Tulczyjew's approach for particles in gauge fields

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Geometry of Jets and Fields
(in honour of Janusz Grabowski's 60th birthday)
Będlewo, 10-15 May, 2015

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- The Tulczyjew triple $T^*T^*X \xleftarrow{\beta} TT^*X \xrightarrow{\alpha} T^*TX$,
- The canonical isomorphism $T^*E^* \cong T^*E$,

came from a talk by Janusz at a workshop organized by Partha Guha in January 2014.

Thank you very much, Janusz, for sharing the great ideas.

I would also thank the warm receptions I received from Janusz, Paweł, and perhaps some other members of the Polish school.

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- *God always geometrizes.*

  — Plato

- *At any particular moment in the history of science, the most important and fruitful ideas are often lying dormant merely because they are unfashionable.*

  — Freeman J. Dyson

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I believe that Tulczyjew's idea about mechanics is one of these ideas.

- In 1974 Tulczyjew introduced a geometric approach to classical mechanics which brings the Hamiltonian and Lagrangian formalisms under a common geometric roof.
- In this approach the dynamics of a particle with configuration space X is determined by a Lagrangian submanifold D of TT^*X (the total tangent space of T^*X), and the description of D by its Hamiltonian $H\colon T^*X \to \mathbb{R}$ (resp. its Lagrangian $L\colon TX \to \mathbb{R}$) yields the Hamilton (resp. Euler-Lagrange) equation.
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- review Dufour's canonical isomorphism of double vector bundles,
- review Tulczyjew's approach to particle dynamics,
- review Sternberg's phase space,
- introduce an extension of Tulczyjew's approach to dynamics of (charged) particles in gauge fields. This is another demonstration of the simple and powerful idea of Tulczyjew.

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A canonical isomorphism

Theorem (J. P. Dufour, 1990)

Let $E \to X$ be a real vector bundle and $E^* \to X$ be its dual vector bundle. Then $T^*E^* \cong T^*E$ canonically as symplectic manifolds.

- The canonical symplectomorphism is a family version of $V^* \times V^{**} \cong V \times V^*$.
- In Tulczyjew's work, $E \to X$ is $TX \to X$, so $E^* \to X$ is $T^*X \to X$ and we have Tulczyjew isomorphism $\kappa : T^*T^*X \cong T^*TX$.

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Tulczyjew's Insight

Tulczyjew discovered (1974) that TT^*X has

- one symplectic structure: $\mathrm{d}_T\omega_X$
- two Liouville structures: $\mathrm{d}_T\theta_X$, $\mathrm{i}_T\omega_X$

as one can see from the Tulczyjew's triangle

$$\begin{array}{ccccc} T^*T^*X & & \xleftarrow{\beta} & & TT^*X \\ & \searrow \kappa & & \swarrow \alpha & \\ & & T^*TX & & \end{array}$$

$\alpha^*\theta_{TX} = \mathrm{d}_T\theta_X$ and $\beta^*\theta_{T^*X} = \mathrm{i}_T\omega_X$. The Lagrangian L: $TX \to \mathbb{R}$ defines a Lagrangian sub manifold $D_L := \mathrm{Im}(\mathrm{d}L)$ and the Hamiltonian H: $T^*X \to \mathbb{R}$ defines a Lagrangian sub manifold $D_H := \mathrm{Im}(-\mathrm{d}H)$. Fact: $\alpha^{-1}(D_L) = \beta^{-1}(D_H)$ if L and H are related by the Legendre transformation. In general, a dynamics is just a Lagrangian sub manifold of TT^*X, which may or may not have a Lagrangian or a Hamiltonian.

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Let c: $I \to T^*X$ be a smooth map, c' be its tangent lift

$$\begin{array}{ccc} & & TT^*X \\ & \overset{c'}{\nearrow} & \downarrow \\ I & \overset{c}{\longrightarrow} & T^*X \end{array}$$

The dynamics is given by a Lagrangian sub-manifold D of TT^*X, where D could be

$$\beta^{-1}(D_H) = \{(q, p, \dot{q}, \dot{p}) : \dot{p} = -\frac{\partial H}{\partial q},\ \dot{q} = \frac{\partial H}{\partial p}\}$$

or

$$\alpha^{-1}(D_L) = \{(q, p, \dot{q}, \dot{p}) : p = \frac{\partial L}{\partial \dot{q}},\ \dot{p} = \frac{\partial L}{\partial q}\}$$

in the sense that the equation of motion can be stated as follows: *the image of c' is inside D*.

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A technical setup

G	a compact connected Lie group
$\mathfrak{g},\ \mathfrak{g}^*$	the Lie algebra of G and its dual
$P \to X$	a principal G-bundle over X
Θ	a fixed principal connection form
F	a Hamiltonian G-space
Ω	the symplectic form on F
$\Phi : F \to \mathfrak{g}^*$	the G-equivariant moment map
$\mathcal{F} \to X$	the associated fiber bundle with fiber F
$\mathcal{F}^\sharp$	the limit of diagram $T^*X \to X \leftarrow \mathcal{F}$

For notational sanity here, we shall use the same notation for both a differential form (or a map) and its pullback under a fiber bundle projection map.

