Poisson Lie 2-algebroids and degenerate Courant algebroids

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Geometry of jets and fields, Bedlewo

Motivation: the correspondence between Courant algebroids and symplectic Lie 2-algebroids

- The standard Courant algebroid structure on $TM \oplus T^*M$, given a smooth manifold M, was discovered by Ted Courant in the late 80's.
- Later, in the late 90's, Liu, Weinstein and Xu defined general Courant algebroids and proved that the bicrossproduct of a Lie bialgebroid is a Courant algebroid.
- A few years later, Roytenberg, and independently Severa found that Courant algebroids were equivalent to symplectic Lie 2-algebroids, or in other words symplectic positively graded manifolds of degree 2 with a compatible homological vector field.

The classical definition of a Courant algebroid

A Courant algebroid over a manifold M is a vector bundle $\mathsf{E} \to M$ with a fibrewise nondegenerate symmetric bilinear form $\langle \cdot\,, \cdot \rangle$, a bracket $[\![\cdot\,, \cdot]\!]$ on the smooth sections $\Gamma(\mathsf{E})$, and an anchor $\rho \colon \mathsf{E} \to TM$, which satisfy the following conditions

1. $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,
2. $\rho(e_1)\langle e_2, e_3 \rangle = \langle [\![e_1, e_2]\!], e_3 \rangle + \langle e_2, [\![e_1, e_3]\!] \rangle$,
3. $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \rho^* \mathbf{d} \langle e_1, e_2 \rangle$

for all $e_1, e_2, e_3 \in \Gamma(\mathsf{E})$.

Back to theorem

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The aim of this talk is to show how the Courant algebroid bracket and the anchor of a Courant algebroid can be retrieved as a kind of "semidirect product" of the Poisson structure and the dual of the Lie structure.

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Outline

Positively graded manifolds

An **N-manifold** or $\mathbb{N}$**-graded manifold** $\mathcal{M}$ of degree n and dimension $(p; r_1, \ldots, r_n)$ is a smooth p-dimensional manifold M endowed with a sheaf $\mathrm{C}^\infty(\mathcal{M})$ of $\mathbb{N}$-graded commutative associative unital $\mathbb{R}$-algebras, whose degree 0 term is $\mathrm{C}^\infty(\mathrm{M})$ and which can locally be written

$$\mathrm{C}^\infty(\mathcal{M})_\mathrm{U} = \mathrm{C}^\infty(\mathrm{U})\left[\xi_1^1, \ldots, \xi_1^{r_1}, \xi_2^1, \ldots, \xi_2^{r_2}, \ldots, \xi_n^1, \ldots, \xi_n^{r_n}\right]$$

with $r_1 + \ldots + r_n$ graded commutative generators ξ_i^j of degree i for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, r_i\}$.

Positively graded manifolds are noncanonically split

For any $\mathbb{N}$-graded manifold $\mathcal{M}$ of degree n and dimension $(p; r_1, \ldots, r_n)$, there exist smooth vector bundles $E_{-1}, E_{-2}, \ldots, E_{-n}$ of ranks $r_1, \ldots, r_n$ over M such that $\mathcal{M}$ is isomorphic (in a noncanonical manner) to the *split* $[n]$-manifold $E_{-1}[-1] \oplus \ldots \oplus E_{-n}[-n]$, which has local basis sections of ${E_{-i}}^*$ as local generators of degree i, for $i = 1, \ldots, n$.

Degree 1 and degree 2 cases

In particular, if $\mathscr{M}$ is an $\mathbb{N}$-graded manifold of base M and of degree 1, then $C^\infty(\mathscr{M})$ is (canonically) isomorphic to $\Gamma(\bigwedge^\bullet E^*)$ for a vector bundle E over M.

If $\mathscr{M}$ is an $\mathbb{N}$-graded manifold of base M and of degree 2, then $C^\infty(\mathscr{M})$ is (noncanonically) isomorphic to $\Gamma(\bigwedge^\bullet E_{-1}^* \otimes S^\bullet E_{-2}^*)$ for two vector bundles E_{-1} and E_{-2} over M.

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Graded vector fields

Let $\mathscr{M}$ be an $[n]$-manifold. A **vector field** of degree j on $\mathscr{M}$ is a graded derivation ϕ of $C^\infty(\mathscr{M})$ such that

$$|\phi(\xi)| = j + |\xi|$$

for a homogeneous element $\xi \in C^\infty(\mathscr{M})$.

