Geometry of PDEs and Hamiltonian systems

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Bedlewo, May, 2015

Lagrangian and Hamiltonian systems in jet bundles

unique treatment of general Lagrangian systems

time independent and time dependent

regular and degenerate

first order and higher order

mechanics and field theory

Aldaya–de Azcárraga, Carathéodory, Cariñena, Crampin,
De Donder, de León et al., Dedecker,
Echeverría-Enríques–Muñoz-Lecanda–Román-Roy,
Ferraris–Francaviglia, Forger et al., Garcia–Muñoz,
Giachetta–Mangiarotti–Sardanashvily, Goldschmidt–Sternberg,
Gotay, Grabowski, Grabowska, Ibort, Kanatchikov, Kastrup,
Kolář, Krupka, Krupková-Rossi, Lepage, Marle, Marsden,
Massa–Pagani, McLean–Norris, Marrero, Olver, Saunders,
Shadwick, Tulczyjew, Vinogradov, Vitagliano, Weyl, . . .

REMINDER

$\pi : Y \to X$ smooth $\quad X$ orientable
$\pi_1 : J^1Y \to X$

Mechanics / ODEs — **Field theory / PDEs**

$$\lambda = L(t, q^i, \dot{q}^i)\, dt \qquad\qquad \lambda = L(x^i, y^\sigma, y^\sigma_j)\, \omega_0$$

Cartan form

$$\theta_\lambda = Ldt + \frac{\partial L}{\partial \dot{q}^i}\, \omega^i \qquad\qquad \theta_\lambda = L\omega_0 + \frac{\partial L}{\partial y^\sigma_j} \omega^\sigma \wedge \omega_j$$

contact forms: $\omega^i = dq^i - \dot{q}^i dt \qquad \omega^\sigma = dy^\sigma - y^\sigma_j dx^j$

$$\omega_0 = dx^1 \wedge \cdots \wedge dx^n$$
$$\omega_j = i_{\partial/\partial x^j} \omega_0$$

Euler–Lagrange equations

$$J^1\gamma^* i_\xi d\theta_\lambda = 0 \quad \text{for every vertical vector field } \xi \text{ on } J^1Y$$

second order equations
solutions = extremals: sections γ of $\pi : Y \to X$

De Donder–Hamilton equations

$$\delta^* i_\xi d\theta_\lambda = 0 \quad \text{for every vertical vector field } \xi \text{ on } J^1Y$$

first order equations
solutions = Hamilton extremals: sections δ of $\pi : J^1Y \to X$

Relationship between Lagrangian and Hamiltonian solutions

Hamilton equations = equations for integral sections of an EDS generated by n-forms (a Pfaffian system for ODEs)

$$\mathcal{D} = \{ i_\xi d\theta_\lambda \} \quad \text{where } \xi \text{ runs over vertical fields on } J^1 Y$$

Euler–Lagrange equations = equations for holonomic integral sections of the same EDS

prolongations of extremals form a subset in the set of Hamilton extremals regular Lagrangians

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0 \qquad \det\left(\frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu_k}\right) \neq 0$$

bijection between extremals and Hamilton extremals:
Euler–Lagrange and Hamilton equations are equivalent

Hamilton–Jacobi equation

$w : Y \supset U \to J^1 Y$ jet field
embedded section: $w \circ \gamma = J^1 \gamma$

$$w^* d\theta_\lambda = 0 \qquad w^* \theta_\lambda = dS$$

field of extremals

Van Hove theorem
embedded section in w satisfying the Hamilton–Jacobi equation is extremal of λ
every extremal of a regular λ can be locally embedded into a field of extremals

Hamiltonian side: Dualization

- proper underlying manifold (in place of T^*Q)
- Legendre map

MECHANICS fibred manifold $\pi : Y \to X$, $\dim X = 1$

$J^1 Y$ the first jet bundle of π $(t, q^i, \dot{q}^i)$

$J^\dagger Y$ the extended dual of the first jet bundle = the manifold of real-valued affine maps on the fibres of $J^1 Y$ (t, q^i, p, p_i) with a choice of a volume element on X

$$J^\dagger Y = T^* Y \quad \text{symplectic manifold}$$

$$\Omega = dp \wedge dt + dp_i \wedge dq^i$$

J^*Y the reduced dual = quotient of $J^\dagger Y$ by constant (on fibres over Y) maps (t, q^i, p_i)

$$\text{quotient map } \rho : J^\dagger Y \to J^* Y$$

given a Lagrangian system on J^1Y, construct a dual Hamiltonian system via Legendre map

