The works of William Rowan Hamilton in Geometric Optics and the Malus-Dupin theorem

Charles-Michel Marle

Université Pierre et Marie Curie

Geometry of Jets and Fields, Będlewo, May 11–16 2015
A conference in honour of Janusz Grabowski

Summary

UPMC PARIS UNIVERSITAS

Introduction

I am very honoured to present a lecture at this conference in honour of Professor Janusz Grabowski.

Introduction

I am very honoured to present a lecture at this conference in honour of Professor Janusz Grabowski.

I am very glad of this opportunity to express to Janusz Grabowski my high esteem, and I hope that he will be interested in my modest work.

Introduction

I am very honoured to present a lecture at this conference in honour of Professor Janusz Grabowski.

I am very glad of this opportunity to express to Janusz Grabowski my high esteem, and I hope that he will be interested in my modest work.

In this talk I will first recall the evolution of ideas of scientists about the nature of *light*, from the XVII-th century up to now, and I will recall the basic concepts of *Geometric Optics*.

Introduction

I am very honoured to present a lecture at this conference in honour of Professor Janusz Grabowski.

I am very glad of this opportunity to express to Janusz Grabowski my high esteem, and I hope that he will be interested in my modest work.

In this talk I will first recall the evolution of ideas of scientists about the nature of *light*, from the XVII-th century up to now, and I will recall the basic concepts of *Geometric Optics*.

Then I will focus on the *Malus-Dupin theorem*. I will present the proof of that theorem due to *William Rowan Hamilton*.

Introduction

I am very honoured to present a lecture at this conference in honour of Professor Janusz Grabowski.

I am very glad of this opportunity to express to Janusz Grabowski my high esteem, and I hope that he will be interested in my modest work.

In this talk I will first recall the evolution of ideas of scientists about the nature of *light*, from the XVII-th century up to now, and I will recall the basic concepts of *Geometric Optics*.

Then I will focus on the *Malus-Dupin theorem*. I will present the proof of that theorem due to *William Rowan Hamilton*.

Finally I will indicate another proof of that theorem, which illustrates its links with *Symplectic Geometry*.

Theories of light, from the XVII-th century up to now

In the XVII-th century the laws which govern reflection and refraction of light were known, but the nature of light remained mysterious: some scientists thought that light was a *flux of very small particles*, while other scientists thought that it was an *oscillation propagating by waves*.

Theories of light, from the XVII-th century up to now

In the XVII-th century the laws which govern reflection and refraction of light were known, but the nature of light remained mysterious: some scientists thought that light was a *flux of very small particles*, while other scientists thought that it was an *oscillation propagating by waves*.

Scientists who believed in a *corpuscular theory of light*:

Theories of light, from the XVII-th century up to now

In the XVII-th century the laws which govern reflection and refraction of light were known, but the nature of light remained mysterious: some scientists thought that light was a *flux of very small particles*, while other scientists thought that it was an *oscillation propagating by waves*.

Scientists who believed in a *corpuscular theory of light*:
René Descartes (1596–1650). Publishes *Dioptrique* (1637), an Appendix to his *Discours de la méthode*.

Theories of light, from the XVII-th century up to now

In the XVII-th century the laws which govern reflection and refraction of light were known, but the nature of light remained mysterious: some scientists thought that light was a *flux of very small particles*, while other scientists thought that it was an *oscillation propagating by waves*.

Scientists who believed in a *corpuscular theory of light*:
René Descartes (1596–1650). Publishes *Dioptrique* (1637), an Appendix to his *Discours de la méthode*.
Isaac Newton (1643–1727). Publishes *Optiks* in 1704. However, he uses *waves* in parallel to particles to explain the phenomenon of *diffraction of light*.

Theories of light, from the XVII-th century up to now

In the XVII-th century the laws which govern reflection and refraction of light were known, but the nature of light remained mysterious: some scientists thought that light was a *flux of very small particles*, while other scientists thought that it was an *oscillation propagating by waves*.

Scientists who believed in a *corpuscular theory of light*:
René Descartes (1596–1650). Publishes *Dioptrique* (1637), an Appendix to his *Discours de la méthode*.
Isaac Newton (1643–1727). Publishes *Optiks* in 1704. However, he uses *waves* in parallel to particles to explain the phenomenon of *diffraction of light*.
Pierre-Simon de Laplace (1749–1827).

Theories of light, from the XVII-th century up to now

In the XVII-th century the laws which govern reflection and refraction of light were known, but the nature of light remained mysterious: some scientists thought that light was a *flux of very small particles*, while other scientists thought that it was an *oscillation propagating by waves*.

Scientists who believed in a *corpuscular theory of light*:
René Descartes (1596–1650). Publishes *Dioptrique* (1637), an Appendix to his *Discours de la méthode*.
Isaac Newton (1643–1727). Publishes *Optiks* in 1704. However, he uses *waves* in parallel to particles to explain the phenomenon of *diffraction of light*.
Pierre-Simon de Laplace (1749–1827).
Siméon Denis Poisson (1781–1840).

Theories of light, from the XVII-th century up to now (2)

Scientists who believed in an *undulatory theory of light*:

Theories of light, from the XVII-th century up to now (2)

Scientists who believed in an *undulatory theory of light*:

Christiaan Huygens (1629-1695). Formulates his theory around 1678 and publishes it in 1690. Stresse the fact that the laws of refraction were discovered by *Willebrord Snell* (1580–1626) before *René Descartes* claimed to have discovered these laws.

Theories of light, from the XVII-th century up to now (2)

Scientists who believed in an *undulatory theory of light*:

Christiaan Huygens (1629-1695). Formulates his theory around 1678 and publishes it in 1690. Stresse the fact that the laws of refraction were discovered by *Willebrord Snell* (1580–1626) before *René Descartes* claimed to have discovered these laws.

Thomas Young (1773–1829) discovers *light interferences* in 1801.

Theories of light, from the XVII-th century up to now (2)

Scientists who believed in an *undulatory theory of light*:

Christiaan Huygens (1629-1695). Formulates his theory around 1678 and publishes it in 1690. Stresse the fact that the laws of refraction were discovered by *Willebrord Snell* (1580–1626) before *René Descartes* claimed to have discovered these laws.

Thomas Young (1773–1829) discovers *light interferences* in 1801.

François Arago (1786–1853). Creates a laboratory at the *Observatoire de Paris* and offers its direction to *Fresnel*. Proposes an experimental setup to measure the velocity of light in air and in water, but these measurments were not made before 1850 by *Léon Foucault* (1819–1868) and a little later by *Hippolyte Fizeau* (1819–1896), who found that the velocity of light was smaller in water. It was an argument in favor of the *undulatory theory of light*.

Theories of light, from the XVII-th century up to now (2)

Scientists who believed in an *undulatory theory of light*:

Christiaan Huygens (1629-1695). Formulates his theory around 1678 and publishes it in 1690. Stresse the fact that the laws of refraction were discovered by *Willebrord Snell* (1580–1626) before *René Descartes* claimed to have discovered these laws.

Thomas Young (1773–1829) discovers *light interferences* in 1801.

François Arago (1786–1853). Creates a laboratory at the *Observatoire de Paris* and offers its direction to *Fresnel*. Proposes an experimental setup to measure the velocity of light in air and in water, but these measurments were not made before 1850 by *Léon Foucault* (1819–1868) and a little later by *Hippolyte Fizeau* (1819–1896), who found that the velocity of light was smaller in water. It was an argument in favor of the *undulatory theory of light*.

Augustin Louis Fresnel (1788–1827) observes that two beams of light polarized in orthogonal directions do not interfere and concludes that the vibrations of light are *transverse* to its direction of propagation.

Theories of light, from the XVII-th century up to now (3)

More scientists who believed in an *undulatory theory of light*:

Theories of light, from the XVII-th century up to now (3)

More scientists who believed in an *undulatory theory of light*:

James Clerk Maxwell (1831–1879). Formulates *Maxwell's equations of electromagnetism* in 1864 and concludes that light is an electromagnetic wave.

Scientists who did not choose between corpuscular and undulatory theories of light:

Theories of light, from the XVII-th century up to now (3)

More scientists who believed in an *undulatory theory of light*:

James Clerk Maxwell (1831–1879). Formulates *Maxwell's equations of electromagnetism* in 1864 and concludes that light is an electromagnetic wave.

Scientists who did not choose between corpuscular and undulatory theories of light:

Pierre de Fermat (1601–1665). States in 1657 his *Principe*: **the path taken by light to go from a point to another point is always such that the time of travel between these points is the shortest possible**. It explains the propagation of light along straight lines in an homogeneous medium, as well as the laws of reflection an refraction. Today we know that in the statement of this Principle the words "the shortest possible" should be replaced by "stationary".

Theories of light, from the XVII-th century up to now (3)

More scientists who believed in an *undulatory theory of light*:

James Clerk Maxwell (1831–1879). Formulates *Maxwell's equations of electromagnetism* in 1864 and concludes that light is an electromagnetic wave.

Scientists who did not choose between corpuscular and undulatory theories of light:

Pierre de Fermat (1601–1665). States in 1657 his *Principe*: **the path taken by light to go from a point to another point is always such that the time of travel between these points is the shortest possible**. It explains the propagation of light along straight lines in an homogeneous medium, as well as the laws of reflection an refraction. Today we know that in the statement of this Principle the words "the shortest possible" should be replaced by "stationary".

William Rowan Hamilton (1805–1865). Uses in his works on *Optics* made during the years 1824–1844 concepts which can be interpreted in both theories.

Theories of light, from the XVII-th century up to now (4)

The undulatory theory of light was triumphant after the establishment of *Maxwell's equations* and the measurments of the velocity of light in water and in air by *Foucault* and *Fizeau*. The serious difficulty caused by the fact that the velocity of electromagnetic waves with respect to any reference frame is the same in all directions was solved by *Albert Einstein* (1979–1955) thanks to his *theory of Relativity* (1905).

