A Quantum Route to Hamilton-Jacobi Theory: Considerations on the Quantum-to-Classical Transition

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Lie-Scheffers System: A Geometric Approach

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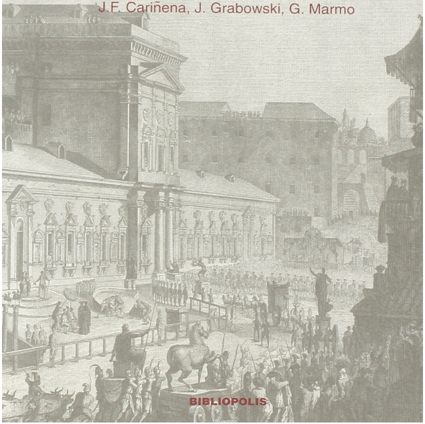

BIBLIOPOLIS

Two quotations from Dirac

The Method of Classical Analogy

> *"... The value of classical analogy in the developments of quantum mechanics depends on the fact that classical mechanics provides a valid description of dynamical systems..."*
>
> *"... Classical mechanics must therefore be a limiting case of quantum mechanics..."*

Two quotations from Dirac

The Hamiltonian form of field dynamics

> *"..In classical dynamics, one has usually supposed that, when one has solved the equation of motion, one has done everything worth doing..."*

Talking about the family of solutions of Hamilton equations which fill a Lagrangian submanifold transversal to the fibers of the cotangent bundle, Dirac says:

> *"... The family does not have any importance from the point of view of Newtonian mechanics; but it is a family which corresponds to one state of motion in the quantum theory, so presumable the family has some deep significance in nature, not yet properly understood..."*

General settings

- $\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$ Hamilton equations

- $H\left(q, \frac{\partial S}{\partial q}\right) = E = \frac{\partial S}{\partial t}$ Hamilton-Jacobi

- $H(\vec{q}, -i\hbar\vec{\nabla}) = i\hbar\frac{\partial}{\partial t}$ Schrödinger

Linear versus non-linear

Lie-Scheffers ideology: superposition rules

$$\Gamma = a^j(t)X_j, \quad [X_j, X_k] = c_{jk}^l X_l$$

$$\mathcal{H}, \quad \mathcal{R}(\mathcal{H}), \quad U(\mathcal{H})$$

$$TU(\mathcal{H}) \equiv [U(\mathcal{H})]^{\mathbb{C}}, \quad T\mathfrak{u}(\mathcal{H}) \rightleftharpoons T^*\mathfrak{u}(\mathcal{H}) \rightleftharpoons T^*\mathfrak{u}^*(\mathcal{H})$$

$T\mathfrak{u}(\mathcal{H})$ is a C*-algebra

GNS-construction: from $T\mathfrak{u}(\mathcal{H})$ to $\mathcal{H}$.
With any C*-algebra on any manifold M, we construct Hilbert spaces.

Second order dynamics on $U(\mathcal{H})$

Quantum systems on $\mathcal{R}(\mathcal{H})$

$$\frac{d}{dt}\left(\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}\right) = \frac{A}{i\hbar}\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} - \frac{1}{i\hbar}\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}A$$

$\mathcal{R}(\mathcal{H})$ homogeneous space of $U(\mathcal{H})$

$T\mathcal{R}(\mathcal{H})$ homogeneous space of $TU(\mathcal{H}) \simeq GL(\mathcal{H})$

Quantum systems on $\mathcal{R}(\mathcal{H}) \equiv$
complete solution of H.J. associated with

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}(\rho_\psi(U^{-1}\dot{U})^2) = \frac{1}{2}\frac{\langle\psi|(U^{-1}\dot{U})^2|\psi\rangle}{\langle\psi|\psi\rangle}$$

Dirac interaction picture

On the group

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}(U^{-1}\dot{U})^2 \quad \longrightarrow \quad \text{E.L. equations } \frac{d}{dt}(U^{-1}\dot{U}) = 0$$

