Physics of a qubit from geometric quantization

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Geometry of Jets and Fields
Będlewo, May 11.

Quantum mechanics

- Hilbert space - $L^2(Q)$, where $Q = \mathbb{R}^n$ is the classical configuration space.
- Observables - self-adjoint operators.
- Evolution of states - governed by a Hamiltonian operator:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}),$$

  where:
  $\hat{p}_k\psi(x) := \frac{\hbar}{i}\frac{d}{dx^k}\psi(x)$ – momentum operator,
  $\hat{x}^k\psi(x) := x^k\psi(x)$ – position operator.
- Momentum representation: $\tilde{\psi}(p)$ – Fourier transform of $\psi(x)$.
- Probability densities: $|\psi(x)|^2$ and $|\tilde{\psi}(p)|^2$.

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- Is the linear (affine) structure of the configuration space Q necessary in quantum mechanics?
- Is the Lebesque measure $\mathrm{d}^n x$ carried by the linear structure of Q necessary for the definition of the appropriate Hilbert space structure:

$$(\varphi|\psi) := \int_Q \overline{\varphi}\, \psi\, \mathrm{d}^n x \ .$$

- Phase space: $\mathcal{P} = T^*Q = \mathbb{R}^{2n}$; symplectic form $\omega = \mathrm{d}p_i \wedge \mathrm{d}x^i$
- Observables - functions on $\mathcal{P}$.
- Evolution - governed by the Hamiltonian vector field X_H, uniquely assigned to any observable H according to:

$$\omega(X_H, \cdot) = -\mathrm{d}H \ .$$

- Example:

$$H = \frac{p^2}{2m} + V(x) \ .$$

Its Hamiltonian vector field:

$$X_H = g^{ij}\frac{1}{m}p_j\partial_{x^i} - \frac{\partial V}{\partial x^i}\partial_{p_i} \ .$$

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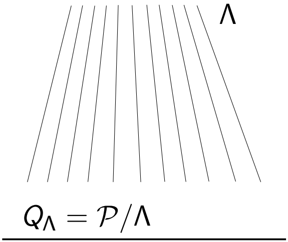

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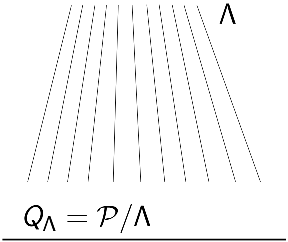

- Geometrically: quantum states represented by wave functions defined on a generalized configuration space $\mathcal{Q}_\Lambda = \mathcal{P}/\Lambda$

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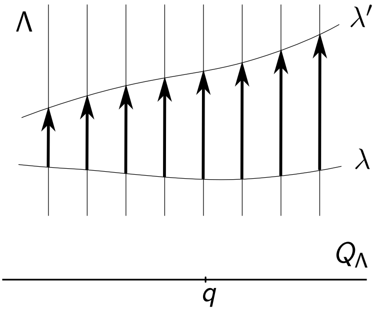
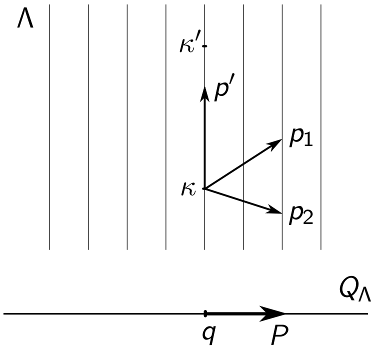


Proof: For $q \in Q_\Lambda$ and $\kappa \in q$ there is a canonical isomorphism:

$$T_\kappa q \simeq T_q^* Q_\Lambda$$

where $\langle P|p'\rangle := \Omega(p_1, p') = \Omega(p_2, p')$.



Each fiber q is an affine space.