Theories of light, from the XVII-th century up to now (4)

The undulatory theory of light was triumphant after the establishment of *Maxwell's equations* and the measurments of the velocity of light in water and in air by *Foucault* and *Fizeau*. The serious difficulty caused by the fact that the velocity of electromagnetic waves with respect to any reference frame is the same in all directions was solved by *Albert Einstein* (1979–1955) thanks to his *theory of Relativity* (1905).

To explain the laws which govern the *photoelectric effect*, discovered by *Heinrich Rudolf Hertz* (1857–1894) around 1886-1887, *Albert Einstein* reintroduces, in 1905, a *corpuscular theory of light*, in which interactions between light and matter occur by *discrete quanta*.

Theories of light, from the XVII-th century up to now (4)

The undulatory theory of light was triumphant after the establishment of *Maxwell's equations* and the measurments of the velocity of light in water and in air by *Foucault* and *Fizeau*. The serious difficulty caused by the fact that the velocity of electromagnetic waves with respect to any reference frame is the same in all directions was solved by *Albert Einstein* (1979–1955) thanks to his *theory of Relativity* (1905).

To explain the laws which govern the *photoelectric effect*, discovered by *Heinrich Rudolf Hertz* (1857–1894) around 1886-1887, *Albert Einstein* reintroduces, in 1905, a *corpuscular theory of light*, in which interactions between light and matter occur by *discrete quanta*.

Today the duality *particle — wave* is an essential aspect of *quantum electrodynamics*, the modern theory of interactions between light and matter.

Geometric Optics

In this section, first I briefly recall the **Main concepts of Geometric Optics**.

Geometric Optics

In this section, first I briefly recall the **Main concepts of Geometric Optics**.

Then I explain what is the **Malus-Dupin theorem**, and I say a few words about its history.

Geometric Optics

In this section, first I briefly recall the **Main concepts of Geometric Optics**.

Then I explain what is the **Malus-Dupin theorem**, and I say a few words about its history.

Next I describe **Hamilton's proof of the Malus-Dupin theorem**.

At the end of this section I present **some other works of Hamilton in Geometric Optics**, specially his *Characteristic Function*, which will be used in his works on *Dynamics*.

1. Main concepts of Geometric Optics (1)

Geometric Optics is a physical theory in which the propagation of light is described in terms of *light rays*. In this theory, the physical space in which we live and in which the light propagates is treated, once a unit of length is chosen, as a *three-dimensional Euclidean affine space* $\mathcal{E}$.

1. Main concepts of Geometric Optics (1)

Geometric Optics is a physical theory in which the propagation of light is described in terms of *light rays*. In this theory, the physical space in which we live and in which the light propagates is treated, once a unit of length is chosen, as a *three-dimensional Euclidean affine space* $\mathcal{E}$. In a transparent homogeneous medium, a *light ray* is described by a segment of an *oriented straight line* drawn in that space. It will be convenient to consider the full *oriented straight line* which bears that segment. *Reflections* on smooth reflecting surfaces, or *refractions* through smooth surfaces separating two transparent media with different refractive indices, which transform an incident light ray into the corresponding reflected or refracted light ray, appear as *transformations*, defined on a part of the set $\mathcal{L}$ of all oriented straight lines in $\mathcal{E}$, with values in $\mathcal{L}$.

1. Main concepts of Geometric Optics (2)

The set $\mathcal{L}$ of all possible oriented straight lines drawn in the three-dimensional Euclidean affine space $\mathcal{E}$ depends on four parameters. We will prove below that $\mathcal{L}$ has the structure of a smooth four-dimensional symplectic manifold[1].

[1] More generally, the space of oriented straight lines in an n-dimensional Euclidean affine space is a $2(n-1)$-dimensional symplectic manifold.

1. Main concepts of Geometric Optics (2)

The set $\mathcal{L}$ of all possible oriented straight lines drawn in the three-dimensional Euclidean affine space $\mathcal{E}$ depends on four parameters. We will prove below that $\mathcal{L}$ has the structure of a smooth four-dimensional symplectic manifold[1].

Geometric Optics can be interpreted both in corpuscular and undulatory theories of light: in a corpuscular theory of light, a light ray is the trajectory of a light particle, while in an undulatory theory of light it is an infinitely thin pencil in which the vibrations of light propagate.

[1] More generally, the space of oriented straight lines in an n-dimensional Euclidean affine space is a $2(n-1)$-dimensional symplectic manifold.

1. Main concepts of Geometric Optics (3)

Let us state some definitions which follow Hamilton's teminology.

1. Main concepts of Geometric Optics (3)

Let us state some definitions which follow Hamilton's teminology.

Definition

The *rank* of a family of rays which smoothly depend on a finite number of parameters is the number of these parameters.

1. Main concepts of Geometric Optics (3)

Let us state some definitions which follow Hamilton's teminology.

Definition

The *rank* of a family of rays which smoothly depend on a finite number of parameters is the number of these parameters.

Examples

The family of light rays emitted by a luminous point in all possible directions, and the family of light rays emitted by a smooth luminous surface, each point of that surface emitting only one ray in the direction orthogonal to the surface, are *rank 2* families.

1. Main concepts of Geometric Optics (4)

Definition

Let R_0 be a ray in a rank 2 family $\mathcal{F}$ of rays. A point $m_0 \in R_0$ is said to be *regular* if it satisfies the following conditions: for each smooth surface $S \subset \mathcal{E}$ which contains m_0 and is transverse to R_0, there exists an open neighbourhood U of R_0 in $\mathcal{F}$ and an open neighbourhood V of m_0 in S such that each ray $R \in U$ meets V at a unique point m, and is such that the map $R \mapsto m$ is a diffeomorphism of U onto V.

1. Main concepts of Geometric Optics (4)

Definition

Let R_0 be a ray in a rank 2 family $\mathcal{F}$ of rays. A point $m_0 \in R_0$ is said to be *regular* if it satisfies the following conditions: for each smooth surface $S \subset \mathcal{E}$ which contains m_0 and is transverse to R_0, there exists an open neighbourhood U of R_0 in $\mathcal{F}$ and an open neighbourhood V of m_0 in S such that each ray $R \in U$ meets V at a unique point m, and is such that the map $R \mapsto m$ is a diffeomorphism of U onto V.

Remark

Non-regular points of rays in a rank 2 family form the *caustic surfaces* of the family of rays. They were studied by Hamlilton as soon as 1824, when he was only 19 years old [4]. Hamilton tacitly assumes that on each ray of a rank 2 family of rays there exists regular points.

1. Main concepts of Geometric Optics (5)

Definition

A rank 2 family $\mathcal{F}$ of rays is said to be *rectangular* if for each ray $R \in \mathcal{F}$ and each regular point $m \in R$, there exists a small piece of smooth surface which contains m which is orthogonal to the ray R at that point and which is crossed orthogonally by all the rays of a neighbourhood of R in $\mathcal{F}$.

1. Main concepts of Geometric Optics (5)

Definition

A rank 2 family $\mathcal{F}$ of rays is said to be *rectangular* if for each ray $R \in \mathcal{F}$ and each regular point $m \in R$, there exists a small piece of smooth surface which contains m which is orthogonal to the ray R at that point and which is crossed orthogonally by all the rays of a neighbourhood of R in $\mathcal{F}$.

Example

The rank 2 family of light rays emitted by a luminous point in all possible directions is *rectangular*: all the points in $\mathcal{E}$ other than the luminous point are regular and the spheres centered on the luminous point are crossed orthogonally by all rays.

1. Main concepts of Geometric Optics (6)

Example

Similarly the rank 2 family of light rays emitted by a luminous smooth surface, each point of that surface emitting only one ray in a direction normal to the surface, is *rectangular*: the surfaces obtained by moving each point of the luminous surface by a given length along the straight line normal to the surface are crossed orthogonally by all light rays, except at non-regular points where these surfaces have singularities. These singular points make the *caustic surfaces* of the family of rays.

1. Main concepts of Geometric Optics (6)

Example

Similarly the rank 2 family of light rays emitted by a luminous smooth surface, each point of that surface emitting only one ray in a direction normal to the surface, is *rectangular*: the surfaces obtained by moving each point of the luminous surface by a given length along the straight line normal to the surface are crossed orthogonally by all light rays, except at non-regular points where these surfaces have singularities. These singular points make the *caustic surfaces* of the family of rays.

Example

Let D_1 and D_2 be two straight lines in the 3-dimensional Euclidean space $\mathcal{E}$, *not both contained in the same plane*. The family of straight lines which meet both D_1 and D_2, oriented from D_1 to D_2, *is not rectangular* (it can be proven with the help of Frobenius theorem).

2. The Malus-Dupin theorem

Theorem (Malus-Dupin theorem)

A rank 2 family of light rays wich is rectangular before entering an optical device with any number of homogeneous and isotropic transparent media of various refraction indices, separated by smooth surfaces of any shapes, and any number of smooth reflecting surfaces of any shapes, remains rectangular in all transparent media of the optical device in which it propagates.

Theorem (Malus-Dupin theorem)

A rank 2 family of light rays wich is rectangular before entering an optical device with any number of homogeneous and isotropic transparent media of various refraction indices, separated by smooth surfaces of any shapes, and any number of smooth reflecting surfaces of any shapes, remains rectangular in all transparent media of the optical device in which it propagates.

Remark

The Malus-Dupin theorem states the conservation of a property (rectangularity of rank 2 families) by transformations of the set $\mathcal{L}$ of light rays associated to reflections or refractions. It implies that an optical device made of homogeneous and isotropic transparent media, with smooth refracting or reflecting surfaces of any shapes, cannot concentrate a rank 2 family of light rays to a point if that family is not already rectangular before entering the device.

3. History of the Malus-Dupin theorem

Étienne Louis Malus (1775–1812) was an officer in the French army, a mathematician and a physicist. He studied the geometric properties of families of oriented straight lines in a 3-dimensional Euclidean space in view of applications to Geometric Optics. He improved *Huygens*' undulatory theory of light, discovered and studied the phenomena of *polarization* of light and *birefringence* in crystal optics.