The Lie bracket on graded vector fields, defined by $[\phi, \psi] = \phi\psi - (-1)^{|\phi||\psi|}\psi\phi$ is graded skew-symmetric and satisfies graded Leibniz and graded Jacobi identities.

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Homological vector fields

A *homological vector field* $\mathcal{Q}$ on a graded manifold $\mathcal{M}$ is a graded vector field of degree 1 that commutes with itself

$$[\mathcal{Q}, \mathcal{Q}] = 2\mathcal{Q} \circ \mathcal{Q} = 0.$$

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Example: Lie 1-algebroid

Take an $\mathbb{N}$-graded manifold of degree 1, i.e. $C^\infty(\mathscr{M}) = \Gamma(\bigwedge^\bullet E^*)$ for a vector bundle E over M and take a trivialising chart $U \subseteq M$ for E. Any homological vector field $\mathcal{Q}$ on $\mathscr{M}$ can locally be written as

$$\mathcal{Q}_U = \sum_{ij} \rho(e_j)(x_i)\varepsilon_i \partial_{x_j} - \sum_{ijk} \langle [e_i, e_j], \varepsilon_k \rangle \varepsilon_i \varepsilon_j \partial_{\varepsilon_k},$$

defining locally a Lie algebroid structure on $E|_U$. This structure is in fact global, and Lie 1-algebroids are equivalent to Lie algebroids. (This is due to Arkady Vaintrob.)

Example: Lie 1-algebroid

Let us describe yet another way to get the Lie algebroid structure from the homological vector field $\mathcal{Q}$.

$e \in \Gamma(E)$ is identified with the graded vector field e of degree -1 that sends $\varepsilon \in \Gamma(E^*)$ to $\langle e, \varepsilon \rangle$ and $f \in C^\infty(M)$ to 0.

Then $[\mathcal{Q}, e](f) = \rho(e)(f)$ and $[[\mathcal{Q}, e], e'] = [e, e']$. The Lie algebroid structure is hence *derived* from the homological vector field.

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Example: Lie 2-algebroid

On a split [2]-manifold $Q[-1] \oplus B^*[-2]$, a homological vector field can be written

$$\begin{aligned}\mathcal{Q} = &\sum_{i,j} \rho_Q(q_i)(x_j)\tau_i\partial_{x_j} - \sum_{i<j}\sum_k \langle [\![q_i, q_j]\!], \tau_k\rangle \tau_i\tau_j\partial_{\tau_k} \\ &+ \sum_{r,k}\langle \partial_B^*\beta_r, \tau_k\rangle b_r\partial_{\tau_k} - \sum_{i<j<k}\sum_l \omega(q_i, q_j, q_k)(b_l)\tau_i\tau_j\tau_k\partial_{b_l} \\ &- \sum_{ijl}\langle \nabla^*_{q_i}\beta_j, b_l\rangle \tau_i b_j \partial_{b_l},\end{aligned}$$

where $\partial_B\colon Q^* \to B$, $\rho_Q\colon Q \to TM$, $[\![\cdot\,,\cdot]\!]\colon \Gamma(Q)\times\Gamma(Q)\to\Gamma(Q)$, $\nabla\colon \Gamma(Q)\times\Gamma(B)\to\Gamma(B)$ and $\omega\in\Omega^3(Q, B^*)$ are the components of a split Lie 2-algebroid.

Split Lie 2-algebroids

A split Lie 2-algebroid $Q \oplus B^* \to M$ is a pair of an anchored vector bundle $(Q \to M, \rho_Q)$ and a vector bundle $B \to M$, together with

1. a vector bundle map $\partial_B^* \colon B^* \to Q$,
2. a skew-symmetric dull bracket $[\![\cdot\,,\cdot]\!] \colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$,
3. a linear connection $\nabla^* \colon \Gamma(Q) \times \Gamma(B^*) \to \Gamma(B^*)$ and
4. a vector valued 3-form $\omega \in \Omega^3(Q, B^*)$,

such that

(i) $\nabla^*_{\partial_B^*(\beta_1)}\beta_2 + \nabla^*_{\partial_B^*(\beta_2)}\beta_1 = 0$,

(ii) $[\![q, \partial_B^*(\beta)]\!] = \partial_B^*(\nabla^*_q \beta)$,

(iii) $\mathrm{Jac}_{[\![\cdot,\cdot]\!]} = -\partial_B^* \circ \omega \in \Omega^3(Q, Q)$,

(iv) $R_{\nabla^*}(q_1, q_2)\beta = \omega(q_1, q_2, \partial_B^*(\beta))$, and

(v) $\mathbf{d}_{\nabla^*}\omega = 0$

for all $\beta, \beta_1, \beta_2 \in \Gamma(B^*)$ and $q, q_1, q_2 \in \Gamma(Q)$.

