Jets and Fields on Lie Algebroids

Geometry of Jets and Fields

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Mechanics on Lie algebroids

(Weinstein 1996, Martínez 2001, ...)

Lie algebroid $E \to M$.

$L \in C^\infty(E)$ or $H \in C^\infty(E^*)$

- $E = TM \to M$ Standard classical Mechanics
- $E = \mathfrak{D} \subset TM \to M$ (integrable) System with holonomic constraints
- $E = TQ/G \to M = Q/G$ System with symmetry
- $E = \mathfrak{g} \to \{e\}$ System on Lie algebras
- $E = M \times \mathfrak{g} \to M$ System on a semidirect products (ej. heavy top)

Symplectic and variational

The theory is symplectic:

$$i_\Gamma \omega_L = dE_L$$

with $\omega_L = -d\theta_L$, $\theta_L = S(dL)$ and $E_L = d_\Delta L - L$.

Here d is the differential on the Lie algebroid $\tau^E_E : \mathcal{T}^E E \to E$.

It is also a variational theory:

- Admissible curves or E-paths
- Variations are E-homotopies
- Infinitesimal variations are

$$\delta x^i = \rho^i_\alpha \sigma^\alpha$$
$$\delta y^\alpha = \dot{\sigma}^\alpha + C^\alpha_{\beta\gamma} y^\beta \sigma^\gamma$$

Time dependent systems

(Martínez, Mestdag and Sarlet 2002)

With suitable modifications one can describe time-dependent systems.

Cartan form

$$\Theta_L = S(dL) + L\,dt.$$

Dynamical equation

$$i_\Gamma\, d\Theta_L = 0 \qquad \text{and} \qquad \langle \Gamma, dt \rangle = 1.$$

Field theory in 1-d space-time

Affgebroids

Martinez, Mestdag and Sarlet 2002

Grabowska, Grabowski and Urbanski 2003

Example: standard case

$$
\begin{array}{ccc}
TM & \xrightarrow{T\pi} & TN \\
\downarrow & & \downarrow \\
M & \xrightarrow[\pi]{} & N
\end{array}
$$

$m \in M$ and $n = \pi(m)$

$$0 \longrightarrow \mathrm{Ver}_m \longrightarrow T_m M \longrightarrow T_n N \longrightarrow 0$$

Set of splittings: $J_m\pi = \{\, \phi\colon T_nN \to T_mM \mid T\pi \circ \phi = \mathrm{id}_{T_nN} \,\}$.

Lagrangian: $L\colon J\pi \to \mathbb{R}$

Example: principal bundle

$$\begin{array}{ccc} TQ/G & \xrightarrow{[T\pi]} & TM \\ \downarrow & & \downarrow \\ Q/G = M & \xrightarrow[\mathrm{id}]{} & M \end{array}$$

$m \in M$

$$0 \longrightarrow \mathrm{Ad}_m \longrightarrow (TQ/G)_m \longrightarrow T_m M \longrightarrow 0$$

Set of splittings: $C_m(\pi)$.

Lagrangian: $L\colon C(\pi) \to \mathbb{R}$

General case

Consider

$$\begin{array}{ccc} E & \xrightarrow{\overline{\pi}} & F \\ \downarrow & & \downarrow \\ M & \xrightarrow[\underline{\pi}]{} & N \end{array}$$

with $\pi = (\overline{\pi}, \underline{\pi})$ epimorphism.

Consider the subbundle $K = \ker(\pi) \to M$.

For $m \in M$ and $n = \underline{\pi}(m)$ we have

$$0 \longrightarrow K_m \longrightarrow E_m \longrightarrow F_n \longrightarrow 0$$

and we can consider the set of splittings of this sequence.

We define the sets

$$\mathcal{L}_m\pi = \{\, w\colon F_n \to E_m \mid w \text{ is linear} \,\}$$
$$\mathcal{J}_m\pi = \{\, \phi \in \mathcal{L}_m\pi \mid \overline{\pi} \circ \phi = \mathrm{id}_{F_n} \,\}$$
$$\mathcal{V}_m\pi = \{\, \psi \in \mathcal{L}_m\pi \mid \overline{\pi} \circ \psi = 0 \,\}.$$

Projections

$\underline{\tilde{\pi}_{10}}\colon \mathcal{L}\pi \to M$ vector bundle

$\underline{\pi_{10}}\colon \mathcal{J}\pi \to M$ affine subbundle

$\underline{\boldsymbol{\pi}_{10}}\colon \mathcal{V}\pi \to M$ vector subbundle

Local expressions

Take $\{e_a, e_\alpha\}$ adapted basis of $\mathrm{Sec}(E)$, i.e. $\{\overline{\pi}(e_a) = \bar{e}_a\}$ is a basis of $\mathrm{Sec}(F)$ and $\{e_\alpha\}$ basis of $\mathrm{Sec}(K)$. Also take adapted coordinates (x^i, u^A) to the bundle $\pi\colon M \to N$.

An element of $\mathcal{L}\pi$ is of the form

$$w = (y_a^b e_b + y_a^\alpha e_\alpha) \otimes e^a$$

Thus we have coordinates $(x^i, u^A, y_a^b, y_a^\alpha)$ on $\mathcal{L}\pi$.

An element of $\mathcal{J}\pi$ is of the form

$$\phi = (e_a + y_a^\alpha e_\alpha) \otimes e^a$$

Thus we have coordinates (x^i, u^A, y_a^α) on $\mathcal{J}\pi$.

