Hamilton-Poincaré field equations

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In honor of Janusz Grabowski

Work in progress with S Capriotti

Janusz, a good friend!

In my home

A big man in a big mountain!

(Teide, Tenerife)

From the Canary group: Congratulations!!

Generalized versions of Tulczyjew triples in first (higher) order classical field theories

K Grabowska, arXiv:1109.2533

K Grabowska, J Grabowski, arXiv:1306.2744

K Grabowska, J Grabowski, P Urbanski, arXiv:1401.6970

K Grabowska, L Vitagliano, arXiv:1406.6503

A motivation

Some interesting systems may be formulated as symmetric Hamiltonian classical field theories

- Classical Poisson sigma models
- Spin glasses (F Gay-Balmaz et al, 2010)
- Charged Molecular strands (David C P Ellis et al, 2010)
- Minimal inmersions in a Riemannian manifold (F Gay-Balmaz et al, 2011)

Plan of the talk

- Prolongations of a vertical vector field and Hamilton-deDonder-Weyl equations
- Prolongations of an invariant vertical vector field and Hamilton-Poincaré field equations
- An example: Minimal inmersions in a Riemannian manifold

M an oriented manifold; vol a volume form on M,
$m = dim M$

(N, g) a Riemannian manifold

$\pi = pr_1 : P = M \times N \to M$ the trivial fibration

$$J^1\pi = T^*M \otimes TN$$

$$\mathcal{U} = \{j^1_x\sigma \;/\; \sigma : U \subseteq M \to N \text{ is a local inmersion, } x \in U\} \subseteq J^1\pi$$

The Lagrangian function

$$L : \mathcal{U} \to \mathbb{R}, \quad L(j^1_x\sigma)\mathrm{vol}(x) = \mathrm{vol}_{\sigma^*(g)}(x)$$

Local expression

$$\mathrm{vol} = d^m x \Rightarrow L(x_i, u^\alpha, u^\alpha_i) = \sqrt{det(g_{\alpha\beta} u^\alpha_i u^\beta_j)}$$

$$\mathcal{V} \subseteq \mathcal{M}^0\pi = TM \otimes T^*N$$

$$\mathcal{V} = \cup_{x\in M, y\in N}\{\psi \in Lin(T_x^*M, T_y^*N) \;/\; \psi \text{ is a monomorphism }\}$$

The Legendre transformation

$$\mathbb{F}L : \mathcal{U} \to \mathcal{V}, \;\; \mathbb{F}L(j_x^1\sigma) = L(j_x^1\sigma)(\flat_{g(\sigma(x))} \circ j_x^1\sigma \circ \sharp_{g(x)})$$

is a diffeomorphism (for $m \geq 2$).

The Hamiltonian function on $\mathcal{V} \subseteq TM \otimes T^*N$

$$H : \mathcal{V} \to \mathbb{R}, \quad \psi \to (m-1)(det(N(\psi)))^{\frac{1}{2(m-1)}}$$

with $N(\psi) = g(\psi(e^i), \psi(e^j))$ if $\mathrm{vol}(x) = e^1 \wedge \cdots \wedge e^m$.

The local expression

$$H(x_i, u^\alpha, p^i_\alpha) = (m-1)(det(g^{\alpha\beta} p^i_\alpha p^j_\beta))^{\frac{1}{2(m-1)}}$$

$$\mathcal{M}\pi = \mathbb{R} \times (TM \otimes T^*N)$$

The extended Hamiltonian function on $\mu^{-1}\mathcal{V} \subseteq \mathcal{M}\pi$

$$F_h : \mu^{-1}\mathcal{V} \to \mathbb{R}, \quad (p, \psi) \to -p - (m-1)(det(N(\psi)))^{\frac{1}{2(m-1)}}$$

$\tilde{\pi}_N : \mathcal{V} \subseteq TM \otimes T^*N \to N$ the canonical projection

Remark: $s : U \subseteq M \to \mathcal{V} \subseteq TM \otimes T^*N \Rightarrow \tilde{\pi}_N \circ s : U \subseteq M \to N$ is an inmersion

H-D-W equations

$s : U \subseteq M \to \mathcal{V} \subseteq TM \otimes T^*N$ is a solution of the Hamilton-deDonder-Weyl equations

$$\Updownarrow$$

$$(V^{1*}(-p - H) \circ h \circ s)\Omega = (h \circ s)^*(d\tilde{V}), \ \ \forall V \in \mathfrak{X}(N) \qquad (1)$$

- V is tangent to $S = (\tilde{\pi}_N \circ s)(M)$

$$\Downarrow$$

Eq (1) holds

- V is normal to $S = (\tilde{\pi}_N \circ s)(M)$

$$\Downarrow$$

Eq (1) holds $\Leftrightarrow$ the mean curvature 1-form of S is zero in the direction of V

H-D-W equations

$s : U \subseteq M \to \mathcal{V} \subseteq TM \otimes T^*N$ is a solution of the Hamilton-deDonder-Weyl equations

$$\Updownarrow$$

$\tilde{\pi}_N \circ s$ is a minimal inmersion on (N, g)

G is a Lie subgroup of the isometry group of (N, g)

Free and proper action of G on N

$$\Downarrow$$

- There exists a Riemannian metric $\hat{g}$ on $\hat{N} = N/G$ such that $\pi_N : N \to \hat{N} = N/G$ is a Riemannian submersion
- The Hamiltonian function is G-invariant

Two problems

- To discuss the relation between the minimal submanifolds of $(\hat{N}, \hat{g})$ and the projections to $\hat{N}$ of the solutions $\hat{s} : M \to \widehat{\mathcal{M}^0\pi} = \mathcal{M}^0\pi/G = TM \otimes T^*N/G$ of the H-P field equations
- To develop a reconstruction process: from the solutions of the H-P field eqs to obtain minimal submanifolds of (N, g).

Remark: Probably, everything will depend on the configuration tensors of the Riemannian submersion $\pi_N : N \to \hat{N} = N/G$

THANKS!

HAPPY BIRTHDAY JANUSZ!