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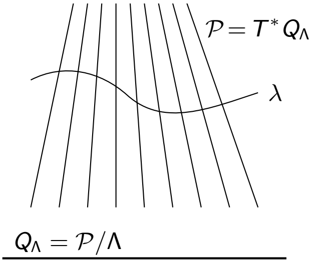

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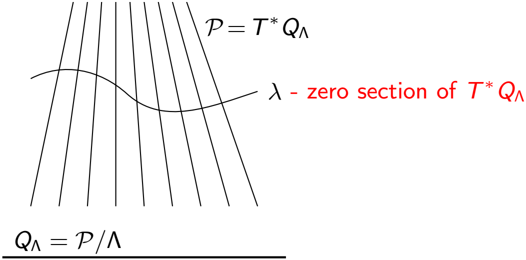

Observable $S_{\lambda',\lambda}$ on $\mathcal{P}$ generates a group of symplectomorphisms:

$$(q, p) \to \big(q, p + t(\lambda' - \lambda)\big)$$

Hilbert space of half-densities

There is no need for a "privileged" measure on the configuration space Q_Λ if we treat wave functions as half-densities and not just scalar functions:

$$(\phi|\psi) := \int_Q \overline{\phi}\,\psi\,\mathrm{d}^n x = \int_Q \overline{\left(\phi\sqrt{\mathrm{d}^n x}\right)}\left(\psi\sqrt{\mathrm{d}^n x}\right) \ .$$

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$L^2(Q_\Lambda)$ – Hilbert space of square-integrable half-forms with scalar product:

$$(\Phi|\Psi) = \int_Q \overline{\phi}\Psi$$

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Naive quantization rule: $\hat{p}_k \psi(x) := \frac{\hbar}{i} \frac{d}{dx^k} \psi(x)$ must be replaced by:

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Quantization of momenta

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Automatically self-adjoint if X-complete!

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If we want to have a polarization-independent description of a quantum state, we must define a quantum counterpart of this change, i.e. a mapping from classical to quantum observables:

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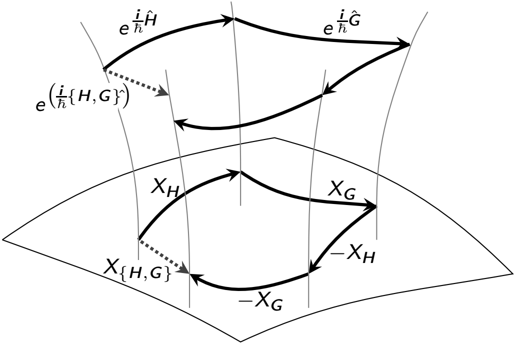
- Linearity ???

The result of the "change of polarization" procedure should not depend upon the way we change it!!!



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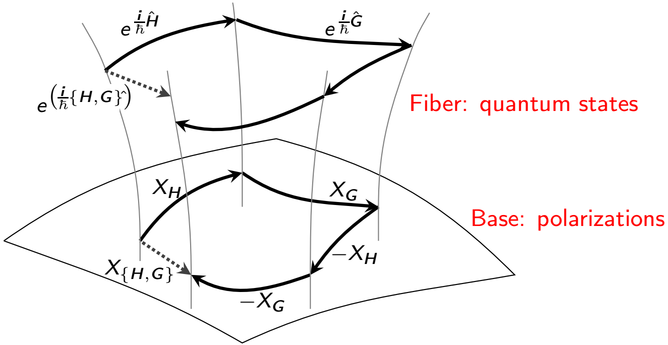

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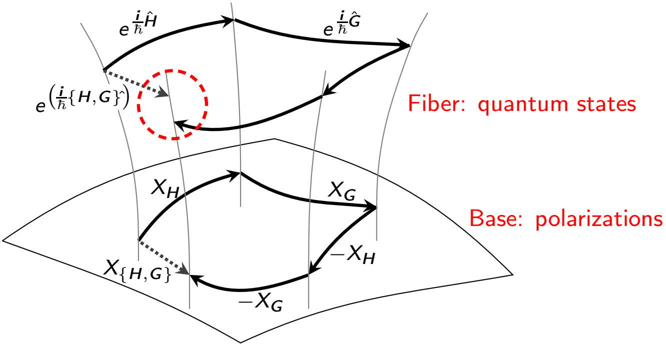

Path-independence requires: $[\hat{H}, \hat{G}] - \{H, G\}^{\wedge} = c \cdot \mathbb{I}$.
Modulo "$c \cdot \mathbb{I}$" because only projective representations considered: global phase never controlled!

Linear Symplectic group

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Small miracle: If $\mathcal{P}$ is a **linear** symplectic space than algebra $\mathcal{F}^2(\mathcal{P})$ of "at most quadratic" observables generates the linear symplectyic group $Sp(\mathcal{P})$ which is uniquely, and **exactly** quantized.