Anchor

We will assume that F and E are Lie algebroids and π is a morphism of Lie algebroids.

$$\rho(\bar{e}_a) = \rho^i_a \frac{\partial}{\partial x^i}$$
$$\rho(e_a) = \rho^i_a \frac{\partial}{\partial x^i} + \rho^A_a \frac{\partial}{\partial u^A}$$
$$\rho(e_\alpha) = \rho^A_\alpha \frac{\partial}{\partial u^A}$$

Total derivative with respect to a section $\eta \in \mathrm{Sec}(F)$

$$\widehat{df \otimes \eta} = \acute{f}_{|a} \eta^a.$$

where

$$\acute{f}_{|a} = \rho^i_a \frac{\partial f}{\partial x^i} + (\rho^A_a + \rho^A_\alpha y^\alpha_a) \frac{\partial f}{\partial u^A}.$$

Bracket

Since π is a morphism

$$[\bar{e}_a, \bar{e}_b] = C^a_{bc}\bar{e}_a$$

$$[e_a, e_b] = C^\gamma_{ab}e_\gamma + C^a_{bc}e_a$$

$$[e_a, e_\beta] = C^\gamma_{a\beta}e_\gamma$$

$$[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta}e_\gamma$$

Affine structure functions:

$$Z^\alpha_{a\gamma} = (d_{e_\gamma}\widehat{e^\alpha)} \otimes \bar{e}_a = C^\alpha_{a\gamma} + C^\alpha_{\beta\gamma}y^\beta_a$$

$$Z^\alpha_{ac} = (d_{e_c}\widehat{e^\alpha)} \otimes \bar{e}_a = C^\alpha_{ac} + C^\alpha_{\beta c}y^\beta_a$$

$$Z^b_{a\gamma} = (d_{e_\gamma}\widehat{e^b)} \otimes \bar{e}_a = 0$$

$$Z^b_{ac} = (d_{e_c}\widehat{e^b)} \otimes \bar{e}_a = C^b_{ac}$$

Variational Calculus

Only for $F = TN$.

Let ω be a fixed volume form on N.

Variational problem: Given a function $L \in C^\infty(\mathcal{J}\pi)$ find those morphisms $\Phi\colon F \to E$ of Lie algebroids which are sections of π and are critical points of the action functional

$$\mathcal{S}(\Phi) = \int_N L(\Phi)\,\omega$$

Variations

A **homotopy** is a morphism of Lie algebroids ,

$$
\begin{array}{ccc}
TI \times F & \xrightarrow{\Psi} & E \\
\downarrow & & \downarrow \\
I \times N & \xrightarrow[\varphi]{} & M
\end{array}
$$

where $I = [0, 1]$, such that $\overline{\pi} \circ \Psi = \mathrm{pr}_2$, satisfying some boundary conditions.

For every $s \in I = [0, 1]$ define the maps

- $\varphi_s \colon N \to M$ by $\varphi_s(n) = \varphi(s, n)$.
- $\phi_s \colon N \to \mathcal{J}\pi$, section of $\pi_1 : \mathcal{J}\pi \to N$ along φ_s by

$$\phi_s(n)(a) = \Psi(0_s, a) \qquad \text{for all } n \in N \text{ and all } a \in F_n.$$

- $\sigma_s \colon N \to E$, section of $E \to N$ along φ_s by

$$\sigma_s(n) = \Psi\left(\frac{\partial}{\partial s}\bigg|_s, 0_n \right)$$

In this way

$$\Psi(\lambda \frac{\partial}{\partial s}\Big|_s, a_n) = \phi_s(a_n) + \lambda \sigma_s(n).$$

- **Interpretation:**
  - ϕ_s is a 1-parameter family of jets, and we say that ϕ_0 is homotopic to ϕ_1
  - σ_s is the section that controls the variation ϕ_s
- **Boundary conditions:**
  - σ_s with compact support.
- **Variational vector field:**

$$\frac{d}{ds}\phi_s(n)\Big|_{s=0} = \rho^A_\alpha \sigma^\alpha \frac{\partial}{\partial u^A} + \left(\sigma^\alpha_{,a} + Z^\alpha_{a\gamma}\sigma^\gamma\right) \frac{\partial}{\partial y^\alpha_a}.$$

Two consequences

- Variations are of the form

$$\delta u^A = \rho^A_\alpha \sigma^\alpha$$
$$\delta y^\alpha_a = \sigma^\alpha_{,a} + Z^\alpha_{a\beta}\sigma^\beta.$$

where σ^α have compact support.

- ϕ_s **is a morphism of Lie algebroids for every** $s \in [0, 1]$**.**

Variational problem

Only for $F = TN$.

Let ω be a fixed volume form on N.

