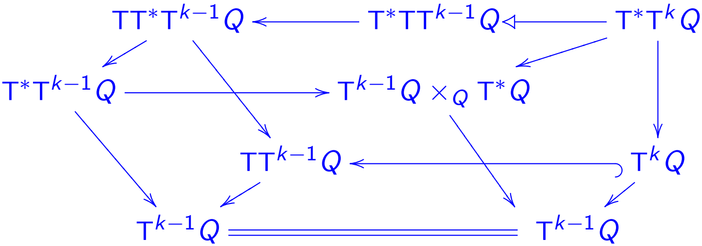

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The Lagrangian function $L = L(q, \dots, \overset{(k)}{q})$ generates the phase dynamics

$$\mathcal{D} = \left\{ (v, p, \dot{v}, \dot{p}) : \; \dot{v}_{i-1} = v_i, \;\; \dot{p}_i + p_{i-1} = \frac{\partial L}{\partial \overset{(i)}{q}}, \dot{p}_0 = \frac{\partial L}{\partial q}, p_{k-1} = \frac{\partial L}{\partial \overset{(k)}{q}} \right\}.$$

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Theorem (Bruce-Grabowska-Grabowski)

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Weighted Lie algebroids out of reductions

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider the subbundle $\mathsf{T}^k\mathcal{G}^{\underline{s}} \subset \mathsf{T}^k\mathcal{G}$ consisting of all higher order velocities tangent to source-leaves. The bundle

$$F_k = A^k(\mathcal{G}) := \mathsf{T}^k\mathcal{G}^{\underline{s}}\Big|_M ,$$

inherits graded bundle structure of degree k as a graded subbundle of $\mathsf{T}^k\mathcal{G}$. Of course, $A = A^1(\mathcal{G})$ can be identified with the Lie algebroid of $\mathcal{G}$.

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The linearisation of $A^k(\mathcal{G})$ is given as

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viewed as a vector bundle over $A^{k-1}(\mathcal{G})$ with respect to the obvious projection of part Z onto $A^{k-1}(\mathcal{G})$, where $\rho : A(\mathcal{G}) \to \mathsf{T}M$ is the standard anchor of the Lie algebroid and $\tau : A^{k-1}(\mathcal{G}) \to M$ is the obvious projection. Moreover, the above bundle is canonically a weighted Lie algebroid, a Lie algebroid prolongation in the sense of Popescu and Martínez.

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The linearisation of $A^k(\mathcal{G})$ is given as

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Lagrangian framework for graded bundles

A weighted Lie algebroid on $D(F_k)$ gives the Tulczyjew triple

Here, the diagram consists of relations, $\hat\varepsilon : \mathsf{T}^*F_k \longrightarrow\!\!\triangleright\, \mathsf{T}^*D(F_k) \to \mathsf{T}D^*(F_k)$, and $\mathrm{Mi}(F_k)$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^kM) = \mathsf{T}^{k-1}M \times_M \mathsf{T}^*M$. Forget the Hamiltonian side.

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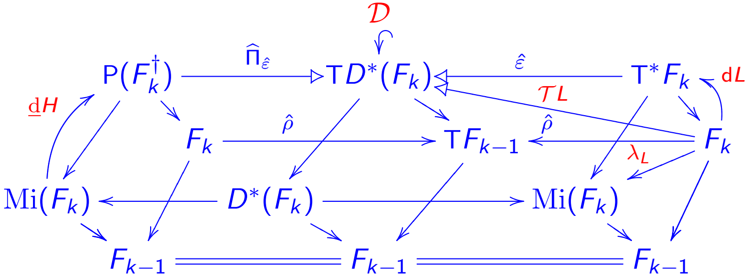

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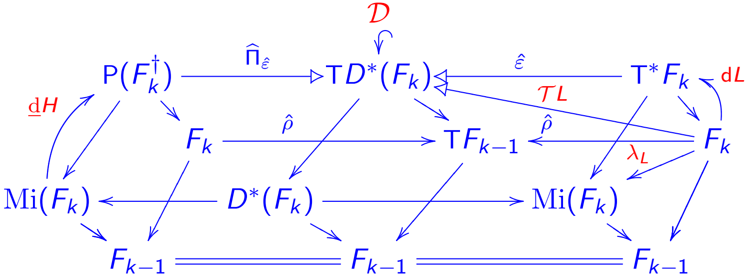