Geometric Optics

3. History of the Malus-Dupin theorem

Étienne Louis Malus (1775–1812) was an officer in the French army, a mathematician and a physicist. He studied the geometric properties of families of oriented straight lines in a 3-dimensional Euclidean space in view of applications to Geometric Optics. He improved *Huygens*' undulatory theory of light, discovered and studied the phenomena of *polarization* of light and *birefringence* in crystal optics.

Malus proved [11] that the family of light rays emitted by a luminous point (which of course is rectangular) remains rectangular after *one* reflection on a smooth reflecting surface, or after *one* refraction across a smooth surface separating two transparent media of different refractive indices. But he wondered whether this property was still true for *several* successive reflections or refractions [12]. Later *Hamilton* pursued *Malus*' work on families of oriented straight lines and gave a full proof of the Malus-Dupin theorem [4, 5].

3. History of the Malus-Dupin theorem (2)

Charles François Dupin (1784–1873) was a French naval engineer and mathematician. His name is attached to several mathematical objects: *Dupin's cyclids*, remarkable surfaces he discovered when he was still a student of *Gaspard Monge* (1746–1818) at the French École Polytechnique, *Dupin's indicatrix* which describes the local shape of a surface near one of its points.

3. History of the Malus-Dupin theorem (2)

Charles François Dupin (1784–1873) was a French naval engineer and mathematician. His name is attached to several mathematical objects: *Dupin's cyclids*, remarkable surfaces he discovered when he was still a student of *Gaspard Monge* (1746–1818) at the French École Polytechnique, *Dupin's indicatrix* which describes the local shape of a surface near one of its points.

According to Wikipedia, he inspired to the poet and novelist *Edgar Allan Poe* (1809–1849) the character of *Auguste Dupin* appearing in the three detective stories: *The murders in the rue Morgue*, *The Mystery of Marie Roget* and *The Purloined Letter* [13].

Geometric Optics
3. History of the Malus-Dupin theorem (2)

Charles François Dupin (1784–1873) was a French naval engineer and mathematician. His name is attached to several mathematical objects: *Dupin's cyclids*, remarkable surfaces he discovered when he was still a student of *Gaspard Monge* (1746–1818) at the French École Polytechnique, *Dupin's indicatrix* which describes the local shape of a surface near one of its points.

According to Wikipedia, he inspired to the poet and novelist *Edgar Allan Poe* (1809–1849) the character of *Auguste Dupin* appearing in the three detective stories: *The murders in the rue Morgue*, *The Mystery of Marie Roget* and *The Purloined Letter* [13].

He obtained a very simple geometric proof of the Malus-Dupin theorem for reflections [3]. For refractions he knew that this theorem was true, but did not publish his proof.

3. History of the Malus-Dupin theorem (3)

According to the editors of Hamilton's Mathematical works ([1]), *Adolphe Quetelet* (1796–1874) and *Joseph Diaz Gergonne* (1771–1859) gave in 1825 a proof of the Malus-Dupin theorem both for reflections and for refractions.

3. History of the Malus-Dupin theorem (3)

According to the editors of Hamilton's Mathematical works ([1]), *Adolphe Quetelet* (1796–1874) and *Joseph Diaz Gergonne* (1771–1859) gave in 1825 a proof of the Malus-Dupin theorem both for reflections and for refractions.

Independently, a little later, the great Irish mathematician *William Rowan Hamilton* (1805–1865), in his famous paper [5], gave a full proof of this theorem. He quotes the previous work of *Malus*, but was not aware of the works of *Dupin*, *Quetelet* and *Gergonne*. Maybe this explains why this theorme called *Théorème de Malus-Dupin* in French textbooks on Optics [2], the Malus-Dupin theorem is called *Malus' theorem* in countries other than France.

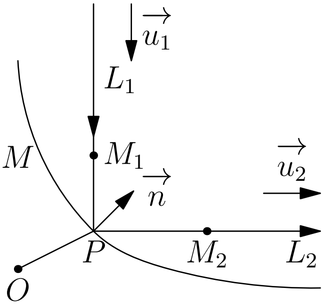


Figure 1. Reflection

Let L_1 be a light ray which meets transversally a smooth mirror M at P. Let L_2 be the corresponding reflected ray. Let $\overrightarrow{u}_1$ et $\overrightarrow{u}_2$ be the unitarry directing vectors of L_1 and L_2 and $\overrightarrow{n}$ the unitary vector normal to the mirror M at P. Let M_1 be a point of L_1, M_2 a point of L_2 and O a point arbitrarily chosen as origin (figure 1).

4. Proof by Hamilton of the Malus-Dupin theorem

a) Reflections

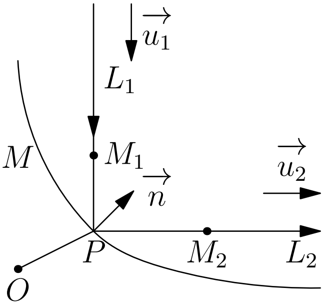


Figure 1. Reflection

Let L_1 be a light ray which meets transversally a smooth mirror M at P. Let L_2 be the corresponding reflected ray. Let $\overrightarrow{u}_1$ et $\overrightarrow{u}_2$ be the unitarry directing vectors of L_1 and L_2 and $\overrightarrow{n}$ the unitary vector normal to the mirror M at P. Let M_1 be a point of L_1, M_2 a point of L_2 and O a point arbitrarily chosen as origin (figure 1).

Any infinitesimal variation of the light ray L_1 implies determined corresponding infinitesimal variations of P, L_2, $\overrightarrow{u}_1$, $\overrightarrow{u}_2$ and $\overrightarrow{n}$.

4. Proof by Hamilton of the Malus-Dupin theorem

a) Reflections (2)

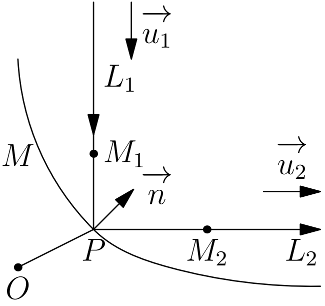


Figure 1. Reflection

For any infinitesimal variation of L_1, we can impose to M_1 an infinitesimal variations in such a way that this point always remain on L_1. And similarly for M_2.

a) Reflections (2)

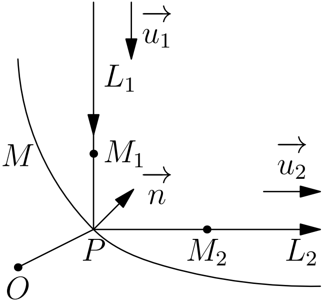


Figure 1. Reflection

For any infinitesimal variation of L_1, we can impose to M_1 an infinitesimal variations in such a way that this point always remain on L_1. And similarly for M_2.

By the laws of reflection, $\overrightarrow{u}_2 - \overrightarrow{u}_1$ and $\overrightarrow{n}$ are collinear. Any infinitesimal variation of P is tangent to the mirror M, therefore orthogonal to $\overrightarrow{n}$. It implies

4. Proof by Hamilton of the Malus-Dupin theorem

a) Reflections (2)

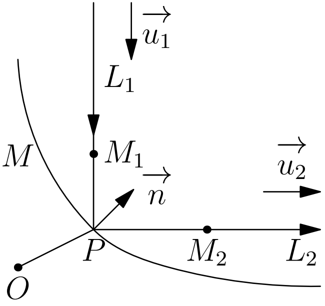


Figure 1. Reflection

For any infinitesimal variation of L_1, we can impose to M_1 an infinitesimal variations in such a way that this point always remain on L_1. And similarly for M_2.

By the laws of reflection, $\overrightarrow{u}_2 - \overrightarrow{u}_1$ and $\overrightarrow{n}$ are collinear. Any infinitesimal variation of P is tangent to the mirror M, therefore orthogonal to $\overrightarrow{n}$. It implies

$$(\overrightarrow{u}_2 - \overrightarrow{u}_1) \cdot \mathrm{d}\overrightarrow{OP} = 0\,. \qquad (*)$$

4. Proof by Hamilton of the Malus-Dupin theorem

a) Reflections (2)

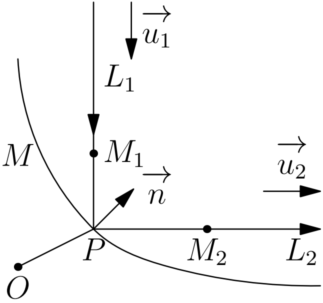


Figure 1. Reflection

For any infinitesimal variation of L_1, we can impose to M_1 an infinitesimal variations in such a way that this point always remain on L_1. And similarly for M_2.

By the laws of reflection, $\overrightarrow{u}_2 - \overrightarrow{u}_1$ and $\overrightarrow{n}$ are collinear. Any infinitesimal variation of P is tangent to the mirror M, therefore orthogonal to $\overrightarrow{n}$. It implies

$$(\overrightarrow{u}_2 - \overrightarrow{u}_1) \cdot \mathrm{d}\overrightarrow{OP} = 0\,. \qquad (*)$$

Let $\overline{M_1P} = \overrightarrow{u}_1 \cdot \overrightarrow{M_1P}$, $\overline{PM_2} = \overrightarrow{u}_2 \cdot \overrightarrow{PM_2}$.

a) Reflections (3)

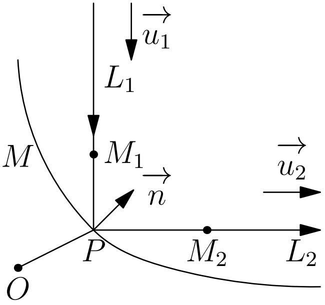


Figure 1. Reflection

An easy calculation shows that for any infinitesimal variation of L_1

$$\overrightarrow{u}_2 \cdot \mathrm{d}\overrightarrow{M}_2 - \overrightarrow{u}_1 \cdot \mathrm{d}\overrightarrow{M}_1 = \mathrm{d}(\overline{M_1 P} + \overline{P M_2})\,. \qquad (**)$$

Assume now that L_1 is an element of a rank 2 rectangular family of light rays $\mathcal{F}$, which varies within that family.