Variational problem: Given a function $L \in C^\infty(\mathcal{J}\pi)$ find those sections $\Phi\colon F \to E$ of π which are a morphism of Lie algebroids and are critical points of the action

$$\mathcal{S}(\Phi) = \int_N L(\Phi)\,\omega$$

Euler-Lagrange equations

Infinitesimal admissible variations are

$$\delta u^A = \rho^A_\alpha \sigma^\alpha$$

$$\delta y^\alpha_a = \sigma^\alpha_{,a} + Z^\alpha_{a\beta}\sigma^\beta .$$

Integrating by parts we get the Euler-Lagrange equations

$$\frac{d}{dx^a}\left(\frac{\partial L}{\partial y^\alpha_a}\right) = \frac{\partial L}{\partial y^\gamma_a} Z^\gamma_{a\alpha} + \frac{\partial L}{\partial u^A}\rho^A_\alpha ,$$

$$u^A_{,a} = \rho^A_a + \rho^A_\alpha y^\alpha_a$$

$$\left(y^\alpha_{a,b} + C^\alpha_{b\gamma} y^\gamma_a\right) - \left(y^\alpha_{b,a} + C^\alpha_{a\gamma} y^\gamma_b\right) + C^\alpha_{\beta\gamma} y^\beta_b y^\gamma_a + y^\alpha_c C^c_{ab} + C^\alpha_{ab} = 0.$$

Prolongation

Given a Lie algebroid $\tau\colon E \to M$ and a submersion $\mu\colon P \to M$ we can construct the E-tangent to P (the prolongation of P with respect to E). It is the vector bundle $\tau^E_P\colon \mathcal{T}^E P \to P$ where the fibre over $p \in P$ is

$$\mathcal{T}^E_p P = \{\, (b, v) \in E_m \times T_p P \mid T\mu(v) = \rho(b) \,\}$$

where $m = \mu(p)$.

Redundant notation: (p, b, v) for the element $(b, v) \in \mathcal{T}^E_p P$.

The bundle $\mathcal{T}^E P$ can be endowed with a structure of Lie algebroid. The anchor $\rho^1\colon \mathcal{T}^E P \to TP$ is just the projection onto the third factor $\rho^1(p, b, v) = v$. The bracket is given in terms of projectable sections (σ, X), (η, Y)

$$[(\sigma, X), (\eta, Y)] = ([\sigma, \eta], [X, Y]).$$

Local basis

Local coordinates (x^i, u^A) on P and a local basis $\{e_\alpha\}$ of sections of E, define a local basis $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$ of sections of $\mathcal{T}^E P$ by

$$\mathcal{X}_\alpha(p) = \left(p, e_\alpha(\pi(p)), \rho^i_\alpha \frac{\partial}{\partial x^i}\Big|_p\right) \qquad \text{and} \qquad \mathcal{V}_A(p) = \left(p, 0, \frac{\partial}{\partial u^A}\Big|_p\right).$$

The Lie brackets of the elements of the basis are

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = C^\gamma_{\alpha\beta}\, \mathcal{X}_\gamma, \qquad [\mathcal{X}_\alpha, \mathcal{V}_B] = 0 \qquad \text{and} \qquad [\mathcal{V}_A, \mathcal{V}_B] = 0,$$

and the exterior differential is determined by

$$dx^i = \rho^i_\alpha \mathcal{X}^\alpha, \qquad\qquad du^A = \mathcal{V}^A,$$
$$d\mathcal{X}^\gamma = -\frac{1}{2} C^\gamma_{\alpha\beta} \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, \qquad\qquad d\mathcal{V}^A = 0,$$

where $\{\mathcal{X}^\alpha, \mathcal{V}^A\}$ is the dual basis corresponding to $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$.

Prolongation of maps

If $\Psi\colon P \to P'$ is a bundle map over $\varphi\colon M \to M'$ and $\Phi\colon E \to E'$ is a morphism over the same map φ then we can define a morphism $\mathcal{T}^{\Phi}\Psi\colon \mathcal{T}^{E}P \to \mathcal{T}^{E'}P'$ by means of

$$\mathcal{T}^{\Phi}\Psi(p, b, v) = (\Psi(p), \Phi(b), T_p\Psi(v)).$$

In particular, for $P = E$ we have the E-tangent to E

$$\mathcal{T}^{E}_{a}E = \{\, (b, v) \in E_m \times T_aE \;\mid\; T\tau(v) = \rho(b) \,\}\,.$$

Repeated jets

- **E-tangent to $\mathcal{J}\pi$.**

Consider $\tau^E_{\mathcal{J}\pi} \colon \mathcal{T}^E\mathcal{J}\pi \to \mathcal{J}\pi$

$$\mathcal{T}^E\mathcal{J}\pi = \left\{ (\phi, a, V) \in \mathcal{J}\pi \times E \times T\mathcal{J}\pi \;\middle|\; T_\phi\underline{\pi_{10}}(V) = \rho(a) \right\}$$

and the projection $\pi_1 = \pi \circ \pi_{10} = (\overline{\pi} \circ \overline{\pi_{10}}, \underline{\pi} \circ \underline{\pi_{10}})$

$$\begin{array}{ccc} \mathcal{T}^E\mathcal{J}\pi & \xrightarrow{\overline{\pi_1}} & F \\ \downarrow & & \downarrow \\ \mathcal{J}\pi & \xrightarrow[\underline{\pi_1}]{} & N \end{array}$$

A repeated jet $\psi \in \mathcal{J}\pi_1$ at the point $\phi \in \mathcal{J}\pi$ is a map $\psi\colon F_n \to \mathcal{T}^E_\phi \mathcal{J}\pi$ such that $\overline{\pi_1} \circ \psi = \mathrm{id}_{F_n}$.