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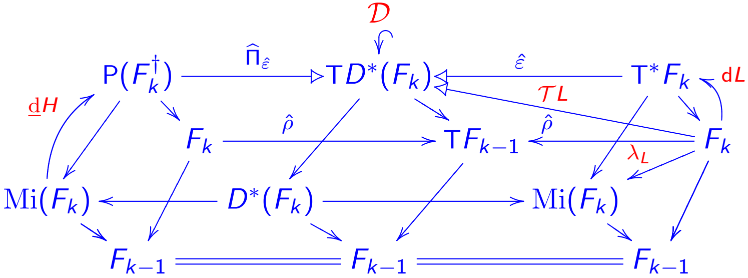

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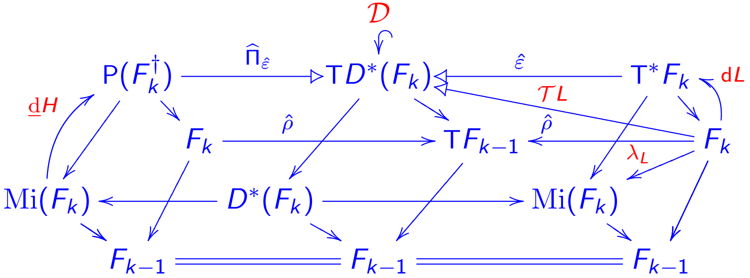

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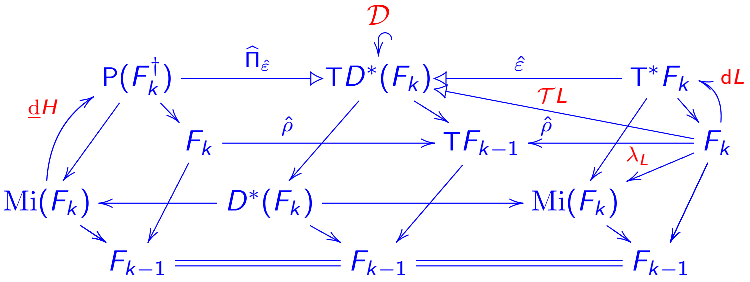

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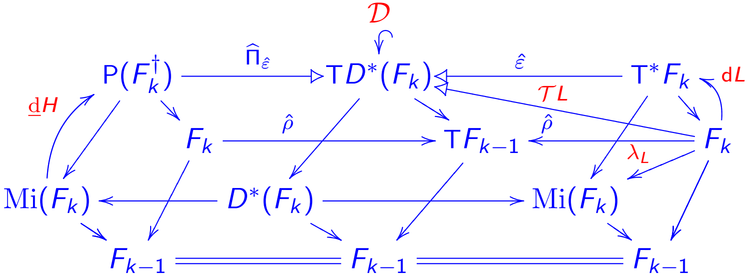

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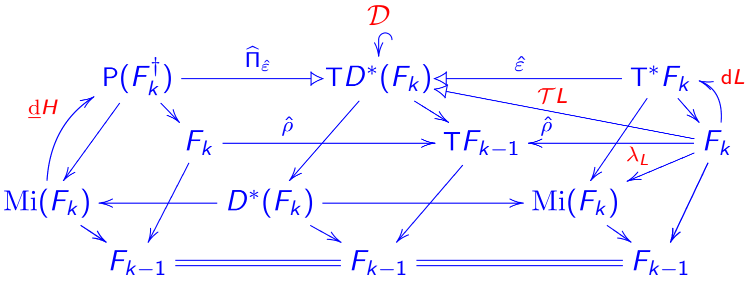

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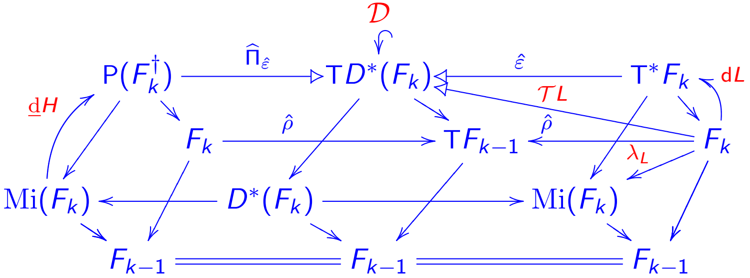

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Example

Let g be a Lie algebra and put $F_2 = g_2 = g[1] \times g[2]$, with coordinates (x^i, z^j) on g_2 and coordinates (x^i, y^j, z^k) on $D(g_2) = g[1] \times g[1] \times g[2]$.