4. Proof by Hamilton of the Malus-Dupin theorem

a) Reflections (3)

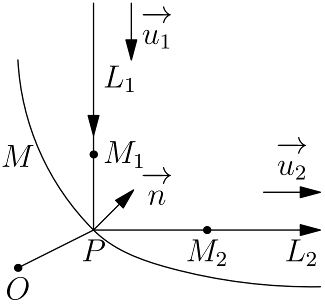


Figure 1. Reflection

An easy calculation shows that for any infinitesimal variation of L_1

$$\overrightarrow{u}_2\cdot\mathrm{d}\overrightarrow{M}_2 - \overrightarrow{u}_1\cdot\mathrm{d}\overrightarrow{M}_1 = \mathrm{d}(\overline{M_1P} + \overline{PM_2})\,. \qquad (**)$$

Assume now that L_1 is an element of a rank 2 rectangular family of light rays $\mathcal{F}$, which varies within that family.

The rectangularity of $\mathcal{F}$ allows us to impose to M_1 to always remain in a small piece of surface crossed orthogonally by the rays of $\mathcal{F}$. Therefore

Geometric Optics

4. Proof by Hamilton of the Malus-Dupin theorem

a) Reflections (3)

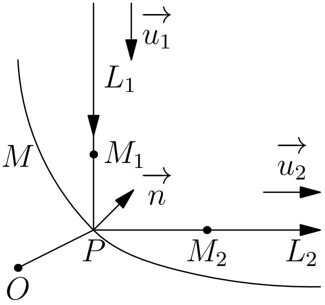


Figure 1. Reflection

An easy calculation shows that for any infinitesimal variation of L_1

$$\overrightarrow{u}_2 \cdot \mathrm{d}\overrightarrow{M}_2 - \overrightarrow{u}_1 \cdot \mathrm{d}\overrightarrow{M}_1 = \mathrm{d}(\overline{M_1 P} + \overline{P M_2})\,. \quad (**)$$

Assume now that L_1 is an element of a rank 2 rectangular family of light rays $\mathcal{F}$, which varies within that family.

The rectangularity of $\mathcal{F}$ allows us to impose to M_1 to always remain in a small piece of surface crossed orthogonally by the rays of $\mathcal{F}$. Therefore

$$\overrightarrow{u}_1 \cdot \mathrm{d}\overrightarrow{OM_1} = 0\,.$$

4. Proof by Hamilton of the Malus-Dupin theorem

a) Reflections (4)

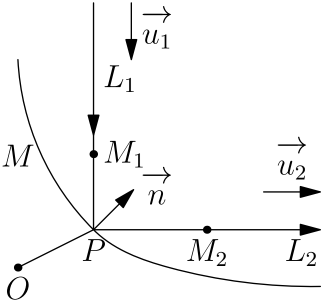


Figure 1. Reflection

For a given incident ray L_1 we choose on the corresponding reflected ray L_2 a regular point M_2, and when L_1 varies, we choose M_2 on the corresponding reflected ray in such a way that $\overline{M_1P} + \overline{PM_2}$ keeps a constant value.

4. Proof by Hamilton of the Malus-Dupin theorem

a) Reflections (4)

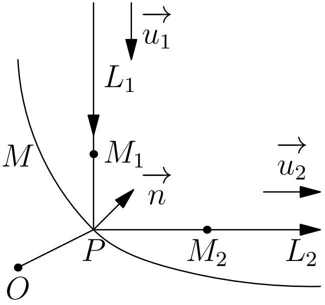


Figure 1. Reflection

For a given incident ray L_1 we choose on the corresponding reflected ray L_2 a regular point M_2, and when L_1 varies, we choose M_2 on the corresponding reflected ray in such a way that $\overline{M_1P} + \overline{PM_2}$ keeps a constant value.

The above seen equality

$$\overrightarrow{u}_2 \cdot \mathrm{d}\overrightarrow{M}_2 - \overrightarrow{u}_1 \cdot \mathrm{d}\overrightarrow{M}_1 = \mathrm{d}(\overline{M_1P} + \overline{PM_2}) \,. \qquad (**)$$

proves that $\overrightarrow{u}_2 . \mathrm{d}\overrightarrow{OM}_2 = 0$. The infinitesimal variations of M_2 therefore draw a small piece of surface orthogonally crossed by the reflected rays. The family of reflected rays is therefore rectangular.

4. Proof by Hamilton of the Malus-Dupin theorem

b) Refractions

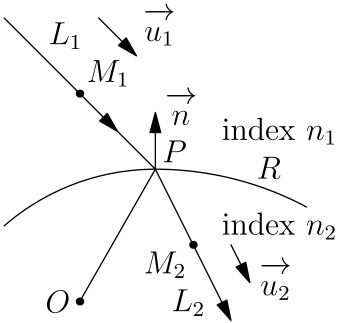


Figure 2. Refraction

In a transparent medium of refractive index n_1 let L_1 be a light ray which meets transversally at P a smooth surface R which separates that medium from another transparent medium of refractive index n_2, under an incidence angle such that there exists a correponding refracted light ray L_2 transverse to R.

4. Proof by Hamilton of the Malus-Dupin theorem

b) Refractions

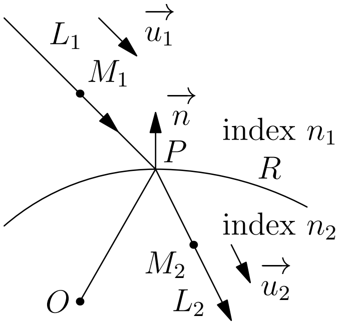


Figure 2. Refraction

In a transparent medium of refractive index n_1 let L_1 be a light ray which meets transversally at P a smooth surface R which separates that medium from another transparent medium of refractive index n_2, under an incidence angle such that there exists a correponding refracted light ray L_2 transverse to R.

The Snell-Descartes laws of refraction shows that equality $(*)$ for reflection must be replaced by

$$(n_1 \overrightarrow{u}_1 - n_2 \overrightarrow{u}_2) \cdot \mathrm{d}\overrightarrow{OP} = 0 \,.$$

4. Proof by Hamilton of the Malus-Dupin theorem

b) Refractions (2)

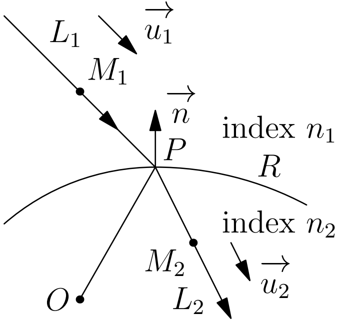


Figure 2. Refraction

Similarly, equality $(**)$ for reflection must be replaced by

$$n_2 \overrightarrow{u}_2 \cdot \mathrm{d}\overrightarrow{OM_2} - n_1 \overrightarrow{u}_1 \cdot \mathrm{d}\overrightarrow{OM_1}$$

$$= \mathrm{d}\left(n_1 \overline{M_1 P} + n_2 \overline{PM_2}\right).$$

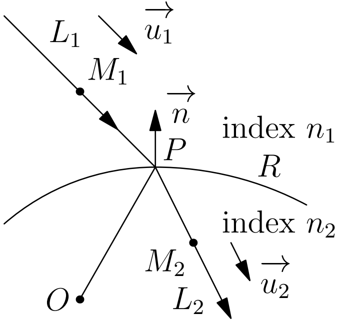


Figure 2. Refraction

Similarly, equality $(**)$ for reflection must be replaced by

$$n_2 \overrightarrow{u}_2 \cdot \mathrm{d}\overrightarrow{OM_2} - n_1 \overrightarrow{u}_1 \cdot \mathrm{d}\overrightarrow{OM_1}$$

$$= \mathrm{d}(n_1 \overline{M_1 P} + n_2 \overline{PM_2})\,.$$

Using this equality, the same argument as that used for a reflection proves that if the incident light rays form a rectangular family, the family of refracted rays too is rectangular.

5. Other works of Hamilton in Geometric Optics

Before proving the Malus-Dupin theorem for reflection, Hamilton deduces from the equality

$$\overrightarrow{u}_2 \cdot \mathrm{d}\overrightarrow{M}_2 - \overrightarrow{u}_1 \cdot \mathrm{d}\overrightarrow{M}_1 = \mathrm{d}(\overline{M_1 P} + \overline{P M_2})$$

two other results, in a way slightly more precise than that theorem.

Before proving the Malus-Dupin theorem for reflection, Hamilton deduces from the equality

$$\overrightarrow{u}_2 \cdot \mathrm{d}\overrightarrow{M}_2 - \overrightarrow{u}_1 \cdot \mathrm{d}\overrightarrow{M}_1 = \mathrm{d}(\overline{M_1 P} + \overline{P M_2})$$

two other results, in a way slightly more precise than that theorem.

1. If by reflection on a smooth mirror a rank 2 family of light rays is focused into a point, before hitting the mirror that family is rectangular.

Before proving the Malus-Dupin theorem for reflection, Hamilton deduces from the equality

$$\overrightarrow{u}_2 \cdot \mathrm{d}\overrightarrow{M_2} - \overrightarrow{u}_1 \cdot \mathrm{d}\overrightarrow{M_1} = \mathrm{d}(\overline{M_1 P} + \overline{P M_2})$$

two other results, in a way slightly more precise than that theorem.

1. If by reflection on a smooth mirror a rank 2 family of light rays is focused into a point, before hitting the mirror that family is rectangular.

2. In a rectangular family of rays, near any regular point of a ray it is possible to determine the shape of a small mirrow which will focus a neigbourhood ot that ray into a point.