Legendre map

$$\mathrm{Leg} : J^1Y \to J^\dagger Y \qquad p = -L + \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i \quad p_i = \frac{\partial L}{\partial \dot{q}^i}$$

reduced Legendre map

$$\mathrm{leg} = \rho \circ \mathrm{Leg} : J^1 Y \to J^* Y \qquad p_i = \frac{\partial L}{\partial \dot{q}^i}$$

regular Lagrangian

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0$$

hyperregular Lagrangian - if there is an extended Legendre map Leg defined globally, and such that the corresponding reduced Legendre map leg is a diffeomorphism

Hamiltonian section $\quad h : J^* Y \to J^\dagger Y$

$$\mathrm{leg}^* \, h^* \Omega = d\theta_\lambda$$

(quite) straightforward generalization to

- higher-order regular Lagrangians in mechanics
- classical field theory - regular Lagrangians

duality restricted to regular Lagrangians

BUT: almost all interesting field Lagrangians are singular :-((

Can the class of variational problems having a dual Hamiltonian description be enlarged?

YES, BUT concepts of regularity and Legendre transformation have to be revisited

AIM: enlarge the class of regular variational problems (with a proper dual Hamiltonian description) as much as possible

Variational equations revisited: a no-Lagragian viewpoint

MOTIVATION

$$L_1 = u_x^2\,, \qquad L_2 = u_x^2 + u_x v_y - u_y v_x$$

$L_1 \sim L_2$ giving the same Euler–Lagrange expressions

L_2 is regular, L_1 is not regular

Hamilton equations:

$\delta^* i_\xi d\theta_{\lambda_2} = 0$ are equivalent with the Euler–Lagrange equations
duality!

$\delta^* i_\xi d\theta_{\lambda_1} = 0$ are not equivalent with the Euler–Lagrange equations
no duality! – constrained in the sense of Dirac

IDEAS

Associate Hamilton equations with the Euler–Lagrange form
$=$ with the class of equivalent Lagrangians

Extend the Euler–Lagrange form to a (proper!) closed $(n+1)$-form
EDS $\quad i_\xi \alpha \quad \forall \xi$

Euler–Lagrange equations – holonomic sections
Hamilton equations – all sections

regularity $=$ property of α
guarantees bijection between solutions (for EDS on $J^1 Y$)

LEPAGE MANIFOLDS

differential equations in jet bundles: dynamical forms

1-contact, ω^σ-generated $(\omega^\sigma = dy^\sigma - y^\sigma_j dx^j)$

$$E = E_\sigma \, \omega^\sigma \wedge \omega_0 \qquad E_\sigma = E_\sigma(x^i, y^\nu, y^\nu_j, y^\nu_{jk})$$

sections γ of π such that E vanishes along $J^2\gamma$ are solutions of a system of m second order partial differential equations of the form

$$E_\sigma\left(x^i, f^\nu, \frac{\partial f^\nu}{\partial x^i}, \frac{\partial^2 f^\nu}{\partial x^i \partial x^j}\right) = 0$$

where f^ν are components of a section, $\gamma = (x^i, f^\nu)$.

Remind:
decomposition of differential forms into contact components
for $(n+1)$-forms on $J^r Y$

$$\pi^*_{r+1,r}\alpha = p_1\alpha + p_2\alpha + \cdots + p_{n+1}\alpha$$

DEFINITION
Lepage $(n+1)$-form
a closed $(n+1)$-form α such that $p_1\alpha$ is a dynamical form.

Lepage manifold of order r
a fibered manifold $\pi : Y \to X$ where $\dim X = n$, equipped with a Lepage $(n+1)$-form defined on $J^r Y$.