The embedding $\iota : g_2 \hookrightarrow D(g_2)$ takes the form $\iota(x, z) = (x, x, z)$ and the vector bundle projection is $\tau(x, y, z) = x$.

The Lie algebroid structure $\varepsilon : \mathsf{T}^* D(g_2) \to \mathsf{T} D^*(g_2)$ reads

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$$\dot{x} = z\,,$$

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These equations are second order and induce the Euler-Lagrange equations on $\mathfrak{g}$ which are of order 3:

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For instance, the 'free' Lagrangian $L(x,z) = \frac{1}{2}\sum_i I_i(z^i)^2$ induces the equations on $\mathfrak{g}$ (c^k_{ij} are structure constants, no summation convention):

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Let us consider a general Lie groupoid $\mathcal{G}$ and a Lagrangian $L : A^k \to \mathbb{R}$ on $A^k = A^k(\mathcal{G})$. We will refer to such systems as a k-th order Lagrangian system on the Lie algebroid $A(\mathcal{G})$. The relevant diagram here is

Here, $D(A^k(\mathcal{G}))$ is the corresponding Lie algebroid prolongation, $\mathcal{D} = \varepsilon \circ r \circ \mathrm{d}L(A^k(\mathcal{G}))$, and λ_L is the Legendre relation.

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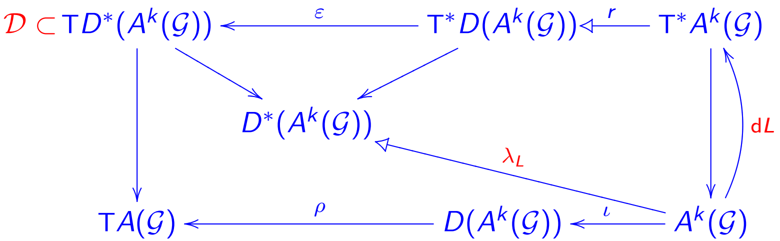

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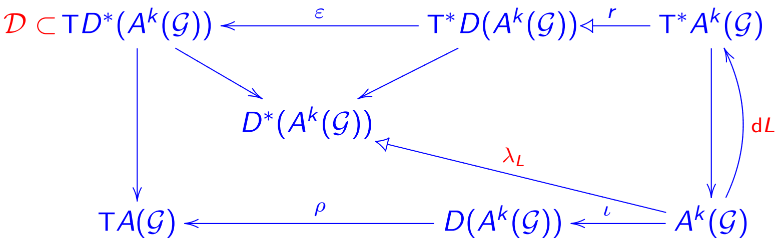

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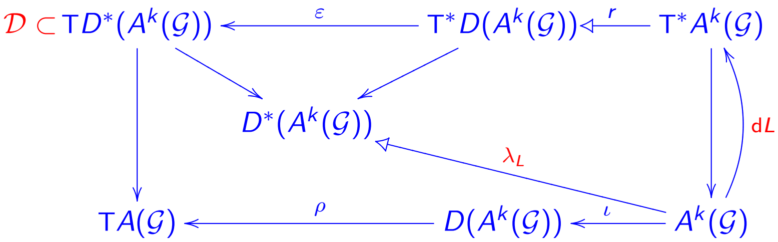

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The above higher order algebroid Euler-Lagrange equations are in complete agrement with the ones obtained by Jóźwikowski & Rotkiewicz, Colombo & de Diego, as well as Martínez. We clearly recover the standard higher Euler–Lagrange equations on $\mathsf{T}^k M$ as a particular example.

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Higher order Lagrangian mechanics on Lie algebroids

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The tip of a javelin

For instance, let L be the Lagrangian, governing the motion of the tip of a javelin defined on $\mathsf{T}^2\mathbb{R}^3$,

$$L(x,y,z)=\frac{1}{2}\left(\sum_{i=1}^{3}(y^i)^2-(z^i)^2\right).$$

We can understand $G=\mathbb{R}^3$ here as a commutative Lie group, and since L is G-invariant, we get immediately the reduction to the graded bundle $\mathbb{R}^3[1]\times\mathbb{R}^3[2]$. The Euler-Lagrange equations on $\mathsf{T}^2\mathbb{R}^3$,

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The Tulczyjew triple for strings

Using the canonical multisymplectic structure on $\wedge^2 \mathrm{T}^* M$, we get the following Tulczyjew triple for multivector bundles, consisting of double graded bundle morphisms:

The way of obtaining the implicit phase dynamics $\mathcal{D}$, as a submanifold of $\wedge^2 \mathrm{T} \wedge^2 \mathrm{T}^* M$, from a Lagrangian $L : \wedge^2 \mathrm{T} M \to \mathbb{R}$ (or from a Hamiltonian $H : \wedge^2 \mathrm{T}^* M \to \mathbb{R}$) is now standard: $\mathcal{D} = \mathcal{T}L(\wedge^2 \mathrm{T} M)$.