Before proving the Malus-Dupin theorem for reflection, Hamilton deduces from the equality

$$\overrightarrow{u}_2 \cdot \mathrm{d}\overrightarrow{M}_2 - \overrightarrow{u}_1 \cdot \mathrm{d}\overrightarrow{M}_1 = \mathrm{d}(\overline{M_1 P} + \overline{P M_2})$$

two other results, in a way slightly more precise than that theorem.

1. If by reflection on a smooth mirror a rank 2 family of light rays is focused into a point, before hitting the mirror that family is rectangular.

2. In a rectangular family of rays, near any regular point of a ray it is possible to determine the shape of a small mirrow which will focus a neigbourhood ot that ray into a point.

For refraction, Hamilton proves the corresponding results before proving the Malus-Dupin theorem.

5. Other works of Hamilton in Geometric Optics (2)

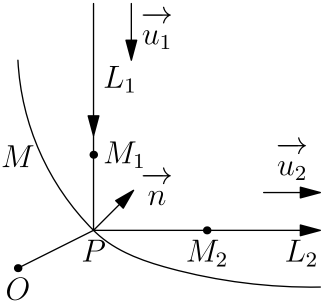


Figure 1. Reflection

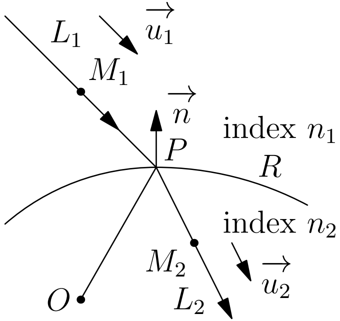


Figure 2. Refraction

5. Other works of Hamilton in Geometric Optics (2)

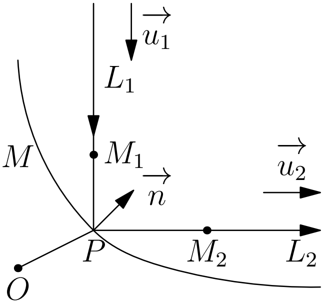


Figure 1. Reflection

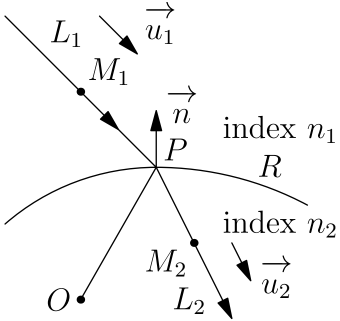


Figure 2. Refraction

The quantity $\overline{M_1P} + \overline{PM_2}$ for reflection, $n_1\overline{M_1P} + n_2\overline{PM_2}$ for reflection, is the *optical length* of the light ray between M_1 and M_2.

Geometric Optics

5. Other works of Hamilton in Geometric Optics (2)

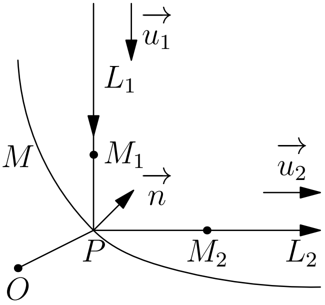


Figure 1. Reflection

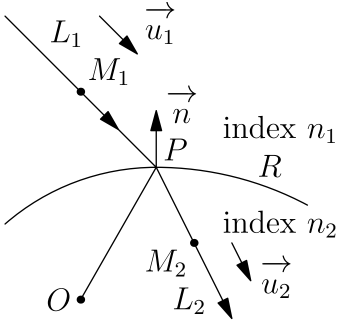


Figure 2. Refraction

The quantity $\overline{M_1P}+\overline{PM_2}$ for reflection, $n_1\overline{M_1P}+n_2\overline{PM_2}$ for reflection, is the *optical length* of the light ray between M_1 and M_2. This observation led Hamilton to define the *characteristic function* of an optical device.

5. Other works of Hamilton in Geometric Optics (3)

The *characteristic function* of an optical system is the main tool used by *Hamilton* in his works on *Geometric Optics*. In [5, 6, 7, 10] he gives successively several more and more general definitions of this concept. The most general is the following: it is a function of two points M_1 and M_2 of the optical system, defined when there exists a possible light path going from M_1 to M_2 obeying the laws of reflection and refraction; its value is then the *optical length* of that light path.

5. Other works of Hamilton in Geometric Optics (3)

The *characteristic function* of an optical system is the main tool used by *Hamilton* in his works on *Geometric Optics*. In [5, 6, 7, 10] he gives successively several more and more general definitions of this concept. The most general is the following: it is a function of two points M_1 and M_2 of the optical system, defined when there exists a possible light path going from M_1 to M_2 obeying the laws of reflection and refraction; its value is then the *optical length* of that light path.

The characteristic function may be *multivalued* and may have *singularities* at non-regular points of a light ray. In [10] *Hamilton* even defines it for a continuous transparent medium, which may be neither homogeneous nor isotropic, with a variable refractive index which may depend on the point and on the direction of light, and even on a *chromatic index* which accounts for the *color* of light. It is then expressed by an *action integral* along the path going from M_1 to M_2.

5. Other works of Hamilton in Geometric Optics (4)

Hamilton proves that when the points M_1 and M_2 remain fixed, the action integral which expresses the value of the *characteristic function* at (M_1, M_2) is *stationary* with respect to infinitesimal variations of the path going from M_1 to M_2. He therefore establishes a link between *Optics* and the *Calculus of variations*, in agreement with the ideas of *Pierre de Fermat*.

5. Other works of Hamilton in Geometric Optics (4)

Hamilton proves that when the points M_1 and M_2 remain fixed, the action integral which expresses the value of the *characteristic function* at (M_1, M_2) is *stationary* with respect to infinitesimal variations of the path going from M_1 to M_2. He therefore establishes a link between *Optics* and the *Calculus of variations*, in agreement with the ideas of *Pierre de Fermat*.

The *characteristic function* is used by *Hamilton* in his famous Essays *On a general method in Dynamics*, parts I and II [8, 9]. It is the integral, along the path of the dynamical system, of the *Poincaré-Cartan* 1-form

$$\sum_{i=1}^{n} p_i \mathrm{d}x^i - H(t, x, p)\mathrm{d}t \,.$$

A symplectic proof of the Malus-Dupin theorem

I am now going to present a proof of the Malus-Dupin theorem in the framework of *Symplectic Geometry*.

A symplectic proof of the Malus-Dupin theorem

I am now going to present a proof of the Malus-Dupin theorem in the framework of *Symplectic Geometry*.

I will first prove that in an homogeneous transparent medium, the set of all possible light rays (that means, in an Euclidean affine 4-dimensional space, the set of all oriented straight lines) has a natural structure of *4-dimensional symplectic manifold*.

A symplectic proof of the Malus-Dupin theorem

I am now going to present a proof of the Malus-Dupin theorem in the framework of *Symplectic Geometry*.

I will first prove that in an homogeneous transparent medium, the set of all possible light rays (that means, in an Euclidean affine 4-dimensional space, the set of all oriented straight lines) has a natural structure of *4-dimensional symplectic manifold*.

Second, I will prove that a *rank 2 rectangular family* of light rays is a *Lagrangian immersed submanifold* of the symplectic manifold of all possible light rays.

A symplectic proof of the Malus-Dupin theorem

I am now going to present a proof of the Malus-Dupin theorem in the framework of *Symplectic Geometry*.

I will first prove that in an homogeneous transparent medium, the set of all possible light rays (that means, in an Euclidean affine 4-dimensional space, the set of all oriented straight lines) has a natural structure of *4-dimensional symplectic manifold*.

Second, I will prove that a *rank 2 rectangular family* of light rays is a *Lagrangian immersed submanifold* of the symplectic manifold of all possible light rays.

Then I will prove that reflections and refractions are *symplectic transformations*.

A symplectic proof of the Malus-Dupin theorem

I am now going to present a proof of the Malus-Dupin theorem in the framework of *Symplectic Geometry*.

I will first prove that in an homogeneous transparent medium, the set of all possible light rays (that means, in an Euclidean affine 4-dimensional space, the set of all oriented straight lines) has a natural structure of *4-dimensional symplectic manifold*.

Second, I will prove that a *rank 2 rectangular family* of light rays is a *Lagrangian immersed submanifold* of the symplectic manifold of all possible light rays.

Then I will prove that reflections and refractions are *symplectic transformations*.

The Malus-Dupin theorem is an easy consequence of these results.

A symplectic proof of the Malus-Dupin theorem
1. The symplectic manifold of light rays

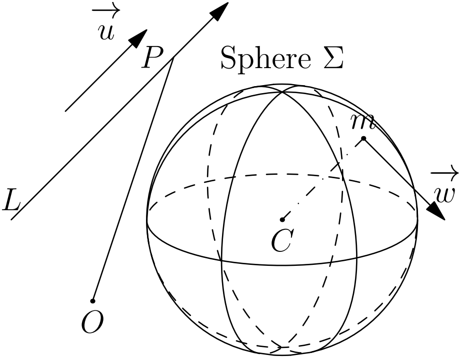


Figure 3. The manifold of light rays

Let O be a fixed point taken as origin and Σ be a sphere of any radius R (for example $R = 1$) centered on a point C. We associate to each oriented straight line L the point $m \in \Sigma$ such that $\overrightarrow{Cm} = \overrightarrow{u}$ and the 1-form $\eta \in T^*_m\Sigma$ defined by

$$\langle \eta, \overrightarrow{w} \rangle = \overrightarrow{OP} \cdot \overrightarrow{w} \,,$$

for all $\overrightarrow{w} \in T_m\Sigma$.