Dull bracket

Let (Q, ρ_Q) be an anchored vector bundle over M. A *skew-symmetric dull bracket* $\Gamma(Q)$ is a skew-symmetric $\mathbb{R}$-bilinear map $[\![\cdot\,,\cdot]\!] : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$ such that

$$\rho_Q[\![q_1, q_2]\!] = [\rho_Q(q_1), \rho_Q(q_2)]$$

and (the Leibniz identity)

$$[\![q_1, f \cdot q_2]\!] = f \cdot [\![q_1, q_2]\!] + \rho_Q(q_1)(f) \cdot q_2$$

for all $q_1, q_2 \in \Gamma(Q)$ and $f \in C^\infty(M)$.

Dual of a skew-symmetric dull bracket

Dualize the bracket in the sense of derivations:

$$\Delta\colon \Gamma(Q)\times\Gamma(Q^*)\to\Gamma(Q^*),$$

$$\langle \Delta_q\tau, q'\rangle = \rho_Q(q)\langle\tau, q'\rangle - \langle [\![q, q']\!], \tau\rangle$$

for all $q, q' \in \Gamma(Q)$ and $\tau\in\Gamma(Q^*)$.

The dual object Δ is called a *skew-symmetric Dorfman* Q-*connection on* Q^*.

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Properties...

$\Delta\colon \Gamma(Q)\times\Gamma(Q^*)\to\Gamma(Q^*)$ has the following properties

1. $\Delta_q(f\tau) = f\cdot\Delta_q\tau + \rho_Q(q)(f)\cdot\tau$,
2. $\Delta_{fq}\tau = f\cdot\Delta_q\tau + \langle q,\tau\rangle\cdot\rho_Q^*\mathbf{d}f$
3. $\langle\Delta_{q_1}\tau, q_2\rangle + \langle\Delta_{q_2}\tau, q_1\rangle = \rho_Q(q_1)\langle\tau, q_2\rangle + \rho_Q(q_2)\langle\tau, q_1\rangle$
4. $\Delta_q(\rho_Q^*\mathbf{d}f) = \rho_Q^*\mathbf{d}(\rho_Q(q)(f))$.

for all $f\in C^\infty(M)$, $q, q_1, q_2\in\Gamma(Q)$ and $\tau\in\Gamma(Q^*)$.

Curvature of a Dorfman connection

The Jacobiator in Leibniz form

$$\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(q_1, q_2, q_3) = [\![\,[\![q_1, q_2]\!], q_3]\!] + [\![q_2, [\![q_1, q_3]\!]\,]\!] - [\![q_1, [\![q_2, q_3]\!]\,]\!]$$

is equivalent to the *curvature* of the Dorfman connection:

$$\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(q_1, q_2, q_3) = \mathrm{R}_\Delta(q_1, q_2)^* q_3$$

for all q_1, q_2, q_3.

Dorfman connections

Lie algebroid brackets are dual to flat, skew-symmetric Dorfman connections, which we call *Dorfman representations*.

Split Lie 2-algebroid brackets are dual to *Dorfman 2-representations.*

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Dorfman 2-representations

Let $(Q \to M, \rho_Q)$ be an anchored vector bundle and B a vector bundle over M. A (Q, ρ_Q)-Dorfman 2-representation on $Q^* \oplus B$ is

1. a vector bundle morphism $\partial_B : Q^* \to B$,
2. a linear connection $\nabla : \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$ such that $\nabla^*_{\partial_B^* \beta_1} \beta_2 + \nabla^*_{\partial_B^* \beta_2} \beta_1 = 0$,
3. a skew-symmetric Dorfman connection $\Delta : \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ such that $\partial_B \circ \Delta_q = \nabla_q \circ \partial_B$ and
4. a vector-valued 2-form $R \in \Omega^2(Q, \mathrm{Hom}(B, Q^*))$ such that
   1. $\partial_B \circ R(q_1, q_2) = R_\nabla(q_1, q_2)$ and $R(q_1, q_2) \circ \partial_B = R_\Delta(q_1, q_2)$,
   2. $R(q_1, q_2)^* q_3 = -R(q_1, q_3)^* q_2$ and
   3. $\mathbf{d}_\Diamond R(q_1, q_2, q_3) = \nabla^*_{\cdot} \left(R(q_1, q_2)^* q_3 \right)$

for all $\xi_1, \xi_2 \in \Gamma(B^*)$ and $q, q_1, q_2, q_3 \in \Gamma(Q)$ and $f \in C^\infty(M)$.