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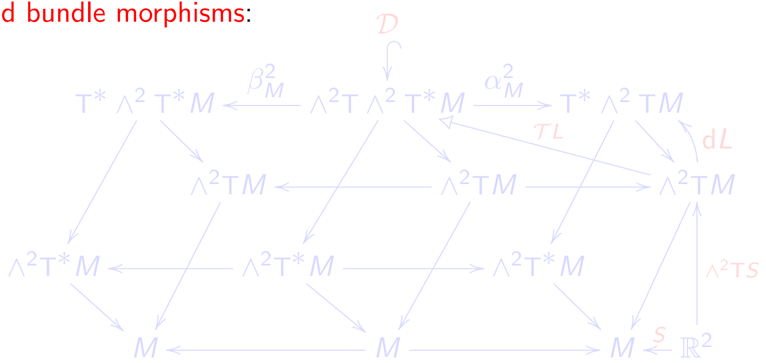

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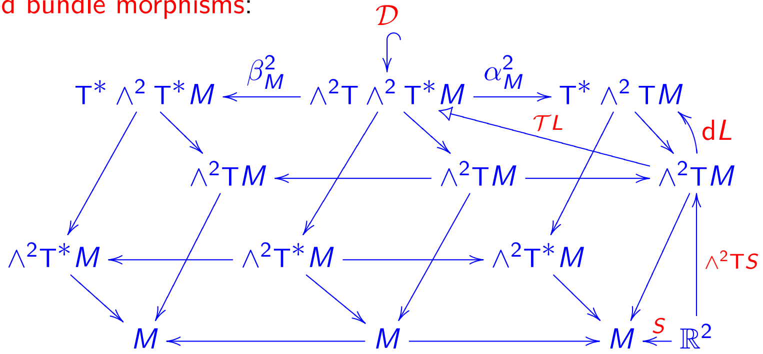

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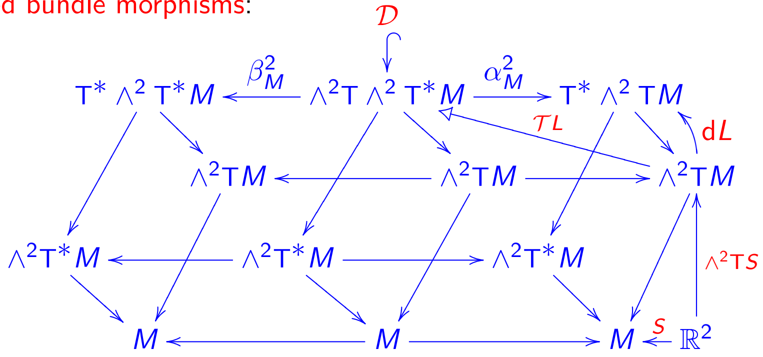

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The Euler-Lagrange equations

A surface $S : (t, s) \mapsto (x^\sigma(t, s))$ in M satisfies the Euler-Lagrange equations if the image by $\mathrm{d}L$ of its prolongation to $\wedge^2 \mathsf{T}M$,

$$(t, s) \mapsto \left(x^\sigma(t, s), \dot{x}^{\mu\nu} = \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t} \right),$$

is α^2_M-related to an admissible surface, i.e. the prolongation of a surface living in the phase space $\wedge^2 \mathsf{T}^* M$ to $\wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M$.
In coordinates, the Euler-Lagrange equations read

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Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the canonically induced 'free' Lagrangian on $\wedge^2 TM$ reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces being graphs $(x, y) \mapsto (x, y, z(x, y))$ provides the well-known equation for minimal surfaces, found already by Lagrange :

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In another form:

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In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the canonically induced 'free' Lagrangian on $\wedge^2 \mathsf{T}M$ reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces being graphs $(x, y) \mapsto (x, y, z(x, y))$ provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0\,.$$

In another form:

$$(1 + z_x^2)z_{yy} - 2z_x z_y z_{xy} + (1 + z_y^2)z_{xx} = 0\,.$$

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THANK YOU FOR YOUR ATTENTION!