Family of generalized solutions of the H.J. on the group

$$\begin{aligned}
&\theta_{\mathcal{L}} = \mathrm{tr}(U^{-1}\dot{U})(U^{-1}dU) \\
&\omega_{\mathcal{L}} = \mathrm{tr}\, d\left[(U^{-1}\dot{U}) \wedge (U^{-1}dU)\right], \qquad d(U^{-1}U) = 0 \\
&U^{-1}\dot{U} = iA, \quad i_X(U^{-1}dU) = iA, \qquad A \text{ Hermitian} \\
&X^*(\theta_{\mathcal{L}}) = i\,\mathrm{tr}(AU^{-1}dU)
\end{aligned}$$

Unfolding nonlinear classical dynamics

Quantization: A procedure to associate linear equations with non-linear ones.

More specifically: Write the equations of motion as a Lie-Scheffers system.

$$i\hbar \frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \xi = \frac{z_1}{z_2} \Rightarrow$$
$$\Rightarrow i\hbar \frac{d}{dt}\xi = H_{12} + (H_{11} - H_{22})\xi - H_{21}\xi^2$$

From an equation on the group we obtain a linear equation on a vector space for any representation of the group.

More on Schrödinger equation and Hamilton-Jacobi equation

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi, \qquad \psi = Ae^{iS/\hbar}$$

We obtain

$$\begin{aligned}
\frac{\partial A}{\partial t} &= -\frac{1}{2m}(A\Delta S + 2\,\mathrm{grad}A\cdot\mathrm{grad}S)\\
\frac{\partial S}{\partial t} &= -\left[\frac{(\mathrm{grad}S)^2}{2m} + V(\vec{r}) - \frac{\hbar^2}{2m}\frac{\Delta A}{A}\right]
\end{aligned}$$

Setting $\vec{u} = \frac{1}{m}\mathrm{grad}S, \quad A^2 = \psi^*\psi = \rho, \quad -\frac{\hbar^2}{2m}\frac{\Delta A}{A} = -\left(\frac{\hbar}{2}\right)^2\frac{1}{m}\frac{\Delta\rho}{\rho}$

$$\begin{aligned}
&\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\vec{u}) = 0\\
&\int_{\mathbb{R}^3}\rho\,d^3x = 1
\end{aligned}$$

$$\text{Hamilton-Jacobi} \Leftarrow \frac{\hbar^2}{2m}\frac{\Delta A}{A} = 0$$

Field theoretical aspect

- $\mathcal{L} = \int_V \rho \left(\frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + V \right) d^3x\, dt, \quad \delta S|_{\partial V} = 0,\ \delta\rho|_{\partial V} = 0$
- $\mathcal{L} = \int_V \rho \left(\frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + \left(\frac{\hbar}{2}\right)^2 \frac{1}{2m} \frac{1}{\rho^2} \nabla\rho \cdot \nabla\rho + V \right) d^3x\, dt$

Fixed end-point variation with respect to S leads to the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \left(\rho \frac{1}{m} \nabla S \right) = 0$$

Fixed end-point variation with respect to ρ leads respectively to

- $\frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + V = 0$
- $\frac{\partial S}{\partial t} + \frac{1}{2m} \nabla S \cdot \nabla S + V + \left(\frac{\hbar}{2}\right)^2 \left(\frac{1}{2m} \frac{1}{\rho^2} \nabla\rho \cdot \nabla\rho - \frac{2}{\rho} \nabla^2 \rho \right) = 0$

The fate of the continuity equation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{u}) = 0$$

Geometric interpretation:

$$dS : Q \times \Lambda \to T^*Q$$

$$dS_*(\theta_0) = \frac{\partial^2 S}{\partial \lambda_j \partial q_k} d\lambda_j \wedge dq_k$$

A symplectic structure.