Lepage $(n+1)$-forms are closed counterparts of Euler–Lagrange forms (variational equations):

THEOREM If α is a Lepage $(n+1)$-form then the dynamical form $E = p_1\alpha$ is locally variational: in a neighbourhood of every point $x \in \mathrm{Dom}\,\alpha$ there exists a Lagrangian L such that E is the Euler–Lagrange form of L.

Equations for the dynamical form arising from a Lepage $(n+1)$-form are Euler–Lagrange equations.

In what follows - first order case: Lepage $(n+1)$-forms on J^1Y.

Structure of first order Lepage $(n+1)$-forms

THEOREM Any Lepage $(n+1)$-form may be written as

$$\alpha = \alpha_E + \eta$$

where η is a closed and at least 2-contact form, and where α_E is closed and completely determined by E.

THEOREM The restriction of α to a suitably small open set U satisfies

$$\alpha|_U = d\Theta_L + d\mu,$$

where Θ_L is the Poincaré–Cartan form of a first-order Lagrangian defined on U, and μ is a 2-contact n-form.

we can regard a Lepage manifold of order one as a fibred manifold, equipped with a family of locally equivalent first order Lagrangians:

$\{L_\iota\}$, each Lagrangian defined on open $U_\iota \subset J^1Y$, such that
$\bigcup_\iota U_\iota = J^1Y$
whenever $U_\iota \cap U_\kappa \neq \emptyset$, around every point of the intersection

$$L_\iota = L_\kappa + d_j\varphi^j$$

for some functions φ^j.

in general no global Lagrangian (even of higher order)
obstructions come from the topology of Y

on a Lepage manifold (π_1, α):

$$\mathcal{D}_\alpha = \{i_\xi \alpha\} \quad \xi \text{ runs over all vertical vector fields on } J^1 Y$$

Euler–Lagrange equations

$$J^1\gamma^* i_\xi \alpha = 0, \quad \forall \text{ vertical vector fields } \xi$$

2nd order PDEs for sections $\gamma = (x^i, f^\sigma)$ of $\pi : Y \to X$ (extremals)
equations for holonomic integral sections of $\mathcal{D}_\alpha$

Hamilton equations

$$\delta^* i_\xi \alpha = 0, \quad \forall \text{ vertical vector fields } \xi$$

1st order PDEs for sections $\delta = (x^i, f^\sigma, g^\sigma_j)$ of $\pi_1 : J^1 Y \to X$
(Hamilton extremals)
equations for all integral sections of $\mathcal{D}_\alpha$

Hamilton and Euler–Lagrange equations are not equivalent, as there might exist Hamilton extremals that are not prolongations of extremals.

on a Lepage manifold, both the Euler–Lagrange equations and the Hamilton equations are independent of a choice of a concrete Lagrangian for E

A new look at the duality problem

LAGRANGIAN SIDE

De Donder–Hamilton equations

We shall be interested in Lepage manifolds where α is at most 2-contact and $\{\omega^\sigma\}$-generated. As we shall see, in this case the Hamilton equations become of De Donder type.

The form α is closed by definition, but rank α need not be maximal, or even constant.

We say that α is regular if corank $\alpha = \dim X$

then: $\operatorname{rank} \alpha = \operatorname{rank} \mathcal{D}_\alpha = m + nm$

with help of the Poincaré Lemma:

THEOREM

(π_1, α) Lepage manifold, suppose α is at most 2-contact and $\{\omega^\sigma\}$-generated. Then for every point in $J^1 Y$ there is a neighbourhood U and functions H and p^j_σ defined on U such that

$$\alpha|_U = -dH \wedge \omega_0 + dp^j_\sigma \wedge dy^\sigma \wedge \omega_j.$$