Sternberg Phase Space

Theorem (Sternberg, 1977)

- *There is a closed real differential two-form Ω_Θ on $\mathcal{F}$ which is of the form $\Omega - \mathrm{d}\langle A, \Phi\rangle$ under a local trivialization of $P \to X$ in which the connection form Θ is represented by the $\mathfrak{g}$-valued differential one-form A on X.*
- *The differential two-form $\omega_\Theta := \omega_X + \Omega_\Theta$ is a symplectic form on $\mathcal{F}^\sharp$, where ω_X is the canonical symplectic form on T^*X, pulled back under $\mathcal{F}^\sharp \to T^*X$, and Ω_Θ is the pullback of Ω_Θ under $\mathcal{F}^\sharp \to \mathcal{F}$.*

- Ω_Θ is the right substitute for Ω when we go from a product bundle with the product connection to a generic bundle.
- If $G = \mathrm{U}(1)$, then $(\mathcal{F}^\sharp, \omega_\Theta) = (T^*X, \omega_X - q_e\,\mathrm{d}A)$ where q_e is the electric charge of the particle.
- In the Hamiltonian formalism, as shown by Sternberg and others, the Sternberg phase space $(\mathcal{F}^\sharp, \omega_\Theta)$ is the right substitute for (T^*X, ω_X) when particles move in a background gauge field.

What is Missing?

- The Lagrangian side of Sternberg's work is missing.
- Tulczyjew's approach for particles in gauge fields is missing.

Since both Sternberg's work and Tulczyjew's work are quite natural, there should be a very natural setting to combine them.
A further setup

$$\mathcal{F}_\sharp \quad \text{the limit of diagram } TX \to X \leftarrow \mathcal{F}$$

$$\begin{array}{ccccc} \mathcal{F}^\sharp & \to & \mathcal{F} & \leftarrow & \mathcal{F}_\sharp \\ \downarrow & & \downarrow & & \downarrow \\ T^*X & \to & X & \leftarrow & TX \end{array}$$

Note that $\mathcal{F}_\sharp \to \mathcal{F}$ is a real vector bundle and its dual is vector bundle $\mathcal{F}^\sharp \to \mathcal{F}$. So $T^*\mathcal{F}^\sharp \cong T^*\mathcal{F}_\sharp$ by Dufour's theorem. So we arrive at a magnetized version of the Tulczyjew triple:

$$T^*\mathcal{F}^\sharp \xleftarrow{\beta_M} T\mathcal{F}^\sharp \xrightarrow{\alpha_M} T^*\mathcal{F}_\sharp$$

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Tulczyjew's Approach for Particles in Gauge Fields

Let $c\colon I \to \mathcal{F}^\sharp$ be a smooth map, c' be its tangent lift

$$\begin{array}{ccc} & & T\mathcal{F}^\sharp \\ & \overset{c'}{\nearrow} & \downarrow \\ I & \xrightarrow{c} & \mathcal{F}^\sharp \end{array}$$

- $T\mathcal{F}^\sharp$ (the substitute of TT^*X) has one symplectic structure and two Liouville structures.
- A dynamics is just a Lagrangian submanifold D of $T\mathcal{F}^\sharp$.
- The Lagrangian for D, if it exists, is a real function L on a submanifold J of $\mathcal{F}_\sharp$. (It is an unconstrained system if $J = \mathcal{F}_\sharp$.)
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- The Hamiltonian side for unconstrained systems is equivalent to Sternberg's work.

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The Lagrangian side, even for unconstrained systems, seems to be new: locally we have

$$\boxed{\begin{array}{rcl} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}^i}\right) & = & \dfrac{\partial L}{\partial q^i} + \dfrac{dq^j}{dt}\langle F_{ji}, \Phi\rangle + \{L, \langle A_i, \Phi\rangle\}_F \\ \dfrac{Dz}{dt} \lrcorner \Omega & = & \partial_F L \end{array}} \tag{1}$$

provided that F is a homogeneous Hamiltonian G-space. Here L: $\mathcal{F}_\sharp \to \mathbb{R}$ is a Lagrangian, and

$$\{f, g\}_F := \Omega^{\alpha\beta} \frac{\partial f}{\partial z^\alpha} \frac{\partial g}{\partial z^\beta}, \quad \partial_F L := \frac{\partial L}{\partial z^\alpha}\, \mathrm{d}z^\alpha.$$

- There is a charge quantization condition: $\frac{\Omega_\Theta}{2\pi}$ represents an integral lattice point of the 2nd cohomology group of $\mathcal{F}$ with real coefficient. This generalizes Dirac's charge quantization condition: $\frac{q_{\mathrm{e}} q_{\mathrm{m}}}{\hbar c} \in \frac{1}{2}\mathbb{Z}$. That is because p: $\mathcal{F}^\sharp \to \mathcal{F}$ is a homotopy equivalence and ω_X is exact, so

$$[\frac{\omega_X + \Omega_\Theta}{2\pi}] = [\frac{\Omega_\Theta}{2\pi}] \in H^2(\mathcal{F}^\sharp, \mathbb{R}).$$

- A great advantage of this formalism, as already demonstrated in the literature, is the study of constrained system.

Happy Birthday, Janusz!