Liouville theorem on $Q \times \Lambda \times \mathbb{R}$ gives the continuity equations.

Generalized Coherent States

Embedding "classical manifolds" into $\mathcal{R}(\mathcal{H})$.

M a manifold with a volume form Ω.

$$m \mapsto |\psi(m)\rangle \in \mathcal{H} \Rightarrow\mapsto \frac{|\psi(m)\rangle\langle\psi(m)|}{\langle\psi(m)|\psi(m)\rangle}$$

Pull-back of Hermitian tensor fields

$$\langle d\psi(m)|d\psi(m)\rangle \to \text{Riemannian + skew-symmetric}$$
$$\text{(2,0) tensors on } M$$

Fisher-Rao quantum information metric.

Tranformations $T : |\psi(m)\rangle \to |\psi(m')\rangle$ defines

$$\phi_T : M \to M,\ m \mapsto m'$$

Generalized Coherent States

Examples: Quantizers and Dequantizers

$$W : m \mapsto W(m) \in \mathfrak{u}(\mathcal{H})$$

$|0\rangle \in \mathcal{H}$ a fiducial state
$|\psi(m)\rangle = W(m)|0\rangle$

$$\mathcal{O}_P(\mathcal{H}) \to \mathcal{F}(M)$$
$$A \mapsto \mathrm{tr}(A\,W(m)) = f_A(m), \quad (f_A * f_B)(m) = \mathrm{tr}(A\,B\,W(m))$$

Remark: When M is a group and W a representation, the $*$-product becomes the convolution product on the group algebra.

If $D : M \to \mathfrak{u}(\mathcal{H})$ is another association such that

$$\operatorname{tr} D(m)W(m') = \delta(m, m'),$$

we define a "quantizer" map

$$A = \int_M f_A(m)D(m)d\mu_\Omega$$

$M = T^*\mathbb{R}^n \quad (q,p) \mapsto D(q,p)$ Weyl system, projective unitary representation

$(f_A * f_B)(q,p)$ Moyal product

Locality versus non-locality of the product.

“Quantum Hamilton-Jacobi”

Replace $Q \times \Lambda$ with operators

$$\widehat{p} = \frac{\partial}{\partial \widehat{q}}\, S(\widehat{q}, \widehat{\lambda}, t), \quad \widehat{P} = -\frac{\partial}{\partial \widehat{\lambda}}\, S(\widehat{q}, \widehat{\lambda}, t)$$

$$H\left(\widehat{q}, \frac{\partial S}{\partial \widehat{q}}, t\right) + \frac{\partial}{\partial t} S(\widehat{q}, \widehat{\lambda}, t) = 0$$

“Well ordering”

$$S(\widehat{q}, \widehat{\lambda}, t) = \sum_{\alpha} f_\alpha(\widehat{q}, t) g_\alpha(\widehat{\lambda}, t)$$

Main ingredient

$$\langle q | S(\widehat{q}, \widehat{\lambda}, t) | \lambda \rangle = S(q, \lambda, t) \langle q | \lambda \rangle$$

$$\Rightarrow \frac{1}{2m}\left[\left(\frac{\partial}{\partial q} S'(q, \lambda, t)\right)^2 - i\hbar \frac{\partial^2}{\partial q^2} S(q, \lambda, t)\right] + V(q) + \frac{\partial}{\partial t} S(q, \lambda, t) = 0$$

$$\psi(q, \lambda, t) = e^{\frac{i}{\hbar} S(q, \lambda, t)}$$

Summarizing

$$\begin{array}{ccc} \text{Nonlinear equations} & \longrightarrow & \text{linear equations} \\ \text{(classical)} & & \text{(quantum)} \end{array}$$

Lie Scheffers: $\Gamma = a^j X_j$

$$g^{-1}\dot{g} = a^j X_j$$

$$U^{-1}(g)\dot{U}(g) = a^j A_j, \quad U(g) \in \mathfrak{u}(\mathcal{H})$$