If, moreover, α is regular then the functions p^j_σ are independent:

$$\mathrm{rank}\left(\frac{\partial p^j_\sigma}{\partial y^\nu_k}\right) = \max = mn.$$

in fibred coordinates

$$\pi_{2,1}^{*}\alpha = E_\sigma \omega^\sigma \wedge \omega_0 + \frac{1}{2}\left(\frac{\partial E_\sigma}{\partial y_j^\nu} - d_k f_{\sigma\nu}^{j,k}\right)\omega^\sigma \wedge \omega^\nu \wedge \omega_j$$
$$+ \left(\frac{\partial E_\sigma}{\partial y_{ij}^\nu} - f_{\sigma\nu}^{i,j}\right)\omega^\sigma \wedge \omega_i^\nu \wedge \omega_j,$$

where $f_{\sigma\nu}^{i,j} = -f_{\sigma\nu}^{j,i} = f_{\nu\sigma}^{j,i}$ are some functions such that $d\alpha = 0$

regularity condition

$$\det\left(\frac{\partial E_\sigma}{\partial y_{ij}^\nu} - f_{\sigma\nu}^{i,j}\right) \neq 0$$

given a regular form α as above

$$(x^i, y^\sigma, y^\sigma_j) \to (x^i, y^\sigma, p^j_\sigma)$$

is a local coordinate transformation on $J^1 Y$
Legendre transformation

by construction, explicit formulas for the Hamiltonian and the momenta come from an integration procedure using the Poincaré Lemma and are determined by α rather than by a particular Lagrangian

$$p^i_\sigma = -y^\nu \int_0^1 \left(\frac{\partial E_\sigma}{\partial y^\nu_i} - d_k f^{i,k}_{\sigma\nu} \right) \circ \chi \, u \, du$$
$$- y^\nu_j \int_0^1 (\left(\frac{\partial E_\sigma}{\partial y^\nu_{ij}} - f^{i,j}_{\sigma\nu} \right) \circ \chi \, u \, du - \frac{\partial f^i}{\partial y^\sigma}$$

THEOREM

There exists(!) L for α such that H and p^j_σ come from L by the "standard" formulas

$$p^j_\sigma = \frac{\partial L}{\partial y^\sigma_j}, \quad H = -L + p^j_\sigma y^\sigma_j.$$

Hamilton equations of α in Legendre coordinates

$$\frac{\partial(y^\sigma \circ \delta)}{\partial x^i} = \frac{\partial H}{\partial p^i_\sigma} \circ \delta, \qquad \frac{\partial(p^i_\sigma \circ \delta)}{\partial x^i} = -\frac{\partial H}{\partial y^\sigma} \circ \delta.$$

De Donder–Hamilton equations

meaning of the regularity condition:

THEOREM

On a Lepage manifold (π_1, α) where α is at most 2-contact, $\{\omega^\sigma\}$-generated and regular, the Euler–Lagrange and Hamilton equations are equivalent.
Explicitly, if γ is an extremal then $J^1\gamma$ is a Hamilton extremal; conversely, every Hamilton extremal is of the form $J^1\gamma$ where γ is an extremal.

Hamilton–Jacobi equation

$w : Y \supset U \to J^1 Y$ jet field
embedded section: $w \circ \gamma = J^1\gamma$

$$w^*\alpha = 0 \qquad w^*\rho = dS$$

field of extremals
Lagrangian submanifolds

Van Hove theorem
embedded section in w satisfying the Hamilton–Jacobi equation is extremal
every extremal of a regular α can be locally embedded into a field of extremals

HAMILTONIAN SIDE

$J^{\dagger}Y$ the affine dual

with a choice of a volume element on X, $J^{\dagger}Y$ is diffeomorphic to the bundle of n-forms on J^1Y, locally generated by $(\omega_0, dy^{\sigma} \wedge \omega_i)$

$$\Omega = dP \wedge \omega_0 + dP^i_{\sigma} \wedge dy^{\sigma} \wedge \omega_i$$

canonical multisymplectic form on $J^{\dagger}Y$

local section h of the projection $\rho : J^{\dagger}Y \to J^*Y$

$\mathcal{H} = -(P \circ h)$ on $\mathrm{Dom}\, h \subset J^*Y$ Hamiltonian

on J^*Y local closed $(n+1)$-form

$$\Omega_h = h^*\Omega = -d\mathcal{H} \wedge \omega_0 + dP^i_{\sigma} \wedge dy^{\sigma} \wedge \omega_i$$

$\mathcal{D}_h = \{i_\xi \Omega_h\}$ ξ runs over vertical vector fields on $J^* Y$

integral sections $\psi : X \to J^* Y$ satisfy

$$\frac{\partial (y^\sigma \circ \psi)}{\partial x^i} = \frac{\partial \mathcal{H}}{\partial P^i_\sigma} \circ \psi, \qquad \frac{\partial (P^i_\sigma \circ \psi)}{\partial x^i} = -\frac{\partial \mathcal{H}}{\partial y^\sigma} \circ \psi.$$

De Donder–Hamilton equations

DEFINITION

De Donder–Hamilton system is Cauchy integrable if the Cauchy problem for the given De Donder–Hamilton equations has, for every initial condition, at least one maximal solution.