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Remainder: this is a **projective** representation of $Sp(\mathcal{P})$. There is no *unitary* representation, unless we pass to the universal covering: the **metaplectic** group $Mp(\mathcal{P})$.

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Theorem 1: Observables which are linear with respect to momenta in any of the above representations span the space $\mathcal{F}(\mathcal{P})$ of all the observables.

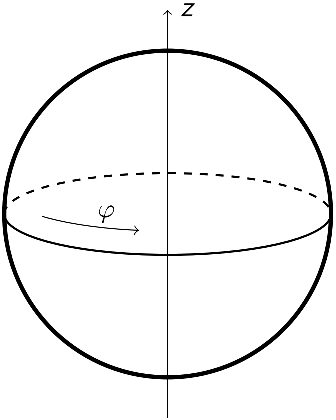
Theorem 2: A unique quantization scheme $\mathcal{F}(\mathcal{P}) \to \mathrm{Op}(\mathcal{H})$ satisfying $\hat{\mathcal{X}} = \frac{\hbar}{i}\mathcal{L}_X$ is the Weyl quantization.

Phase space of a spin

- Classical phase space of an angular momentum $\vec{s}$ is a S^2-sphere of radius s.
- Its symplectic structure: volume form on S^2.

Volume form

$$\omega = s \sin\vartheta \mathrm{d}\vartheta \wedge \mathrm{d}\varphi$$



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$$\xi := s(\cos\vartheta + 1)$$

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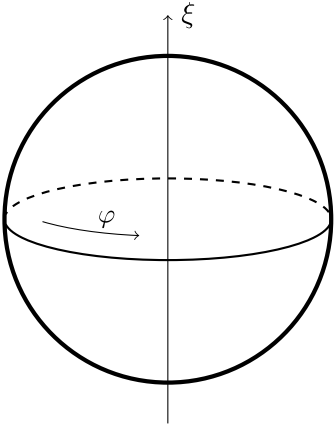

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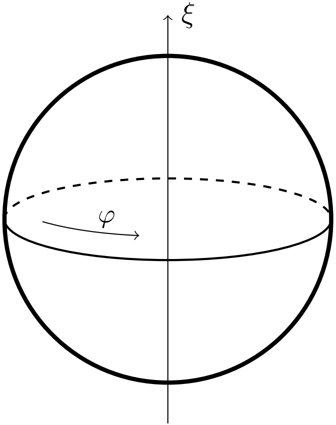

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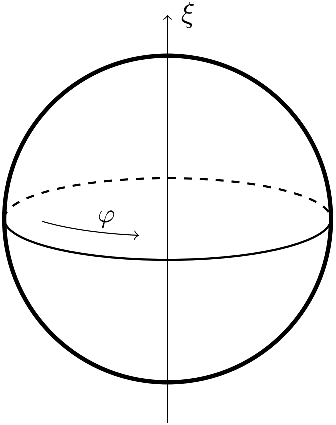

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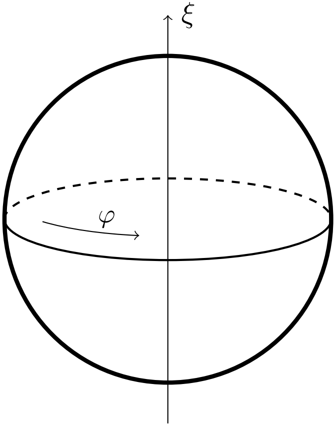
$$\tilde{\psi}(\varphi) = \int \exp\left(-\frac{i}{\hbar}\xi\varphi\right)\psi(\xi)\mathrm{d}\xi$$



Periodicity - a naive approach

Because momentum φ periodic, position ξ - quantized.

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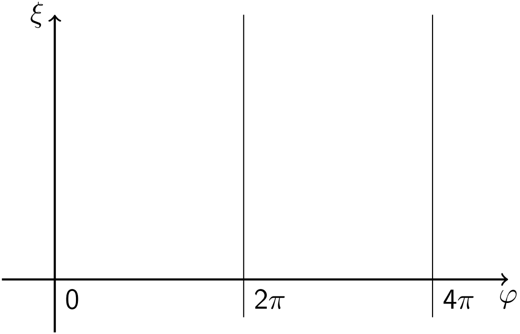