2-term representations up to homotopy

Let A be a Lie algebroid and $E_0 \oplus E_1$ a 2-term graded vector bundle. A 2-representation of A on $E_0 \oplus E_1$ is

1. a map $\partial\colon E_0 \to E_1$,
2. two A-connections, ∇^0 and ∇^1 on E_0 and E_1, respectively, such that $\partial \circ \nabla^0 = \nabla^1 \circ \partial$,
3. an element $R \in \Omega^2(A, \mathrm{Hom}(E_1, E_0))$ such that $R_{\nabla^0} = R \circ \partial$, $R_{\nabla^1} = \partial \circ R$ and $\mathbf{d}_{\nabla^{\mathrm{Hom}}} R = 0$, where ∇^{Hom} is the connection induced on $\mathrm{Hom}(E_1, E_0)$ by ∇^0 and ∇^1.

Example

Let $(\mathsf{E} \to M, \rho_{\mathsf{E}}, [\![\cdot\,,\cdot]\!]_{\mathsf{E}}, \langle\cdot\,,\cdot\rangle)$ be a Courant algebroid. Choose a linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ that preserves the pairing.

1. $\partial_{TM} = \rho_{\mathsf{E}}\colon \mathsf{E} \to TM$.
2. $\Delta\colon \Gamma(\mathsf{E}) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$, $\Delta_e e' = [\![e, e']\!]_{\mathsf{E}} + \nabla_{\rho_{\mathsf{E}}(e')} e$ is a Dorfman connection.
3. The map $\nabla^{bas}\colon \Gamma(\mathsf{E}) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\nabla^{bas}_e X = [\rho_{\mathsf{E}}(e), X] + \rho_{\mathsf{E}}(\nabla_X e)$ is a linear connection.
4. The *basic curvature* $R^{bas}_\Delta \in \Omega^2(\mathsf{E}, \mathrm{Hom}(TM, \mathsf{E}))$ is

$$\begin{aligned} R^{bas}_\Delta(e_1, e_2)X = &-\nabla_X [\![e_1, e_2]\!]_{\mathsf{E}} + [\![\nabla_X e_1, e_2]\!]_{\mathsf{E}} + [\![e_1, \nabla_X e_2]\!]_{\mathsf{E}} \\ &+ \nabla_{\nabla^{bas}_{e_2} X} e_1 - \nabla_{\nabla^{bas}_{e_1} X} e_2 - \beta^{-1}\langle \nabla_{\nabla^{bas}_{\cdot} X} e_1, e_2\rangle \end{aligned}$$

for all $e_1, e_2 \in \Gamma(\mathsf{E})$ and $X \in \mathfrak{X}(M)$.

Degree −2 Poisson bracket on a [2]-manifold

A Poisson [2]-manifold is a [2]-manifold endowed with a Poisson bracket of degree −2.

A Poisson bracket of degree −2 is graded skew-symmetric and satisfies $|\{\xi, \eta\}| = |\xi| + |\eta| - 2$ and graded Leibniz and Jacobi identities.

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Split Poisson [2]-manifolds are equivalent to self-dual 2-term representations up to homotopy

Theorem (J.L. 2015)

A split Poisson $[2]$*-manifold* $(\mathcal{M} = \mathrm{Q}[-1] \oplus \mathrm{B}^*[-2], \{\cdot\,,\cdot\})$ *defines as follows a Lie algebroid structure on* B*, a VB-morphism* $\partial_{\mathrm{Q}} \colon \mathrm{Q}^* \to \mathrm{Q}$ *and a 2-term representation up to homotopy* $(\nabla, \nabla^*, \mathrm{R})$ *of* B *on* $\partial_{\mathrm{Q}} \colon \mathrm{Q}^* \to \mathrm{Q}$*:*