LEGENDRE MAP

We have constructed a universal Hamiltonian bundle, canonically associated with a jet bundle $\pi_1 : J^1Y \to X$.

connection between an abstract Hamiltonian system and a concrete variational system on a Lepage manifold - Legendre maps

extended Legendre map

$$\mathrm{Leg}_\alpha : J^1Y \to J^\dagger Y \qquad \mathrm{Leg}_\alpha^* \,\Omega = \alpha$$

reduced Legendre map

$$\mathrm{leg}_\alpha : J^1Y \to J^*Y \qquad \mathrm{leg}_\alpha^* \,\Omega_h = \alpha$$

duality equations - DEFINITION of the Legendre maps!

If α satisfies the regularity condition

$$\det\left(\frac{\partial E_\sigma}{\partial y^\nu_{ij}} - f^{i,j}_{\sigma\nu}\right) \neq 0$$

we can choose Legendre coordinates on J^1Y.

In Legendre coordinates on J^1Y, and the canonical coordinates on J^*Y, leg_α is represented by the identity mapping.

THEOREM
If α is regular then every extended Legendre map is an immersion and every corresponding reduced Legendre map is a local diffeomorphism.

α is called hyper-regular if there is an extended Legendre map Leg_α defined globally, and such that the corresponding reduced Legendre map leg_α is a diffeomorphism.

global Hamiltonian section

$$h = \mathrm{Leg}_\alpha \circ \mathrm{leg}_\alpha^{-1}$$

global Hamiltonian $\mathcal{H} = -(P \circ h)$

If α is regular - local h and $\mathcal{H}$.

DUALITY THEOREM

For a hyper-regular α on J^1Y

(1) $\mathrm{Leg}_\alpha^* \Omega = \mathrm{leg}_\alpha^* h^*\Omega = \alpha$.

(2) $\mathrm{rank}\, h^*\Omega = \mathrm{rank}\, \mathcal{D}_h = \mathrm{rank}\, \alpha = \mathrm{rank}\, \mathcal{D}_\alpha = m + nm$.

(3) $\mathrm{leg}_\alpha^* \mathcal{D}_h = \mathcal{D}_\alpha$.

(4) If $\psi : X \to J^*Y$ is an integral section of $\mathcal{D}_h$ then $\mathrm{leg}_\alpha^{-1} \circ \psi = J^1\gamma$ where γ is a section of $\pi : Y \to X$, and it is an integral section of $\mathcal{D}_\alpha$.

(5) Every integral section of $\mathcal{D}_\alpha$ is of the form $J^1\gamma$, and $\psi = \mathrm{leg}_\alpha \circ J^1\gamma$ is an integral section of $\mathcal{D}_h$.

APPLICATIONS - field theories in physics

regularity condition for α - free parameters

$$\det\left(\frac{\partial E_\sigma}{\partial y^\nu_{jk}} - f^{jk}_{\sigma\nu}\right) \neq 0$$

correspond to different Lepage $(n+1)$-forms α associated to E

a choice! of a regular Hamiltonian system

- electromagnetic field
- Yang-Mills field
- Dirac field
- gravity (Hilbert Lagrangian)

no Dirac constraints
new Hamiltonians

EXAMPLE Einstein equations - gravitational field

Hilbert Lagrangian - scalar curvature R

second order, (conventionally) not regular(!)

"energy" H momenta p^i_σ on J^2Y

physical meaning still unclear/problematic

Hamilton equations - Dirac formalism(!)