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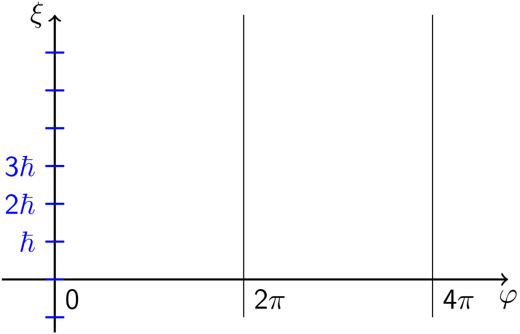

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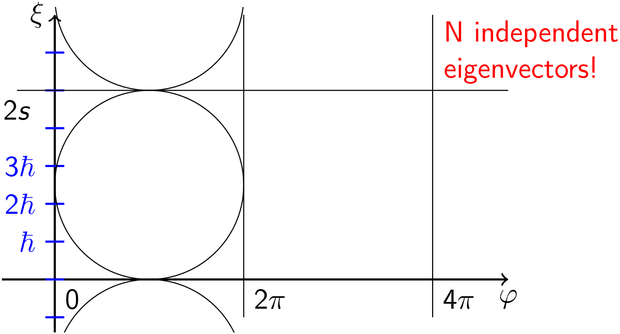

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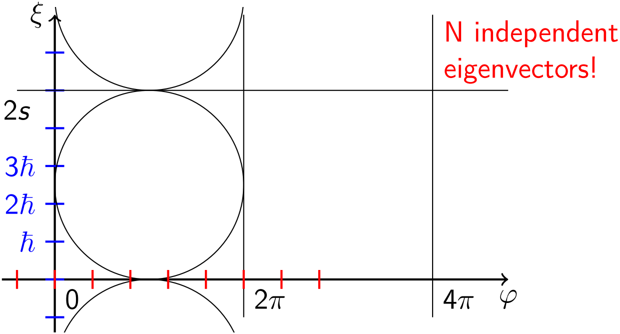

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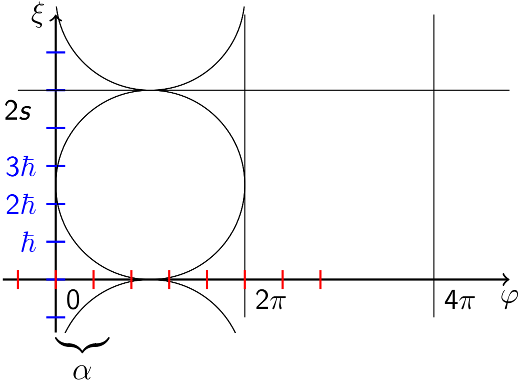

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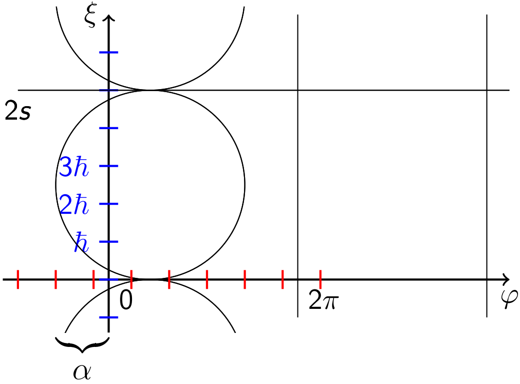

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- But quantum states retrieved from different segments of the φ-axis are the same: they differ by a constant phase factor only!

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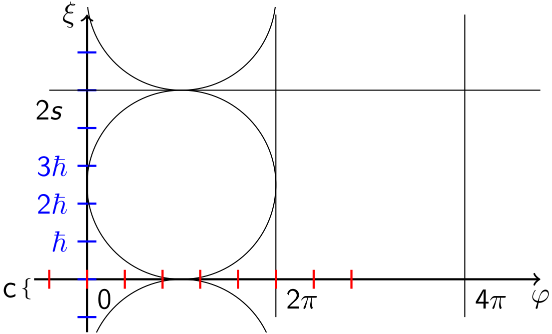

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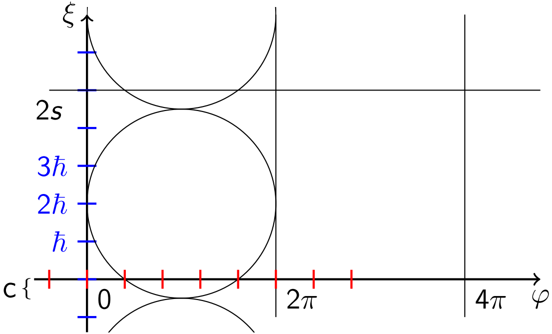