Explicitely ψ is of the form $\Psi = (\phi, \zeta, V)$ with

- $\phi, \zeta \in \mathcal{J}\pi$ and $V \in T_\phi \mathcal{J}\pi$,
- $\underline{\pi_{10}}(\phi) = \underline{\pi_{10}}(\zeta)$,
- $V\colon F_n \to T_\phi \mathcal{J}\pi$ satisfying

$$T\underline{\pi_{10}} \circ V = \rho \circ \zeta.$$

Locally

$$\psi = (\mathcal{X}_a + \Psi^\alpha_a \mathcal{X}_\alpha + \Psi^\alpha_{ab} \mathcal{V}^b_\alpha) \otimes \bar{e}^a.$$

Contact forms

An element $(\phi, a, V) \in \mathcal{T}^E \mathcal{J}\pi$ is horizontal if $a = \phi(\pi(a))$;

$$Z = a^b(\mathcal{X}_b + y_b^\beta \mathcal{X}_\beta) + V_b^\beta \mathcal{V}_\beta^b.$$

An element $\mu \in \mathcal{T}^{*E} \mathcal{J}\pi$ is vertical if it vanishes on horizontal elements.

A **contact 1-form** is a section of $\mathcal{T}^{*E} \mathcal{J}\pi$ which is vertical at every point. They are spanned by

$$\theta^\alpha = \mathcal{X}^\alpha - y_a^\alpha \mathcal{X}^a.$$

The module generated by contact 1-forms is the **contact module** $\mathcal{M}^c$

$$\mathcal{M}^c = \langle \theta^\alpha \rangle.$$

The differential ideal generated by contact 1-forms is the **contact ideal** $\mathcal{I}^c$.

$$\mathcal{I}^c = \langle \theta^\alpha, d\theta^\alpha \rangle$$

Second order jets

- A jet $\psi \in \mathcal{J}_\phi \pi_1$ is **semiholonomic** if $\psi^\star \theta = 0$ for every θ in $\mathcal{M}^c$.

  The jet $\psi = (\phi, \zeta, V)$ is semiholonomic ifand only if $\phi = \zeta$.

- A jet $\psi \in \mathcal{J}_\phi \pi_1$ is **holonomic** if $\psi^\star \theta = 0$ for every θ in $\mathcal{I}^c$.

  The jet $\psi = (\phi, \zeta, V)$ is semiholonomic if and only if $\phi = \zeta$ and $\mathcal{M}^\gamma_{ab} = 0$, where

$$\mathcal{M}^\gamma_{ab} = y^\gamma_{ab} - y^\gamma_{ba} + C^\gamma_{b\alpha} y^\alpha_a - C^\gamma_{a\beta} y^\beta_b - C^\gamma_{\alpha\beta} y^\alpha_a y^\beta_b + y^\gamma_c C^c_{ab} + C^\gamma_{ab}.$$

Jet prolongation of sections

A bundle map $\Phi = (\overline{\Phi}, \underline{\Phi})$ section of π is equivalent to a bundle map $\check{\Phi} = (\check{\overline{\Phi}}, \underline{\Phi})$ from N to $\mathcal{J}\pi$ section of π_1

$$\check{\overline{\Phi}}(n) = \overline{\Phi}\Big|_{F_n}.$$

The jet prolongation of Φ is the section $\Phi^{(1)} \equiv \mathcal{T}^{\Phi}\check{\Phi}$ of π_1.

In coordinates

$$\Phi^{(1)} = (\mathcal{X}_a + \Phi^{\alpha}_{a}\mathcal{X}_{\alpha} + \acute{\Phi}^{\alpha}_{b|a}\mathcal{V}^{b}_{\alpha}) \otimes \bar{e}^{a}.$$

Theorem: Let $\Psi \in \mathrm{Sec}(\pi_1)$ be such that the associated map $\check{\Psi}$ is a semiholonomic section and let $\check{\Phi}$ be the section of $\underline{\pi_1}$ to which it projects. Then

1. The bundle map Ψ is admissible if and only if Φ is admissible and $\Psi = \Phi^{(1)}$.
2. The bundle map Ψ is a morphism of Lie algebroids if and only if $\Psi = \check{\Phi}^{(1)}$ and Φ is a morphism of Lie algebroids.

Corollary: Let Φ an admissible map and a section of π. Then Φ is a morphism if and only if $\Phi^{(1)}$ is holonomic.

Lagrangian formalism

$L \in C^\infty(\mathcal{J}\pi)$ Lagrangian, $\omega \in \bigwedge^r F$ 'volume' form.

- **Canonical form.**

For every $\phi \in \mathcal{J}_m\pi$

$$h_\phi(a) = \phi(\overline{\pi}(a)) \qquad \text{and} \qquad v_\phi(a) = a - \phi(\overline{\pi}(a))$$

They define the map $\vartheta\colon \underline{\pi_{10}}^*E \to \underline{\pi_{10}}^*E$ by

$$\vartheta(\phi, a) = v_\phi(a).$$

Vertical lifting.

As in any affine bundle

$$\psi^V_\phi f = \frac{d}{dt} f(\phi + t\psi)\Big|_{t=0}, \qquad \psi \in \mathcal{V}_m\pi, \quad \phi \in \mathcal{J}_m\pi.$$

Thus we have a map $\xi^V \colon \underline{\pi_{10}}^*(\mathcal{L}\pi) \to \mathcal{T}^E\mathcal{J}\pi$

$$\xi^V(\phi, \varphi) = (\phi, (v_\phi \circ \varphi)^V_\phi).$$

Vertical endomorphism.