1. $\{f_1, f_2\} = \{f, \tau\} = 0$,
2. $\{\tau_1, \tau_2\} = \langle \tau_2, \partial_{\mathrm{Q}}(\tau_1) \rangle$,
3. $\{b, f\} = \rho_{\mathrm{B}}(b)(f)$ *with an anchor* $\rho_{\mathrm{B}} \colon \mathrm{B} \to TM$,
4. $\{b, \tau\} = \nabla_b \tau$ *with a linear* B*-connection* ∇ *on* Q^*,
5. $\{b_1, b_2\} = [b_1, b_2] - \mathrm{R}(b_1, b_2)$ *with* $[\cdot, \cdot]$ *a Lie algebroid bracket on* B *and* $\mathrm{R} \in \Omega^2(\mathrm{B}, \mathrm{Hom}(\mathrm{Q}; \mathrm{Q}^*))$.

The 2-term representation up to homotopy is self-dual*:*

$$\partial_{\mathrm{Q}}^* = \partial_{\mathrm{Q}} \quad \textit{and} \quad \mathrm{R}^* = -\mathrm{R}.$$

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Conversely, any self-dual 2-representation defines a split Poisson [2]-manifold.

Example

Let $(\mathsf{E} \to M, \rho_{\mathsf{E}}, [\![\cdot\,,\cdot]\!]_{\mathsf{E}}, \langle\cdot\,,\cdot\rangle)$ be a Courant algebroid. Choose a linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ that preserves the pairing.

The 2-representation $(\mathrm{Id}_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{E}, \nabla, \nabla, R_\nabla)$ is then self-dual.

We get so for each metric connection ∇ a split Poisson manifold $(\mathsf{E}[-1] \oplus T^*M[-2], \{\cdot\,,\cdot\}_\nabla)$.

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Poisson Lie 2-algebroids

Let $(\mathscr{M}, \{\cdot\,,\cdot\})$ be a Poisson [2]-manifold. Assume that $\mathscr{M}$ has in addition a Lie 2-algebroid structure $\mathcal{Q}$. Then $(\mathscr{M}, \mathcal{Q}, \{\cdot\,,\cdot\})$ is a **Poisson Lie 2-algebroid** if the homological vector field preserves the Poisson structure:

$$\mathcal{Q}\{\xi_1, \xi_2\} = \{\mathcal{Q}(\xi_1), \xi_2\} + (-1)^{|\xi_1|}\{\xi_1, \mathcal{Q}(\xi_2)\}$$

for all $\xi_1, \xi_2 \in C^\infty(\mathscr{M})$.

A characterisation of split Poisson Lie 2-algebroids

Theorem (J.L. 2015)

$(\mathscr{M} = \mathrm{Q}[-1] \oplus \mathrm{B}^*[-2], \mathcal{Q}, \{\cdot\,,\cdot\})$ is a Poisson Lie 2-algebroid if and only if

1. $\partial_{\mathrm{Q}}(\Delta_q \tau) = \nabla_{\partial_{\mathrm{B}}\tau} q + [\![q, \partial_{\mathrm{Q}}\tau]\!] + \partial_{\mathrm{B}}^*\langle \tau, \nabla_{\cdot} q\rangle$,
2. $\partial_{\mathrm{B}}(\nabla_b \tau) = [b, \partial_{\mathrm{B}}\tau] + \nabla_{\partial_{\mathrm{Q}}\tau} b$,
3. $\partial_{\mathrm{B}} R(b_1, b_2) q = -\nabla_q [b_1, b_2] + [\nabla_q b_1, b_2] + [b_1, \nabla_q b_2] + \nabla_{\nabla_{b_2} q} b_1 - \nabla_{\nabla_{b_1} q} b_2$,
4. $\partial_{\mathrm{Q}} R(q_1, q_2) b = -\nabla_b [\![q_1, q_2]\!] + [\![q_1, \nabla_b q_2]\!] + [\![\nabla_b q_1, q_2]\!] + \nabla_{\nabla_{q_2} b} q_1 - \nabla_{\nabla_{q_1} b} q_2 + \partial_{\mathrm{B}}^*\langle R(\cdot, b) q_1, q_2\rangle$, *and*
5. $\mathbf{d}_{\nabla^{\mathrm{B}}} \omega_R = \mathbf{d}_{\nabla^{\mathrm{Q}}} \omega_{\mathrm{B}} \in \Omega^2(\mathrm{B}, \bigwedge^3 \mathrm{Q}^*) = \Omega^3(\mathrm{Q}, \bigwedge^2 \mathrm{B}^*)$.