Generators of the group $SO(3)$ acting on $\mathbb{S}^2$:

$$Z = s\cos\theta \;\; ; \;\; X = s\sin\theta\cos\varphi \;\; ; \;\; Y = s\sin\theta\sin\varphi$$

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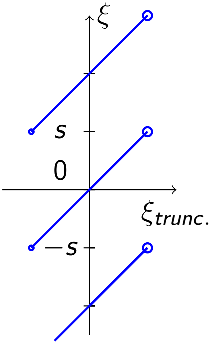

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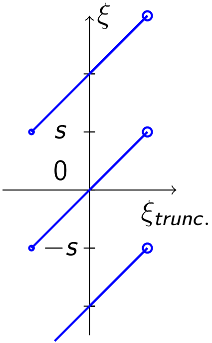
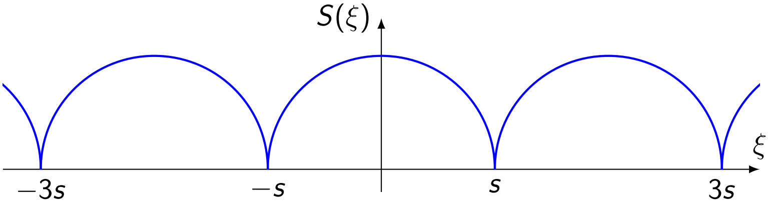
$$s^2\sin^2\theta = s^2(1-\cos^2\theta) = s^2 - \xi_{trunc.}^2$$

$$\begin{cases} X = \sqrt{s^2 - \xi_{trunc.}^2}\cos\varphi = S(\xi)\cos\varphi \\ Y = \sqrt{s^2 - \xi_{trunc.}^2}\sin\varphi = S(\xi)\sin\varphi \\ Z = \xi_{trunc.} \end{cases}$$





Projective vs. unitary representation

Miracle: Weyl quantization of these generators preserves the Lie algebra structure: $[\hat{X}, \hat{Y}] = i\hbar\hat{Z}$ and cyclic. This structure integrates to the correct projective representation of $SO(3)$.

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Denote $N = 2\ell + 1$. N-integer, $\ell = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$

- For N-odd (ℓ - integer) the above projective representation can be lifted to a unitary representation of $SO(3)$.
- For N-even (ℓ - half-integer) it can be lifted to a unitary representation of the double covering of $SO(3)$, i.e. $SU(2)$.

Weyl quantization

For a factorizable function $f(x,p) = f_x(x) f_p(p)$ Weyl quantization can be expressed by simple formulas:

$$f(\hat{x},\hat{p})\psi(x) = \int \mathrm{d}\beta \frac{1}{h} \tilde{f}_p(\beta) f_x(x+\frac{1}{2}\beta)\psi(x+\beta)$$

$$f(\hat{x},\hat{p})\tilde{\psi}(p) = \int \mathrm{d}\alpha \frac{1}{h} \tilde{f}_x(\alpha) f_p(p-\frac{1}{2}\alpha)\tilde{\psi}(p-\alpha)$$

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The Z function quantizes easily in this scheme:

$$\hat{Z}\psi(\xi) = \xi_{trunc.} \cdot \psi(\xi)$$

X and Y functions give more complex formulas:

$$\hat{X}\psi(\xi) = \int \mathrm{d}\alpha \frac{1}{h}\frac{h}{2}(\delta_{\hbar}(\alpha) + \delta_{-\hbar}(\alpha))S(\xi + \frac{\alpha}{2})\psi(\xi + \alpha)$$
$$= \frac{1}{2}\left[S(\xi + \frac{\hbar}{2})\psi(\xi + \hbar) + S(\xi - \frac{\hbar}{2}))\psi(\xi - \hbar)\right]$$
$$\hat{Y}\psi(\xi) = \int \mathrm{d}\alpha \frac{1}{h}\frac{h}{2i}(\delta_{\hbar}(\alpha) - \delta_{-\hbar}(\alpha))S(\xi + \frac{\alpha}{2})\psi(\xi + \alpha)$$
$$= \frac{1}{2i}\left[S(\xi + \frac{\hbar}{2})\psi(\xi + \hbar) - S(\xi - \frac{\hbar}{2})\psi(\xi - \hbar)\right]$$