Every $\nu \in \mathrm{Sec}(E^*)$ defines $S_\nu \colon \mathcal{T}^E\mathcal{J}\pi \to \mathcal{T}^E\mathcal{J}\pi$

$$S_\nu(\phi, a, V) = \xi^V(\phi, a \otimes \nu) = (\phi, 0, v_\phi(a) \otimes \nu).$$

In coordinates

$$S = \theta^\alpha \otimes \bar{e}_a \otimes \mathcal{V}_\alpha^a.$$

Finally

$$S_\omega = \theta^\alpha \wedge \omega_a \otimes \mathcal{V}_\alpha^a.$$

Cartan forms.

$$\Theta_L = S_\omega(dL) + L\omega$$

$$\Omega_L = -d\Theta_L$$

In coordinates

$$\Theta_L = \frac{\partial L}{\partial y^\alpha_a}\theta^\alpha \wedge \omega_a + L\omega$$

Euler-Lagrange equations.

A solution of the field equations is a morphism $\Phi \in \mathrm{Sec}(\pi)$ such that

$$\Phi^{(1)\star}(i_X \Omega_L) = 0$$

for all π_1-vertical section $X \in \mathrm{Sec}(\mathcal{T}^E \mathcal{J}\pi)$.

More generally one can consider the **De Donder** equations

$$\Psi^\star(i_X \Omega_L) = 0.$$

If L is regular then $\Psi = \Phi^{(1)}$.

In coordinates we get the Euler-Lagrange partial differential equations

$$\acute{u}^A_{|a} = \rho^A_a + \rho^A_\alpha y^\alpha_a$$

$$y^\gamma_{a|b} - y^\gamma_{b|a} + C^\gamma_{b\alpha} y^\alpha_a - C^\gamma_{a\beta} y^\beta_b - C^\gamma_{\alpha\beta} y^\alpha_a y^\beta_b + y^\gamma_c C^c_{ab} + C^\gamma_{ab} = 0$$

$$\left(\frac{\partial L}{\partial y^\alpha_a}\right)'_{|a} + \frac{\partial L}{\partial y^\alpha_a} C^b_{ba} - \frac{\partial L}{\partial y^\gamma_a} Z^\gamma_{a\alpha} - \frac{\partial L}{\partial u^A} \rho^A_\alpha = 0,$$

Standard case

In the standard case, we consider a bundle $\underline{\pi}\colon M \to N$, the standard Lie algebroids $F = TN$ and $E = TM$ and the tangent map $\overline{\pi} = T\underline{\pi}\colon TM \to TN$. Then we have that $\mathcal{J}\pi = J^1\underline{\pi}$.

If we take a (non-coordinate) basis of vector fields, our equations provide an expression of the standard Euler-Lagrange and Hamiltonian field equations written in pseudo-coordinates.

In particular, one can take an Ehresmann connection on the bundle $\underline{\pi}\colon M \to N$ and use an adapted local basis

$$\bar{e}_i = \frac{\partial}{\partial x^i} \qquad \text{and} \qquad \begin{cases} e_i = \dfrac{\partial}{\partial x^i} + \Gamma_i^A \dfrac{\partial}{\partial u^A} \\ e_A = \dfrac{\partial}{\partial u^A}. \end{cases}$$

We have the brackets

$$[e_i, e_j] = -R^A_{ij} e_A, \qquad [e_i, e_B] = \Gamma^A_{iB} e_A \qquad \text{and} \qquad [e_A, e_B] = 0,$$

where we have written $\Gamma^B_{iA} = \partial \Gamma^B_i / \partial u^A$ and where R^A_{ij} is the curvature tensor of the nonlinear connection we have chosen. The components of the anchor are $\rho^i_j = \delta^i_j$, $\rho^A_i = \Gamma^A_i$ and $\rho^A_B = \delta^A_B$ so that the Euler-Lagrange equations are

$$\begin{aligned}
&\frac{\partial u^A}{\partial x^i} = \Gamma^A_i + y^A_i \\
&\frac{\partial y^A_i}{\partial x^j} - \frac{\partial y^A_j}{\partial x^i} + \Gamma^A_{jB} y^B_i - \Gamma^A_{iB} y^B_j = R^A_{ij} \\
&\frac{d}{dx^i}\left(\frac{\partial L}{\partial y^A_i}\right) - \Gamma^B_{iA} \frac{\partial L}{\partial y^B_i} = \frac{\partial L}{\partial u^A}.
\end{aligned}$$

Time-dependent Mechanics

Consider a Lie algebroid $\tau_M^E \colon E \to M$ and the standard Lie algebroid $\tau_{\mathbb{R}} \colon T\mathbb{R} \to \mathbb{R}$. We consider the Lie subalgebroid $K = \ker(\pi)$ and define

$$A = \left\{ a \in E \;\middle|\; \overline{\pi}(a) = \frac{\partial}{\partial t} \right\}.$$

Then A is an affine subbundle modeled on K and the 'bidual' of A is $(A^\dagger)^* = E$. Moreover, the Lie algebroid structure on E defines by restriction a Lie algebroid structure on the affine bundle A (i.e. an affgebroid).

Conversely, let A be an affine bundle with a Lie algebroid structure. Then the vector bundle $E \equiv (A^\dagger)^*$ has an induced Lie algebroid structure. If $\tilde{\rho}$ is the anchor of this bundle then the map $\overline{\pi}$ defined by $\overline{\pi}(z) = T\pi(\tilde{\rho}(z))$ is a morphism. Moreover we have that $A = \left\{ a \in E \;\middle|\; \overline{\pi}(a) = \frac{\partial}{\partial t} \right\}$ as above.

We have a canonical identification of A with $\mathcal{J}\pi$.