Lepage manifolds for relativity

$\pi : Y \to X$, $\dim X = 4$, Y bundle of metrics over X

$\lambda = R\omega_g = R\sqrt{|\det g|}\omega_0$

$\alpha = d\Theta_\lambda$ on J^1Y, regular !!!

$$\alpha = -dH \wedge \omega_0 + dP^{rs,i} dg_{rs} \wedge \omega_i$$

$$P^{rs,i} = \sqrt{|\det g|}\Big(-\tfrac{1}{2} g^{rs}(g^{pq}\Gamma^i_{pq} - g^{iq}\Gamma^p_{pq}) + g^{rq}g^{ps}\Gamma^i_{pq} - \tfrac{1}{2}(g^{ir}g^{qs} + g^{is}g^{qr})\Gamma^p_{pq}\Big)$$

$$H = \frac{1}{6}\frac{1}{\sqrt{|\det g|}}(g_{jk}g_{ab}g_{rs} - 4g_{aj}g_{kb}g_{rs} + 4g_{rj}g_{kb}g_{as})P^{ab,j}P^{rs,k}$$

Legendre transformation

$$(x^i, g_{rs}, g_{rs,j}) \to (x^i, g_{rs}, P^{rs,j}) \quad \text{on } J^1Y$$

Hamilton equations

$$\frac{\partial H}{\partial g_{rs}} + \frac{\partial P^{rs,k}}{\partial x^k} = 0, \qquad \frac{\partial H}{\partial P^{rs,k}} - \frac{\partial g_{rs}}{\partial x^k} = 0$$

EXAMPLE Maxwell equations - electromagnetic field

Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(y^\sigma_\nu y^\nu_\sigma - g^{\sigma\nu}g_{\mu\rho}y^\mu_\sigma y^\rho_\nu)$$

$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$, $y^\sigma = g^{\sigma\nu}A_\nu$, $g = \mathrm{diag}(-1,1,1,1)$ Lorentz m.

$\det\left(\dfrac{\partial^2 L}{\partial y^\sigma_\mu \partial y^\rho_\nu}\right) = 0$ conventionally degenerate - Dirac formalism (!)

THEOREM

Lepage manifold for Maxwell equations: $(J^1 Y, \alpha)$ with

$$\text{regular} \quad \alpha = d\Theta_{\hat{\lambda}}, \quad \hat{L} = L - 2(\mathrm{Tr}\, A^2 - (\mathrm{Tr}\, A)^2)$$

independent momenta
"true Hamiltonian" $\hat{H} = H + 2(\mathrm{Tr}\, A^2 - (\mathrm{Tr}\, A)^2)$

Geometric meaning of Hamilton–De Donder equations

strong relationship with Ehresmann connections on the fibred manifold $\tau : J^*Y \to X$.

$\hat{\Gamma}$ Ehresmann connection (jet field) on $\tau : J^*Y \to X$
horizontal projector

$$\Gamma = dx^j \otimes \left(\frac{\partial}{\partial x^j} + \Gamma_j^\sigma \frac{\partial}{\partial y^\sigma} + \Gamma^i_{\sigma j} \frac{\partial}{\partial P^i_\sigma} \right)$$

integral section $\hat{\Gamma} \circ \psi = J^1\psi$
in coordinates - equations

$$\frac{\partial \psi^\sigma}{\partial x^j} = \Gamma_j^\sigma \,, \qquad \frac{\partial \psi^i_\sigma}{\partial x^j} = \Gamma^i_{\sigma j}$$

THEOREM
If an Ehresmann connection $\hat{\Gamma}$ on $\tau : J^*Y \to X$ satisfies the compatibility condition

$$i_\Gamma \Omega_h = (n-1)\Omega_h$$

then any integral section of $\hat{\Gamma}$ is a solution of Hamilton-De Donder equations.

field of Hamilton–De Donder extremals

compatible connections are non-unique - however, we know all of them!