A symplectic proof of the Malus-Dupin theorem
1. The symplectic manifold of light rays

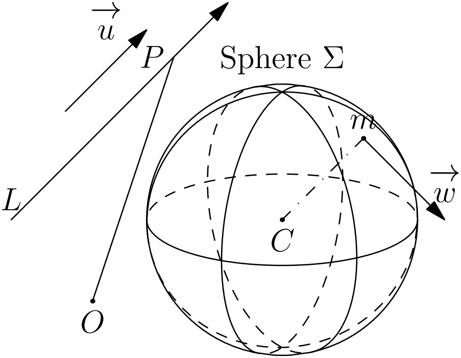


Figure 3. The manifold of light rays

Let O be a fixed point taken as origin and Σ be a sphere of any radius R (for example $R = 1$) centered on a point C. We associate to each oriented straight line L the point $m \in \Sigma$ such that $\overrightarrow{Cm} = \overrightarrow{u}$ and the 1-form $\eta \in T^*_m\Sigma$ defined by

$$\langle \eta, \overrightarrow{w} \rangle = \overrightarrow{OP} \cdot \overrightarrow{w} \,,$$

for all $\overrightarrow{w} \in T_m\Sigma$.

We have denoted by $\overrightarrow{u}$ the unit vector parallel to L with the same orientation.

A symplectic proof of the Malus-Dupin theorem
1. The symplectic manifold of light rays (2)

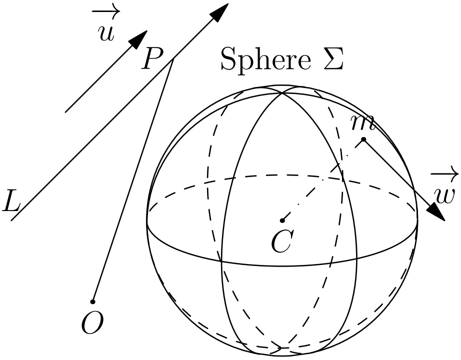


Figure 3. The manifold of light rays

The map $L \mapsto \eta$ defined by

$$\langle \eta, \overrightarrow{w} \rangle = \overrightarrow{OP} \cdot \overrightarrow{w}$$

for all $\overrightarrow{w} \in T_m\Sigma$ is a 1–1 map from the set $\mathcal{L}$ of oriented straight lines onto the cotangent bundle $T^*\Sigma$, which can be used to transfer on $\mathcal{L}$ the topology and the geometric structure of $T^*\Sigma$.

A symplectic proof of the Malus-Dupin theorem
1. The symplectic manifold of light rays (2)

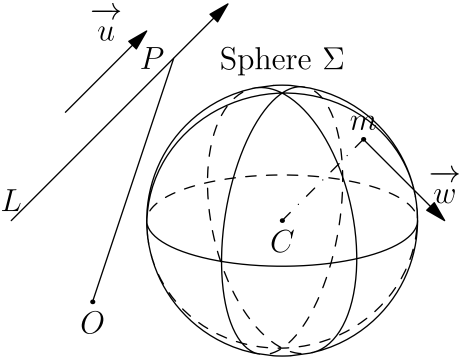


Figure 3. The manifold of light rays

The map $L \mapsto \eta$ defined by

$$\langle \eta, \overrightarrow{w} \rangle = \overrightarrow{OP} \cdot \overrightarrow{w}$$

for all $\overrightarrow{w} \in T_m\Sigma$ is a 1–1 map from the set $\mathcal{L}$ of oriented straight lines onto the cotangent bundle $T^*\Sigma$, which can be used to transfer on $\mathcal{L}$ the topology and the geometric structure of $T^*\Sigma$.

The topology, the smooth manifold structure and the affine bundle structure obtained on $\mathcal{L}$ do not depend on the choices of O and C.

UPMC PARIS UNIVERSITAS

A symplectic proof of the Malus-Dupin theorem
1. The symplectic manifold of light rays (2)

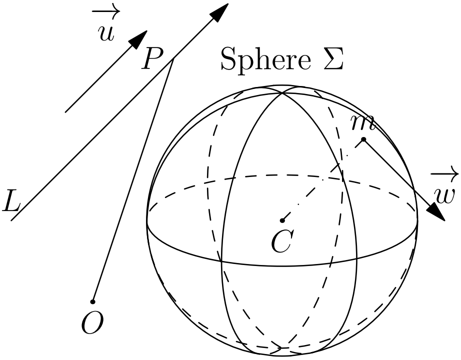


Figure 3. The manifold of light rays

The map $L \mapsto \eta$ defined by

$$\langle \eta, \vec{w} \rangle = \overrightarrow{OP} \cdot \vec{w}$$

for all $\vec{w} \in T_m\Sigma$ is a 1–1 map from the set $\mathcal{L}$ of oriented straight lines onto the cotangent bundle $T^*\Sigma$, which can be used to transfer on $\mathcal{L}$ the topology and the geometric structure of $T^*\Sigma$.

The topology, the smooth manifold structure and the affine bundle structure obtained on $\mathcal{L}$ do not depend on the choices of O and C. However, the vector bundle structure obtained on $\mathcal{L}$ and the pull-back of the Liouville form on $T^*\Sigma$ *depend on the choice of O*.

A symplectic proof of the Malus-Dupin theorem
1. The symplectic manifold of light rays (3)

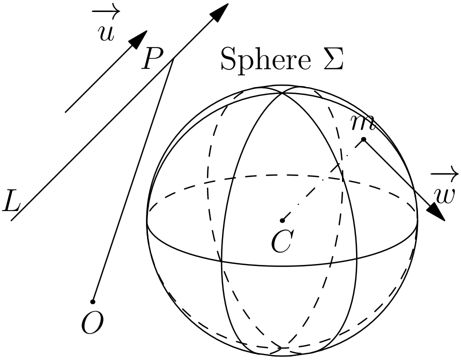


Figure 3. The manifold of light rays

Let λ_Σ be the Liouville 1-form on $T^*\Sigma$. Although its pull-back by the map $L \mapsto \eta$ depends on the choice of O, the pull-back of $\mathrm{d}\lambda_\Sigma$ *does not depend* on that choice, and therefore is a *natural* symplectic form $\omega_{\mathcal{L}}$ on $\mathcal{L}$.

A symplectic proof of the Malus-Dupin theorem
1. The symplectic manifold of light rays (3)

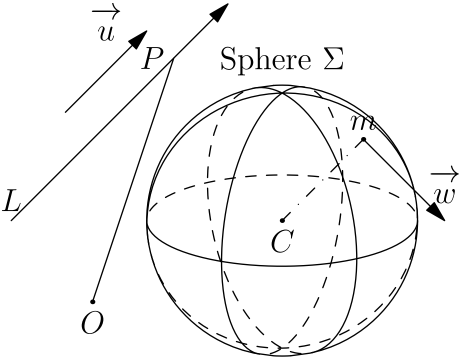


Figure 3. The manifold of light rays

Let λ_Σ be the Liouville 1-form on $T^*\Sigma$. Although its pull-back by the map $L \mapsto \eta$ depends on the choice of O, the pull-back of $\mathrm{d}\lambda_\Sigma$ *does not depend* on that choice, and therefore is a *natural* symplectic form $\omega_{\mathcal{L}}$ on $\mathcal{L}$. Let (p_1, p_2, p_3) and (x_1, x_3, x_3) be the components of $\overrightarrow{OP}$ and of $\overrightarrow{u} = \overrightarrow{CM}$ in an orthonormal basis. Then

A symplectic proof of the Malus-Dupin theorem
1. The symplectic manifold of light rays (3)

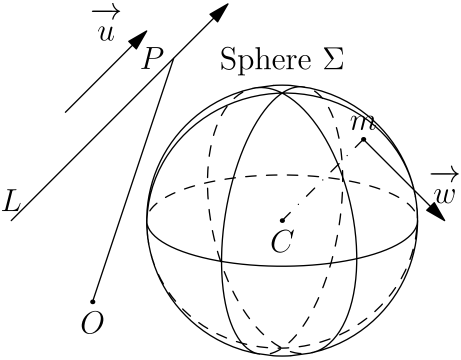


Figure 3. The manifold of light rays

Let λ_Σ be the Liouville 1-form on $T^*\Sigma$. Although its pull-back by the map $L \mapsto \eta$ depends on the choice of O, the pull-back of $\mathrm{d}\lambda_\Sigma$ *does not depend* on that choice, and therefore is a *natural* symplectic form $\omega_{\mathcal{L}}$ on $\mathcal{L}$. Let (p_1, p_2, p_3) and (x_1, x_3, x_3) be the components of $\overrightarrow{OP}$ and of $\overrightarrow{u} = \overrightarrow{CM}$ in an orthonormal basis. Then

$$\omega_{\mathcal{L}}(L) = \sum_{i=1}^{3} \mathrm{d}p_i \wedge dx_i = \mathrm{d}(\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u}) = \mathrm{d}\,\overrightarrow{OP} \wedge \mathrm{d}\,\overrightarrow{u}\,.$$

A symplectic proof of the Malus-Dupin theorem
1. The symplectic manifold of light rays (4)

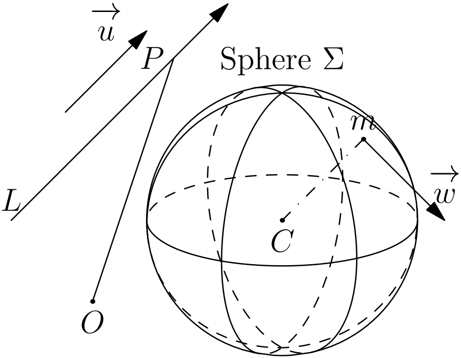


Figure 3. The manifold of light rays

We remark that the 1-form $\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u}$ depends on the choice of O, but that $\omega_{\mathcal{L}} = \mathrm{d}(\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u})$ does not depend of that choice, nor on the choice of the point P on the light ray L.