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The Dorfman 2-representation and the self-dual 2-representation found earlier are compatible and define a Poisson Lie 2-algebroid structure on $\mathsf{E}[-1] \oplus T^*M[-2]$.

Example

Let $(\mathsf{E} \to M, \rho_{\mathsf{E}}, [\![\cdot\,,\cdot]\!]_{\mathsf{E}}, \langle\cdot\,,\cdot\rangle)$ be a Courant algebroid. Choose a linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ that preserves the pairing.

The Dorfman 2-representation and the self-dual 2-representation found earlier are compatible and define a Poisson Lie 2-algebroid structure on $\mathsf{E}[-1] \oplus T^*M[-2]$.

Conversely, how do we recover the Courant algebroid structure from the Dorfman 2-representation and the self-dual 2-representation?

The degenerate Courant algebroid on the core

Theorem (J.L. 2015)

Let $(\mathscr{M} = Q[-1] \oplus B^*[-2], \mathcal{Q}, \{\cdot\,,\cdot\})$ *be a split Poisson Lie 2-algebroid. Then* Q^* *inherits the structure of a degenerate Courant algebroid over* M*, with the anchor* $\rho_Q \partial_Q = \rho_B \partial_B$*, the map* $\mathscr{D} = \rho_Q^* \mathbf{d} \colon C^\infty(M) \to \Gamma(Q^*)$*, the pairing defined by* $\langle \tau_1, \tau_2 \rangle_{Q^*} = \langle \tau_1, \partial_Q \tau_2 \rangle$ *and the bracket defined by* $[\![\tau_1, \tau_2]\!]_{Q^*} = \Delta_{\partial_Q \tau_1} \tau_2 - \nabla_{\partial_B \tau_2} \tau_1$ *for all* $\tau_1, \tau_2 \in \Gamma(Q^*)$*.*

Back to the definition

Given a Poisson Lie 2-algebroid, this structure does not depend on the choice of a splitting, and the map $\partial_B \colon Q^* \to B$ *preserves the brackets and the anchors.*

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Example

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Then

$$\Delta_{e_1} e_2 - \nabla_{\rho(e_2)} e_1 = [\![e_1, e_2]\!]_{\mathsf{E}}.$$

Symplectic Lie 2-algebroids and Courant algebroids

Theorem (J.L. 2015)

Let $\mathscr{M}$ be a symplectic Lie 2-algebroid over a base manifold M. *Choose any splitting* $\mathscr{M} \simeq Q[-1] \oplus T^*M[-2]$ *of the underlying symplectic* [2]*-manifold.*

Then

1. $Q \simeq Q^*$ *via* ∂_Q *and* $\langle \tau_1, \partial_Q \tau_2 \rangle$ *is nondegenerate.*
2. *The map* $\rho_Q \circ \partial_Q = \partial_{TM}$ *defines an anchor on* Q^*.
3. *The bracket* $[\![\cdot\,,\cdot]\!]_{Q^*}$ *defined on* $\Gamma(Q^*)$ *by* $[\![\tau_1, \tau_2]\!]_{Q^*} = \Delta_{\partial_Q \tau_1} \tau_2 - \{\partial_{TM} \tau_2, \tau_1\}$ *does not depend on the choice of the splitting.*

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Quick review of double vector bundles

A double vector bundle is a commutative square

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_Q}\downarrow & & \downarrow{\scriptstyle q_B} \\ Q & \xrightarrow[q_Q]{} & M \end{array}$$

of vector bundles such that the structure maps of the vertical bundles define morphisms of the horizontal bundles.

Take a triple Q, B, C of vector bundles over a smooth manifold M. Then the fibre-product $Q \times_M B \times_M C$ has a vector bundle structure over Q given by $(q, b, c) +_Q (q, b', c') = (q, b + b', c + c')$, and similarly a vector bundle structure over B.

Quick review of double vector bundles

Consider such a decomposed double vector bundle

$$\mathrm{D} = \mathrm{Q} \times_{\mathrm{M}} \mathrm{B} \times_{\mathrm{M}} \mathrm{C}.$$

For each $b \in \Gamma(\mathrm{B})$, we have a *linear section* $\tilde{b} \in \Gamma_{\mathrm{Q}}(\mathrm{D})$,

$$\tilde{b}(q_m) = (q_m, b(m), 0^{\mathrm{C}}_m)$$

and for each $c \in \Gamma(\mathrm{C})$, we have a *core section* $c^\dagger \in \Gamma_{\mathrm{Q}}(\mathrm{D})$,

$$c^\dagger(q_m) = (q_m, 0^{\mathrm{B}}_m, c(m)).$$

Theorem (J.L.2015)

The category of positively graded manifolds of degree 2 is equivalent to the category of metric double vector bundles.