The morphism condition is just the admissibility condition so that the Euler-Lagrange equations are

$$\frac{du^A}{dt} = \rho_0^A + \rho_\alpha^A y^\alpha$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) = \frac{\partial L}{\partial y^\gamma}(C_{0\alpha}^\gamma + C_{\beta\alpha}^\gamma y^\beta) + \frac{\partial L}{\partial u^A}\rho_\alpha^A,$$

where we have written $x^0 \equiv t$ and $y_0^\alpha \equiv y^\alpha$.

Example: The autonomous case

We have two Lie algebroids $\tau_N^F\colon F \to N$ and $\tau_Q^G\colon G \to Q$ over different bases and we set $M = N \times Q$ and $E = F \times G$, where the projections are both the projection over the first factor $\underline{\pi}(n, q) = n$ and $\overline{\pi}(a, k) = a$. The anchor is the sum of the anchors and the bracket is determined by the brackets of sections of F and G (a section of F commutes with a section of G). We therefore have that

$$\rho_a^\alpha = 0, \qquad C_{ab}^\alpha = 0 \qquad \text{and} \qquad C_{a\beta}^\alpha = 0.$$

A jet at a point (n, q) is of the form $\phi(a) = (a, \zeta(a))$, for some map $\zeta\colon F_n \to G_q$. We can identify $\mathcal{J}\pi$ with the set of linear maps from a fibre of F to a fibre of G.

This is further justified by the fact that a map $\Phi\colon F \to G$ is a morphism of Lie algebroids if and only if the section $(\mathrm{id}, \Phi)\colon F \to F \times G$ of π is a morphism of Lie algebroids.

The affine functions $Z^\gamma_{a\alpha}$ reduce to $Z^\gamma_{a\alpha} = C^\gamma_{\beta\alpha} y^\beta_a$ and thus the Euler-Lagrange equations are

$$\left(\frac{\partial L}{\partial y^\alpha_a}\right)'_{|a} + C^b_{ba}\left(\frac{\partial L}{\partial y^\alpha_a}\right) = \frac{\partial L}{\partial y^\gamma_a} C^\gamma_{\beta\alpha} y^\beta_a + \frac{\partial L}{\partial u^A}\rho^A_\alpha.$$

In the more particular and common case where $F = TN$ we can take a coordinate basis, so that we also have $C^c_{ab} = 0$. Therefore the Euler-Lagrange partial differential equations are

$$\begin{aligned}
&\frac{\partial u^A}{\partial x^a} = \rho^A_\alpha y^\alpha_a \\
&\frac{d}{dx^a}\left(\frac{\partial L}{\partial y^\alpha_a}\right) = \frac{\partial L}{\partial y^\gamma_a} C^\gamma_{\beta\alpha} y^\beta_a + \frac{\partial L}{\partial u^A}\rho^A_\alpha, \\
&\frac{\partial y^\alpha_a}{\partial x^b} - \frac{\partial y^\alpha_b}{\partial x^a} + C^\alpha_{\beta\gamma} y^\beta_b y^\gamma_a = 0,
\end{aligned}$$

Autonomous Classical Mechanics

When moreover $F = T\mathbb{R} \to \mathbb{R}$ then we recover Weinstein's equations for a Lagrangian system on a Lie algebroid

$$\begin{aligned} \frac{du^A}{dt} &= \rho^A_\alpha y^\alpha \\ \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) &= \frac{\partial L}{\partial y^\gamma} C^\gamma_{\beta\alpha} y^\beta + \frac{\partial L}{\partial u^A}\rho^A_\alpha, \end{aligned}$$

where, as before, we have written $x^0 \equiv t$ and $y^\alpha_0 \equiv y^\alpha$.

Example: Chern-Simons

Let $\mathfrak{g}$ be a Lie algebra with an ad-invariant metric $\boldsymbol{k}$.

$\{\epsilon_\alpha\}$ basis of $\mathfrak{g}$ and $C^\alpha_{\beta\gamma}$ the structure constants

The symbols $C_{\alpha\beta\gamma} = k_{\alpha\mu} C^\mu_{\beta\gamma}$ are skewsymmetric.

Let N be a 3-dimensional manifold and consider the Lie algebroid $E = TN \times \mathfrak{g} \to N$

$$\tau(v_n, \xi) = n \qquad \rho(v_n, \xi) = v_n \qquad [(X, \xi), (Y, \zeta)] = ([X, Y], [\xi, \zeta]).$$

A basis for sections of E is given by $e_\alpha(n) = (n, \epsilon_\alpha)$.

As before $F = TN \to N$, and $\overline{\pi}(v_n, \xi) = v_n$ and $\underline{\pi} = \mathrm{id}_N$.

A section Φ of π is of the form $\Phi(v) = (v, A^\alpha(v)\epsilon_\alpha)$ for some 1-forms A^α on N.
In other words $\Phi^\star e^\alpha = A^\alpha = y^\alpha_a dx^a$.

The Lagrangian density for Chern-Simons theory is

$$L\,dx^1 \wedge dx^2 \wedge dx^3 = \frac{1}{3!}\,C_{\alpha\beta\gamma}\,A^\alpha \wedge A^\beta \wedge A^\gamma.$$

in other words $L = C_{\alpha\beta\gamma} y_1^\alpha y_2^\beta y_3^\gamma$.

No admissibility conditions (no coordinates u^A).