THEOREM
The family of Ehresmann connections $\hat{\Gamma}$ on $\tau : J^*Y \to X$ compatible with Ω_h is locally described by the horizontal projectors

$$\Gamma = dx^j \otimes \left(\frac{\partial}{\partial x^j} + \frac{\partial \mathcal{H}}{\partial P^j_\sigma} \frac{\partial}{\partial y^\sigma} - \left(\frac{1}{n} \delta^i_j \frac{\partial \mathcal{H}}{\partial y^\sigma} + F^i_{\sigma j} \right) \frac{\partial}{\partial P^i_\sigma} \right),$$

where for every σ, the $(F^i_{\sigma j})$ is an arbitrary $(n \times n)$-matrix on U, traceless at each point of U.

We have a family of Ehresmann connections such that every local section of any of these connections is a Hamilton–De Donder extremal.

In particular, for every integrable connection $\hat{\Gamma}$ (in the sense of Frobenius complete integrability) maximal Hamilton–De Donder extremals form a n-dimensional foliation of $\mathrm{Dom}\,\hat{\Gamma} \subset J^*Y$.

QUESTION:
are all solutions of Ω_h (at least locally) included?

YES !

every solution of Hamilton–De Donder equations is locally an integral section of some compatible connection, which, moreover, is maximal, in the sense that it is defined on the domain $U \subset J^*Y$ of h

THEOREM
If Dom $h = U \subset J^*Y$ then there is a connection $\hat{\Gamma}_0$ on U satisfying $i_{\Gamma_0}\Omega_h = (n-1)\Omega_h$.
If h is a global section then $\hat{\Gamma}_0$ is a global connection.

THEOREM
h a section of ρ defined on $U \subset J^*Y$
W a nonempty open subset of $\tau(U) \subset X$.
If ψ is a local section of $\tau : J^*Y \to X$ defined on W and satisfying

$$\psi^*(i_\xi \Omega_h) = 0 \quad \text{for every vertical vector field } \xi \text{ on } (\pi^*)^{-1}(W)$$

then for each $x \in W$ there is a connection $\hat{\Gamma}$ defined on $U \subset J^*Y$ and satisfying

$$i_\Gamma \Omega_h = (n-1)\Omega_h$$

s.t. for some neighbourhood N of $\psi(x)$ the restriction $\psi|_N$ is an integral section of $\hat{\Gamma}$.

CONCLUSIONS

Every solution of Hamilton-De Donder equations can be locally embedded in a field of Hamilton-De Donder extremals.

$\hat{\Gamma}$ flat, compatible with Ω_h: n-dimensional foliation, the leaves are solutions of the Hamilton-De Donder equations.
It follows that the Cauchy problem has at least one maximal solution for any given initial condition corresponding to the unique maximal integral manifold passing through that point.

Sufficient condition for Cauchy integrability: the existence of a flat Ehresmann connection compatible with Ω_h.

Jacobi theorem

Lepage manifold
α at most 2-contact, $\{\omega^\sigma\}$-generated and regular

duality, $1-1$ correspondence of solutions of Euler–Lagrange and Hamilton equations

solutions of Euler–Lagrange equations:
integral sections of compatible semispray connections
$\Gamma : J^1Y \to J^2Y$

Hamilton–Jacobi equation
local sections $w : Y \to J^1Y \quad w^*\alpha = 0$
Van Hove theorem - fields of extremals

how to find fields of extremals?

THEOREM

Γ an integrable semispray connection, compatible with α
$\{a^1, \ldots, a^{nm}\}$ a set of independent first integrals on $W \subset J^1Y$
If

$$\det\left(\frac{\partial a^K}{\partial y^\sigma_j}\right) \neq 0$$

on W then

$$H_\Xi = \mathrm{span}\{da^1, \ldots, da^{nm}\}$$

is a horizontal distribution for a local jet connection
$\Xi : J^1\pi \supset W \to J^1\pi_{1,0}$ such that every integral section of Ξ is a field of extremals of α.

Talk based on:

O. Krupková and A. Vondra, On some intergation methods for connections on fibered manifilds, Proc. Conf. DGA, Opava, 1993

O. Krupková, Hamiltonian field theory, J. Geom. Phys., 2002

O. Rossi and D.J. Saunders, Lagrangian and Hamiltonian duality, J. Math. Sci, to appear

O. Rossi and D.J. Saunders, Dual jet bundles, Hamiltonian systems and connections, Diff. Geom. Appl., 2014

t h a n k   y o u

:-)


Happy
Birthday