A symplectic proof of the Malus-Dupin theorem
1. The symplectic manifold of light rays (4)

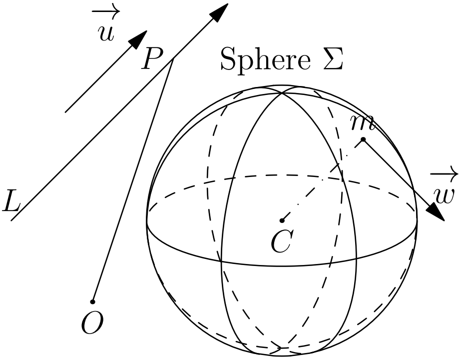


Figure 3. The manifold of light rays

We remark that the 1-form $\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u}$ depends on the choice of O, but that $\omega_{\mathcal{L}} = \mathrm{d}(\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u})$ does not depend of that choice, nor on the choice of the point P on the light ray L. In other words, $\mathrm{d}(\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u})$ is a 2-form defined on the 5-dimensional manifold of *pointed light rays* (set of light rays on which a point is chosen). Since it does not depend on the choice of a point on the light ray, the form $\omega_{\mathcal{L}} = \mathrm{d}(\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u})$ can be considerd as defined on $\mathcal{L}$.

A symplectic proof of the Malus-Dupin theorem
2. Another expression of $\omega_{\mathcal{L}}$

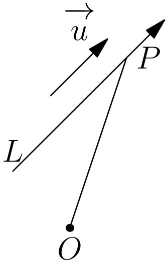


Figure 4. The form $\omega_{\mathcal{L}}$

The symplectic form on $\mathcal{L}$

$$\omega_{\mathcal{L}}(L) = \mathrm{d}(\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u})$$

can be expressed as

$$\omega_{\mathcal{L}}(L) = -\mathrm{d}(\overrightarrow{u} \cdot \mathrm{d}\,\overrightarrow{OP})\,.$$

A symplectic proof of the Malus-Dupin theorem
2. Another expression of $\omega_{\mathcal{L}}$

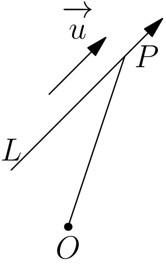


Figure 4. The form $\omega_{\mathcal{L}}$

The symplectic form on $\mathcal{L}$

$$\omega_{\mathcal{L}}(L) = \mathrm{d}(\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u})$$

can be expressed as

$$\omega_{\mathcal{L}}(L) = -\mathrm{d}(\overrightarrow{u} \cdot \mathrm{d}\,\overrightarrow{OP})\,.$$

Indeed, we have

$$\mathrm{d}(\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u}) = \mathrm{d}\big(\mathrm{d}(\overrightarrow{OP} \cdot \overrightarrow{u}) - \overrightarrow{u} \cdot \mathrm{d}\,\overrightarrow{OP}\big)\,.$$

A symplectic proof of the Malus-Dupin theorem
2. Another expression of $\omega_{\mathcal{L}}$

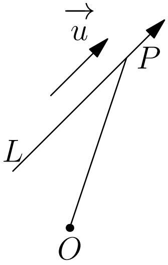


Figure 4. The form $\omega_{\mathcal{L}}$

The symplectic form on $\mathcal{L}$

$$\omega_{\mathcal{L}}(L) = \mathrm{d}(\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u})$$

can be expressed as

$$\omega_{\mathcal{L}}(L) = -\mathrm{d}(\overrightarrow{u} \cdot \mathrm{d}\,\overrightarrow{OP})\,.$$

Indeed, we have

$$\mathrm{d}(\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u}) = \mathrm{d}\big(\mathrm{d}(\overrightarrow{OP} \cdot \overrightarrow{u}) - \overrightarrow{u} \cdot \mathrm{d}\,\overrightarrow{OP}\big)\,.$$

Since $\mathrm{d} \circ \mathrm{d} = 0$,

$$\omega_{\mathcal{L}}(L) = \mathrm{d}(\overrightarrow{OP} \cdot \mathrm{d}\,\overrightarrow{u}) = -\mathrm{d}(\overrightarrow{u} \cdot \mathrm{d}\,\overrightarrow{OP})\,.$$

A symplectic proof of the Malus-Dupin theorem
3. Rectangular families are Lagrangian immersions

Proposition

A rank 2 family of light rays is rectangular if and only if it is an immersed (maybe not embedded) Lagrangian submanifold of the symplectic manifold $(\mathcal{L}, \omega_{\mathcal{L}})$ of all light rays.

Proposition

A rank 2 family of light rays is rectangular if and only if it is an immersed (maybe not embedded) Lagrangian submanifold of the symplectic manifold $(\mathcal{L}, \omega_{\mathcal{L}})$ of all light rays.

Proof.

For each L_0 in a rank 2 family $\mathcal{F}$ of light rays there exists a smooth map $L : k = (k_1, k_2) \mapsto L(k)$, defined on an open neighbourhood U of $(0,0)$ in $\mathbb{R}^2$, with values in $\mathcal{F}$, such that $L(0,0) = L_0$. For each $k \in U$ we choose a point $P(k) \in L(k)$ in such a way that the map $k \mapsto \big(P(k), \overrightarrow{u}(k)\big)$ is smooth, $\overrightarrow{u}(k)$ being the unit vector parallel to $L(k)$ with the same orientation. Then, O being a fixed point,

$$
\begin{aligned}
L^*\omega &= \mathrm{d}\big(\overrightarrow{OP(k)} \cdot \mathrm{d}\overrightarrow{u}(k)\big) = \mathrm{d}\Big(\mathrm{d}\big(\overrightarrow{OP(k)} \cdot \overrightarrow{u}(k)\big) - \overrightarrow{u}(k) \cdot \mathrm{d}\,\overrightarrow{OP(k)}\Big) \\
&= -\mathrm{d}\big(\overrightarrow{u}(k) \cdot \mathrm{d}\,\overrightarrow{OP(k)}\big)\,.
\end{aligned}
$$

A symplectic proof of the Malus-Dupin theorem
3. Rectangular families are Lagrangian immersions (2)

Proof.

(continued)
The rank 2 family $\mathcal{F}$ is an immersed submanifold of $\mathcal{L}$. We see that $\mathcal{L}$ is Lagrangian in a neighbourhood of L_0 if and only if the 1-form $\overrightarrow{u}(k)\cdot \mathrm{d}\,\overrightarrow{OP(k)}$ is closed, or, the problem being local, if and only if there exists a smooth function $k=(k_1,k_2)\mapsto F(k)$ such that

$$\overrightarrow{u}(k)\cdot \mathrm{d}\,\overrightarrow{OP(k)} = \mathrm{d}F(k)\,. \qquad (*)$$

A symplectic proof of the Malus-Dupin theorem
3. Rectangular families are Lagrangian immersions (2)

Proof.

(continued)

The rank 2 family $\mathcal{F}$ is an immersed submanifold of $\mathcal{L}$. We see that $\mathcal{L}$ is Lagrangian in a neighbourhood of L_0 if and only if the 1-form $\overrightarrow{u}(k)\cdot \mathrm{d}\,\overrightarrow{OP(k)}$ is closed, or, the problem being local, if and only if there exists a smooth function $k=(k_1,k_2)\mapsto F(k)$ such that

$$\overrightarrow{u}(k)\cdot \mathrm{d}\,\overrightarrow{OP(k)} = \mathrm{d}F(k)\,. \qquad (*)$$

The vector $\overrightarrow{u}(k)$ being unitary, for any $c\in\mathbb{R}$

$$\mathrm{d}F(k) = \overrightarrow{u}(k)\cdot \mathrm{d}\Big(\big(F(k)+c\big)\overrightarrow{u}(k)\Big)\,.$$

If F satisfies $(*)$, it satisfies too for any $c\in\mathbb{R}$

$$\overrightarrow{u}(k)\cdot \mathrm{d}\Big(\overrightarrow{P}(k) - \big(F(k)+c\big)\overrightarrow{u}(k)\Big) = 0\,. \qquad (**)$$

A symplectic proof of the Malus-Dupin theorem
3. Rectangular families are Lagrangian immersions (3)

Proof.

(continued) Let us assume that $\mathcal{F}$ is Lagrangian near L_0, let F be a smooth function which satisfies $(*)$. Let Q_0 be a regular point of L_0. There exists $c \in \mathbb{R}$ such that $\overrightarrow{OQ_0} = \overrightarrow{OP(0,0)} - (F(0,0)+c)\,\overrightarrow{u}(0,0)$. The points near Q_0 being regular on rays near L_0 which bear them, the variations of $\overrightarrow{OP(k)} - (F(k)+c)\,\overrightarrow{u}(k)$ for k near $(0,0)$ generate a smooth surface which, by $(**)$, is crossed orthogonally by the rays $L(k)$ for all k near enough $(0,0)$. Therefore $\mathcal{F}$ is rectangular near L_0.

A symplectic proof of the Malus-Dupin theorem

3. Rectangular families are Lagrangian immersions (3)

Proof.

(continued) Let us assume that $\mathcal{F}$ is Lagrangian near L_0, let F be a smooth function which satisfies $(*)$. Let Q_0 be a regular point of L_0. There exists $c \in \mathbb{R}$ such that $\overrightarrow{OQ_0} = \overrightarrow{OP(0,0)} - (F(0,0) + c)\overrightarrow{u}(0,0)$. The points near Q_0 being regular on rays near L_0 which bear them, the variations of $\overrightarrow{OP(k)} - (F(k) + c)\overrightarrow{u}(k)$ for k near $(0,0)$ generate a smooth surface which, by $(**)$, is crossed orthogonally by the rays $L(k)$ for all k near enough $(0,0)$. Therefore $\mathcal{F}$ is rectangular near L_0.

Conversely, we assume that $\mathcal{F}$ is rectangular near L_0. Each regular point in L_0 is contained in a small piece of smooth surface crossed orthogonally by L_0 and by the rays $L(k)$ for k near enough $(0,0)$. That surface is drawn by points $P(k) - F(k)\overrightarrow{u}(k)$, with F smooth. Since F satisfies $(*)$, $\mathcal{F}$ is Lagrangian near L_0.