Morphism conditions $\acute{y}_{i|j}^\alpha - \acute{y}_{j|i}^\alpha + C^\alpha_{\beta\gamma} y_j^\beta y_i^\gamma = 0$, can be written

$$dA^\alpha + \frac{1}{2} C^\alpha_{\beta\gamma} A^\beta \wedge A^\gamma = 0.$$

The Euler-Lagrange equations reduce to

$$\begin{aligned}\frac{d}{dx^a}\frac{\partial L}{\partial y_a^\alpha} - \frac{\partial L}{\partial y_a^\gamma} C^\gamma_{\beta\alpha} y_a^\beta = C_{\alpha\beta\gamma}\Big[\quad & (y^\beta_{2|1} - y^\beta_{1|2} + C^\beta_{\mu\nu} y_1^\mu y_2^\nu) y_3^\gamma + \\ & + (y^\beta_{1|3} - y^\beta_{3|1} + C^\beta_{\mu\nu} y_3^\mu y_1^\nu) y_2^\gamma + \\ & + (y^\beta_{3|2} - y^\beta_{2|3} + C^\beta_{\mu\nu} y_2^\mu y_3^\nu) y_1^\gamma \quad \Big] = 0,\end{aligned}$$

which vanish identically in view of the morphism condition.

The conventional Lagrangian density for the Chern-Simons theory is

$$L'\omega = k_{\alpha\beta}\left(A^{\alpha}\wedge dA^{\beta}+\frac{1}{3}C^{\beta}_{\mu\nu}A^{\alpha}\wedge A^{\mu}\wedge A^{\nu}\right),$$

and the difference between L' and L is a multiple of the morphism condition

$$L'\omega - L\omega = k_{\alpha\mu}A^{\mu}\left[dA^{\alpha}+\frac{1}{2}C^{\alpha}_{\beta\gamma}A^{\beta}\wedge A^{\gamma}\right].$$

Therefore both Lagrangians coincide on the set $\mathcal{M}(\pi)$ of morphisms, which is the set where the action is defined.

Example: Poisson Sigma model

As an example of autonomous theory, we consider a 2-dimensional manifold N and it tangent bundle $F = TN$. On the other hand, consider a Poisson manifold (Q, Λ). Then the cotangent bundle $G = T^*Q$ has a Lie algebroid structure, where the anchor is $\rho(\sigma) = \Lambda(\sigma, \cdot\,)$ and the bracket is $[\sigma, \eta] = d^{TQ}_{\rho(\sigma)}\eta - d^{TQ}_{\rho(\eta)}\sigma - d^{TQ}\Lambda(\sigma, \eta)$, where d^{TQ} is the ordinary exterior differential on Q.

The Lagrangian density for the Poisson Sigma model is $\mathcal{L}(\phi) = -\frac{1}{2}\phi^\star\Lambda$. In coordinates (x^1, x^2) on N and (u^A) in Q we have that $\Lambda = \frac{1}{2}\Lambda^{JK}\frac{\partial}{\partial u^J} \wedge \frac{\partial}{\partial u^K}$. A jet at the point (n, q) is a map $\phi\colon T_nN \to T^*_qQ$, locally given by $\phi = y_{Ki}du^K \otimes dx^i$. Thus we have local coordinates (x^i, u^K, y_{Ki}) on $\mathcal{J}\pi$. The local expression of the Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\Lambda^{JK}A_J \wedge A_K = -\frac{1}{2}\Lambda^{JK}y_{J1}y_{K2}\,dx^1 \wedge dx^2.$$

where we have written $A_K = \Phi^\star(\partial/\partial u^K) = y_{Ki}dx^i$.

A long but straightforward calculation shows that for the Euler-Lagrange equation

$$\frac{d}{dx^a}\left(\frac{\partial L}{\partial y_a^\alpha}\right) = \frac{\partial L}{\partial y_a^\gamma} C^\gamma_{\beta\alpha} y_a^\beta + \frac{\partial L}{\partial u^A}\rho^A_\alpha$$

the right hand side vanishes while the left hand side reduces to

$$\frac{1}{2}\Lambda^{LJ}\left(y_{L2|1} - y_{L1|2} + \frac{\partial \Lambda^{MK}}{\partial u^L} y_{M1} y_{K2}\right) = 0.$$

In view of the morphism condition, we see that this equation vanishes. Thus the field equations are just

$$\begin{aligned}
&\frac{\partial u^J}{\partial x^a} + \Lambda^{JK} y_{Ka} = 0 \\
&\frac{\partial y_{Ja}}{\partial x^b} - \frac{\partial y_{Jb}}{\partial x^a} + \frac{\partial \Lambda^{KL}}{\partial u^J} y_{Kb} y_{La} = 0,
\end{aligned}$$

or in other words

$$\begin{aligned}
&d\phi^J + \Lambda^{JK} A_K = 0 \\
&dA_J + \frac{1}{2}\Lambda^{KL}_{,J} A_K \wedge A_L = 0.
\end{aligned}$$

The conventional Lagrangian density for the Poisson Sigma model (Strobl) is $\mathcal{L}' = \mathrm{tr}(\overline{\Phi} \wedge T\underline{\Phi}) + \frac{1}{2}\Phi^\star\Lambda$, which in coordinates reads $\mathcal{L}' = A_J \wedge d\phi^J + \frac{1}{2}\Lambda^{JK} A_J \wedge A_K$. The difference between $\mathcal{L}'$ and $\mathcal{L}$ is a multiple of the admissibility condition $d\phi^J + \Lambda^{JK} A_K$;

$$\mathcal{L}' - \mathcal{L} = A_J \wedge (d\phi^J + \Lambda^{JK} A_K).$$

Therefore both Lagrangians coincide on admissible maps, and hence on morphisms, so that the actions defined by them are equal.