Splittings of a positively graded manifold of degree 2 correspond to maximally isotropic decompositions of the corresponding metric double vector bundle.

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & B \\ \downarrow & & \downarrow \\ Q & \longrightarrow & M \end{array}$$

Theorem (J.L.2015)

The category of positively graded Poisson manifolds of degree 2 is equivalent to the category of self-dual VB-algebroids.

Splittings of a positively graded Poisson manifold of degree 2 correspond to self-dual 2-representations, which correspond to maximally isotropic decompositions of the corresponding self-dual VB-algebroid.

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & B \\ \downarrow & & \downarrow \\ Q & \longrightarrow & M \end{array}$$

(The equivalence of decomposed VB-algebroids with 2-representations is due to Gracia-Saz and Mehta.)

Theorem (Li-Bland 2012, J.L.2015)

The category of Lie 2-algebroids is equivalent to the category of VB-Courant algebroids.

Splittings of a Lie 2-algebroid correspond to Dorfman 2-representations, which correspond to maximally isotropic decompositions of the corresponding VB-Courant algebroid.

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ Q & \longrightarrow & M \end{array}$$

Theorem (Li-Bland 2012, J.L.2015)

The category of Poisson Lie 2-algebroids is equivalent to the category of LA-Courant algebroids.

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & B \\ \downarrow & & \downarrow \\ Q & \longrightarrow & M \end{array}$$

Symplectic Lie 2-algebroids correspond to tangent doubles of Courant algebroids.

$$\begin{array}{ccc} TE & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array}$$

One further application

Consider a double Lie algebroid

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & M \end{array}$$

The direct sum

$$\begin{array}{ccc} D^*_A \oplus D^*_B & \longrightarrow & C^* \\ \downarrow & & \downarrow \\ A \oplus B & \longrightarrow & M \end{array}$$

is a VB-Courant algebroid.

One further application

A decomposition of D gives rise to two 2-representations which form a *matched pair* (Gracia-Saz, J.L., Mackenzie, Mehta).

A decomposition of D naturally induces a maximally isotropic decomposition of $D_A^* \oplus D_B^*$.

The corresponding split Lie 2-algebroid is the bicrossproduct $(A \oplus B)[-1] \oplus C^*[-2]$ of a matched pair of 2-representations!

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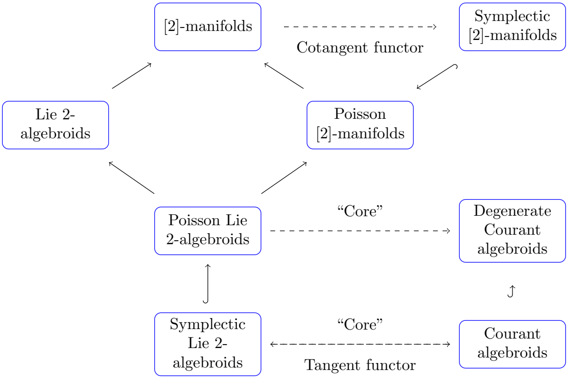
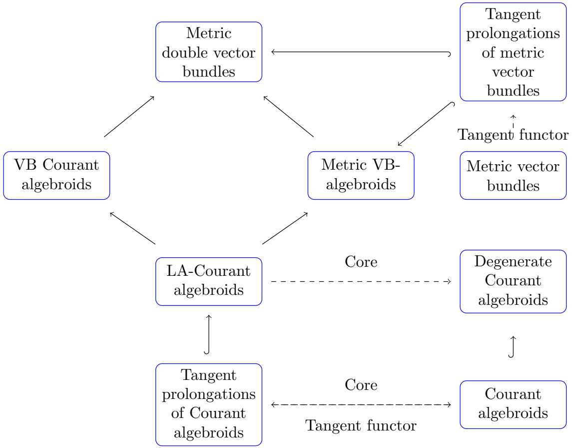
Diagrammatic table of the supergeometric objects in this talk.



Diagrammatic table of the (classical, double) geometric objects in this talk.



Thank you for your attention!

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