Introducing ℓ, by $N = 2\ell + 1$ $(= 2s/\hbar)$ (This gives: $s = (\ell + \frac{1}{2})\hbar$) and with correct positioning of the spheres we obtain:

$$\hat{X}\,|\ell, m\rangle = \frac{\hbar}{2}\sqrt{(\ell + m + \frac{1}{2})(\ell - m - \frac{1}{2})}\,|\ell, m+1\rangle + \frac{\hbar}{2}\sqrt{(\ell + m - \frac{1}{2})(\ell - m + \frac{1}{2})}\,|\ell, m-1\rangle$$

Where $m \in \{-\ell, -\ell + 1, \cdots, \ell - 1, \ell\}$.

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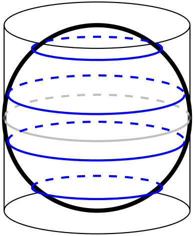
Where $m \in \{-\ell, -\ell + 1, \cdots, \ell - 1, \ell\}$.

This way we obtain the standard spin algebra.

For any choice of z-axis and a 0-meridian quantum states quantum can be described by wave functions in "position representation":



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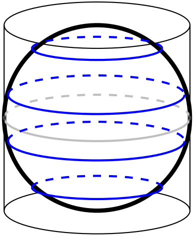

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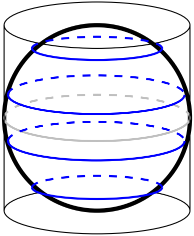

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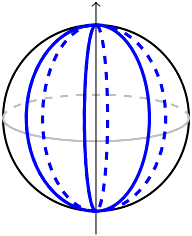
In momentum representation:



$$\tilde{\psi}(\varphi) = \sum_{k=0}^{N-1} \tilde{\psi}_k \delta(\varphi - \frac{2\pi k}{N})$$

The state can also be represented in terms of the **Wigner function**:

$$W(\varphi, \xi) := \hat{M} \int \overline{\tilde{\psi}}(\varphi + \eta)\tilde{\psi}(\varphi - \eta) e^{-\frac{i}{\hbar} 2\xi\eta} \mathrm{d}\eta$$

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Distributional formulation

$$\langle W(\varphi,\xi),\Phi(\varphi,\xi)\rangle = \hat{M}\int \overline{\tilde{\psi}}(\varphi+\eta)\tilde{\psi}(\varphi-\eta)e^{-\frac{i}{\hbar}2\xi\eta}\Phi(\varphi,\xi)\mathrm{d}\eta\mathrm{d}\varphi\mathrm{d}\xi$$

$\Phi(\varphi,\xi)\in C_0^\infty(\mathbb{R}^2)$ - test function

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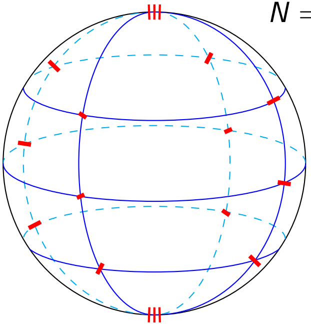
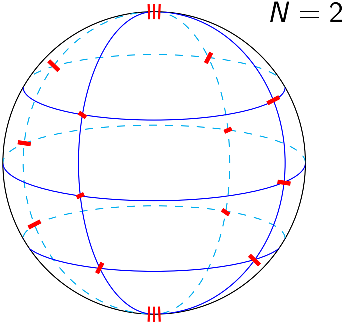
The Wigner function is insensitive to changes of the global phase!

Wigner function



$N = 2$

Wigner function



Averaging it with respect to the action of $SO(3)$, we obtain a smooth function on $\mathbb{S}^2$.

$$W_{av.}(\varphi, \xi) = \frac{1}{4\pi} (1 + f(\varphi, \xi)) \quad \text{where} \quad \int_{\mathbb{S}^2} f \, \mathrm{d}\sigma = 0$$

Conjecture: For a given spin $\ell\hbar$ functions f, corresponding to all mixed states of the system, span the space of all multi-pole functions up to $2^{2\ell}$-poles:

$\ell = \frac{1}{2}$	$N = 2$	3 dipole functions
$\ell = 1$	$N = 3$	+5 dipole functions
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Possible applications: quantum informatics.