In more generality, one can consider a presymplectic Lie algebroid, that is, a Lie algebroid with a closed 2-form Ω, and the Lagrangian density $L = -\frac{1}{2}\Phi^\star\Omega$. The Euler-Lagrange equations vanish as a consequence of the morphism condition and the closure of Ω so that we again get a topological theory. In this way one can generalize the theory for Poisson structures to a theory for Dirac structures.

Hamiltonian formalism

Consider the affine dual of $\mathcal{J}\pi$ considered as the bundle $\underline{\pi_{10}}^\dagger \colon \mathcal{J}^\dagger\pi \to M$ with fibre over m

$$\mathcal{J}^\dagger\pi = \{\, \lambda \in (E_m^*)^{\wedge r} \mid i_{k_1} i_{k_2}\lambda = 0 \text{ for all } k_1, k_2 \in K_m \,\}$$

We have a canonical form Θ in $\mathcal{T}^E\mathcal{J}^\dagger\pi$, given by

$$\Theta_\lambda = (\pi_{10}^\dagger)^\star\lambda.$$

Explicitly

$$\Theta_\lambda(Z_1, Z_2, \ldots, Z_r) = \lambda(a_1, a_2, \ldots, a_r),$$

for $Z_i = (\lambda, a_i, V_i)$.

The differential of Θ is a multisymplectic form

$$\Omega = -d\Theta.$$

For a section h of the projection $\mathcal{J}^\dagger\pi \to \mathcal{V}^*\pi$ we consider the Liouville-Cartan forms

$$\Theta_h = (\mathcal{T}h)^\star\Theta \qquad \text{and} \qquad \Omega_h = (\mathcal{T}h)^\star\Omega$$

We set the Hamilton equations

$$\Lambda^\star(i_X\Omega_h) = 0,$$

for a morphism Λ.

In coordinates we get the Hamiltonian field pdes

$$\acute{u}^A_{|a} = \rho^A_a + \rho^A_\alpha \frac{\partial H}{\partial \mu^a_\alpha}$$

$$\left(\frac{\partial H}{\partial \mu^a_\alpha}\right)'_{|b} - \left(\frac{\partial H}{\partial \mu^b_\alpha}\right)'_{|a} + C^\alpha_{\beta\gamma}\frac{\partial H}{\partial \mu^b_\beta}\frac{\partial H}{\partial \mu^a_\gamma} + C^\alpha_{b\gamma}\frac{\partial H}{\partial \mu^a_\gamma} - C^\alpha_{a\gamma}\frac{\partial H}{\partial \mu^b_\gamma} = C^\alpha_{ab}$$

$$\acute{\mu}^c_{\alpha|c} x^i + \mu^b_\alpha C^c_{bc} = -\rho^A_\alpha \frac{\partial H}{\partial u^A} + \mu^c_\gamma \left(C^\gamma_{c\alpha} + C^\gamma_{\beta\alpha}\frac{\partial H}{\partial \mu^c_\beta}\right).$$

Legendre transformation

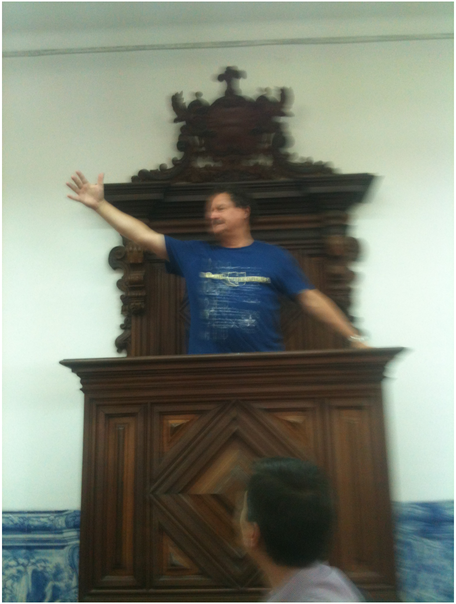
There is a Legendre transformation $\widehat{\mathfrak{F}}_{\mathcal{L}} : \mathcal{J}\pi \to \mathcal{J}^{\dagger}\pi$ defined by affine approximation of the Lagrangian as in the standard case. We have similar results:

- $\Theta_L = (\mathcal{T}\widehat{\mathfrak{F}}_{\mathcal{L}})^{\star}\Theta$
- $\Omega_L = (\mathcal{T}\widehat{\mathfrak{F}}_{\mathcal{L}})^{\star}\Omega$
- For hyperregular Lagrangian L: if Φ is a solution of the Euler-Lagrange equations then $\Lambda = \mathcal{T}\mathfrak{F}_{\mathcal{L}} \circ \Phi^{(1)}$ is a solution of the Hamiltonian field equations. Conversely if Λ is a solution of the Hamiltonian field equations, then there exists a solution Φ of the Euler-Lagrange equations such that $\Lambda = \mathcal{T}\mathfrak{F}_{\mathcal{L}} \circ \Phi^{(1)}$.

For singular systems there is a 'unified Lagrangian-Hamiltonian formalism'.

And of course, we cannot forget ... Tulczyjew triples.

Congratulations Janusz!



ACTIEF INTERIM

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The End