Theorem

In an homogeneous transparent medium, reflection on a smooth surface is a symplectomorphism of an open subset of $(\mathcal{L}, \omega_{\mathcal{L}})$ *on another open subset of that symplectic manifold.*

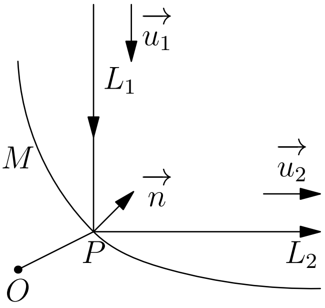


Figure 4. Reflection is a symplectomorphism

Proof. Let L_1 be a light ray which meets transversally a smooth mirror M at a point P, L_2 be the corresponding reflected ray, $\overrightarrow{u}_1$ and $\overrightarrow{u}_2$ the unitary directing vectors of L_1 and L_2 (figure 4). We have

$$\omega_{\mathcal{L}}(L_1) = -\mathrm{d}(\overrightarrow{u}_1 \cdot \mathrm{d}\,\overrightarrow{OP}),$$

$$\omega_{\mathcal{L}}(L_2) = -\mathrm{d}(\overrightarrow{u}_2 \cdot \mathrm{d}\,\overrightarrow{OP}).$$

A symplectic proof of the Malus-Dupin theorem
4. Reflections are symplectomorphisms (2)

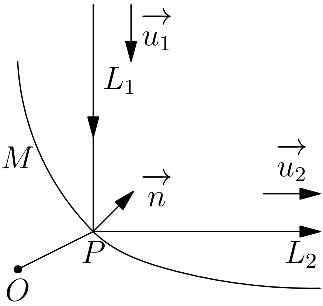


Figure 4. Reflection is a symplectomorphism

To prove that a reflection is a symplectomorphism amounts to prove that

$$\mathrm{d}\big((\overrightarrow{u}_2 - \overrightarrow{u}_1)\cdot \mathrm{d}\,\overrightarrow{OP}\big) = 0\,.$$

This is true because we even have

$$\big((\overrightarrow{u}_2 - \overrightarrow{u}_1)\cdot \mathrm{d}\,\overrightarrow{OP}\big) = 0\,.$$

because, using the laws of reflection, we see that for any infinitesimal variation of P, the vectors $\overrightarrow{u}_2 - \overrightarrow{u}_1$ and $\mathrm{d}\,\overrightarrow{OP}$ are orthogonal.

Theorem

Refraction across a smooth surface R which separates two transparent media of refractive indices n_1 and n_2 is a symplectomorphism of an open subset of $(\mathcal{L}, n_1\omega_{\mathcal{L}})$ on an open subset of $(\mathcal{L}, n_1\omega_{\mathcal{L}})$.

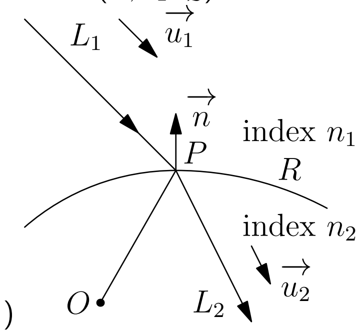


Figure 5. Refraction is a symplectomorphism

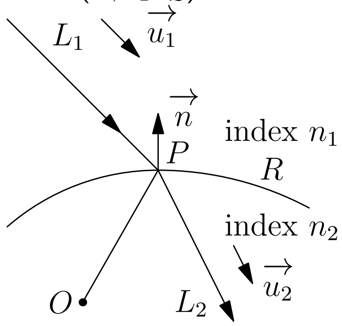
Proof. As for reflections, it is enough to prove that for any infinitesimal variation of P,

$$\mathrm{d}\big((n_2\overrightarrow{u}_2 - n_1\overrightarrow{u}_1)\cdot \mathrm{d}\,\overrightarrow{OP}\big) = 0\,.$$

5. Refractions are symplectomorphisms

Theorem

Refraction across a smooth surface R which separates two transparent media of refractive indices n_1 and n_2 is a symplectomorphism of an open subset of $(\mathcal{L}, n_1\omega_{\mathcal{L}})$ on an open subset of $(\mathcal{L}, n_1\omega_{\mathcal{L}})$.



)

Figure 5. Refraction is a symplectomorphism

Proof. As for reflections, it is enough to prove that for any infinitesimal variation of P,

$$\mathrm{d}\big((n_2\overrightarrow{u}_2 - n_1\overrightarrow{u}_1)\cdot \mathrm{d}\,\overrightarrow{OP}\big) = 0\,.$$

This is a consequence of the laws of refraction since the vectors $n_2\overrightarrow{u}_2 - n_1\overrightarrow{u}_1$ and $\mathrm{d}\,\overrightarrow{OP}$ are orthogonal.

6. Proof of the Malus-Dupin theorem

Since a reflection on a smooth mirror or a refraction through a smooth surface which separates two transparent media of different refraction indices are symplectomorphisms,

6. Proof of the Malus-Dupin theorem

Since a reflection on a smooth mirror or a refraction through a smooth surface which separates two transparent media of different refraction indices are symplectomorphisms, since a transformation composed of several symplectomorphisms is a symplectomorphism, Since a reflection on a smooth mirror or a refraction through a smooth surface which separates two transparent media of different refraction indices are symplectomorphisms, since a transformation composed of several symplectomorphisms is a symplectomorphism, since a Lagrangian immersed submanifold of $\mathcal{L}$ is a rank 2 rectangular family of light rays,

6. Proof of the Malus-Dupin theorem

Since a reflection on a smooth mirror or a refraction through a smooth surface which separates two transparent media of different refraction indices are symplectomorphisms, since a transformation composed of several symplectomorphisms is a symplectomorphism, since a Lagrangian immersed submanifold of $\mathcal{L}$ is a rank 2 rectangular family of light rays, and since the image by a symplectomorphism of a Lagrangian immersed submnifold is a Lagrangian immersed submanifold, we can formulate the Malus-Dupin theorem.

A symplectic proof of the Malus-Dupin theorem

6. Proof of the Malus-Dupin theorem

Since a reflection on a smooth mirror or a refraction through a smooth surface which separates two transparent media of different refraction indices are symplectomorphisms, since a transformation composed of several symplectomorphisms is a symplectomorphism, since a Lagrangian immersed submanifold of $\mathcal{L}$ is a rank 2 rectangular family of light rays, and since the image by a symplectomorphism of a Lagrangian immersed submnifold is a Lagrangian immersed submanifold, we can formulate the Malus-Dupin theorem.

Theorem (Malus-Dupin theorem)

A rank 2 family of light rays wich is rectangular before entering an optical device with any number of homogeneous and isotropic transparent media of various refraction indices, separated by smooth surfaces of any shapes, and any number of smooth reflecting surfaces of any shapes, remains rectangular in all transparent media of the optical device in which it propagates.

Thanks

My warmest thanks to the organizers for inviting me to present a talk.

Thanks

My warmest thanks to the organizers for inviting me to present a talk.

My best wishes to Janusz Grabowski for a happy birthday and many more years of beautiful scientific achievements!

Thanks

My warmest thanks to the organizers for inviting me to present a talk.

My best wishes to Janusz Grabowski for a happy birthday and many more years of beautiful scientific achievements!

And many thanks to all who patiently listened to me.

References I

[1] A. W. Conway and J. L. Synge, *Appendix Editors* to *Sir William Rowan Hamilton mathematical Works, vol. I* pp. 463–464. Cambridge University Press, London, 1931.

[2] J.-M. Courty, *Notes de cours de l'Université Pierre et Marie Curie, L3 Physique, PGA Chapitre VI page 48*, http://www.edu.upmc.fr/physique/phys325/Documents/Poly_II_Chap_6.pdf

[3] C. Dupin, *Applications de la géométrie*, Mémoire présenté à l'Académie des Sciences en 1816, publié à Paris en 1822.

[4] W. R. Hamilton, *On Caustics, Part First*. Manuscript, 1824. In *Sir William Rowan Hamilton mathematical Works*, vol. I, chapter XV, Cambridge University Press, London, 1931.

References II

[5] W. R. Hamilton, *Theory of systems of rays, Part First and Part Second (1827)*. Part first: Trans. Royal Irish Academy, 15 (1828), pp. 69–174. Part Second: manuscript. In *S*ir William Rowan Hamilton mathematical Works, vol. I, chapter I, Cambridge University Press, London, 1931.

[6] W. R. Hamilton, *Supplement to an essay on the theory of systems of rays (1830)*. Trans. Royal Irish Academy, 16 (1830), pp. 1–61. In *S*ir William Rowan Hamilton mathematical Works, vol. I, chapter II, Cambridge University Press, London, 1931.

[7] W. R. Hamilton, *Second supplement to an essay on the theory of systems of rays (1830)*. Trans. Royal Irish Academy, 16 (1831), pp. 93–125. In *S*ir William Rowan Hamilton mathematical Works, vol. I, chapter III, Cambridge University Press, London, 1931.

References III

[8] W. R. Hamilton, *On a general method in Dynamics*. Read April 10, 1834, Philosophical Transactions of the Royal Society, part II for 1834, pp. 247–308. In *S*ir William Rowan Hamilton mathematical Works, vol. IV, Cambridge University Press.

[9] W. R. Hamilton, *Second essay on a general method in Dynamics*. Read January 15, 1835, Philosophical Transactions of the Royal Society, part I for 1835, pp. 95–144. In *S*ir William Rowan Hamilton mathematical Works, vol. IV, Cambridge University Press.

[10] W. R. Hamilton, *Third supplement to an essay on the theory of systems of rays (1830)*. Trans. Royal Irish Academy, 17 (1837), pp. 1–144. In *S*ir William Rowan Hamilton mathematical Works, vol. I, chapter IV, Cambridge University Press, London, 1931.

References IV

[11] É. L. Malus, Journal de l'École Polytechnique 7 (1808), pp. 1–44 and 84–129.

[12] É. L. Malus, *Traité d'optique*. Mémoires présentés à l'Institut par divers savans, 2 (1811), pp. 214–302.

[13] Wikipedia, *Charles Dupin*, `http://fr.wikipedia.org/wiki/Charles_Dupin`