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# Noether theorems in a general setting. Reducible graded Lagrangians

**Basic reference:** G.Sardanashvily, Noether theorems in a general setting, *arXiv*: 1411.2910.

**Main publications on the subject:**

D.Bashkirov, G.Giachetta, L.Mangiarotti, G.Sardanashvily, Noether's second theorem in a general setting. Reducible gauge theory, ***J. Phys. A*** **38** (2005) 5329.

D.Bashkirov, G.Giachetta, L.Mangiarotti, G.Sardanashvily, The antifield Koszul – Tate complex of reducible Noether identities, ***J. Math. Phys.*** **46** (2005) 103513.

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D.Bashkirov, G.Giachetta, L.Mangiarotti, G.Sardanashvily, The KT-BRST complex of degenerate Lagrangian systems, ***Lett. Math. Phys.*** **83** (2008) 237.

G.Giachetta, L.Mangiarotti, G.Sardanashvily, On the notion of gauge symmetries of generic Lagrangian field theory, ***J. Math. Phys*** **50** (2009) 012903.

G.Sardanashvily, Graded Lagrangian formalism, ***Int. J. Geom. Methods. Mod. Phys.*** **10** (2013) 1350016.

# Main Theses

Here we are not concerned with the rich history of Noether theorems, and refer for this subject to a brilliant book: Y. Kosmann-Schwarzbach, *The Noether Theorems. Invariance and the Conservation Laws in the Twentieth Century* (Springer, 2011).

• Second Noether theorems are formulated in a general case of reducible degenerate Grassmann-graded Lagrangian theory of even and odd variables on graded bundles.

• Such Lagrangian theory is characterized by a hierarchy of non-trivial Noether and higher-stage Noether identities which are described as elements of homology groups of some chain complex.

• If a certain homology regularity condition holds, one can associate to a reducible degenerate Lagrangian the exact Koszul – Tate chain complex possessing the boundary operator whose nilpotentness is equivalent to all complete non-trivial Noether and higher-stage Noether identities.

• Second Noether theorems associate to the above-mentioned Koszul – Tate complex a certain cochain sequence whose ascent operator consists of the gauge and higher-order gauge symmetries of a Lagrangian system.

• If gauge symmetries are algebraically closed, this operator is extended to the nilpotent BRST operator which brings the above mentioned cochain sequence into the BRST complex and thus provides a BRST extension of an original Lagrangian system.

# Problems

• A key problem is that any Euler – Lagrange operator satisfies Noether identities, which therefore must be separated into the trivial and non-trivial ones.

• These Noether identities can obey first-stage Noether identities, which in turn are subject to the second-stage ones, and so on. Thus, there is a hierarchy of Noether identities and higher-stage Noether identities which also must be separated into the trivial and non-trivial ones

• This hierarchy of Noether and higher-stage Noether identities is described in the framework of a Grassmann-graded homology complex. Therefore Lagrangian theory of Grassmann-graded even and odd variables should be considered from the beginning.

# Graded Lagrangian formalism

Lagrangian theory of even variables on a smooth manifold $X$, $\dim X = n > 1$, conventionally is formulated ***in terms of fibre bundles and jet manifolds***. However, different geometric models of odd variables either on graded manifolds or supermanifolds are discussed. Both graded manifolds and supermanifolds are phrased in terms of sheaves of graded commutative algebras. Graded manifolds are characterized by sheaves on smooth manifolds, while supermanifolds are constructed by gluing of sheaves on supervector spaces.

• We follow the ***Serre – Swan theorem*** extended to graded manifolds. It states that, if a graded commutative $C^\infty(X)$-ring is generated by a projective $C^\infty(X)$-module of finite rank, it is isomorphic to a ring of graded functions on a graded manifold whose body is $X$. Since higher-stage Noether identities of a Lagrangian system on a manifold $X$ form graded $C^\infty(X)$-modules, ***we describe odd variables in terms of graded manifolds***.

• A graded manifold is characterized by a body manifold $Z$ and a structure sheaf $\mathfrak{A}$ of Grassmann algebras on $Z$. Its sections form a graded commutative structure $C^\infty(Z)$-ring $\mathcal{A}$ of graded functions on $(Z,\mathfrak{A})$. The differential calculus on a graded manifold is the ***Chevalley – Eilenberg differential calculus*** over its structure ring. By virtue of ***Batchelor's theorem***, there is a vector bundle $E \to Z$ such that a structure ring of $(Z,\mathfrak{A})$ is isomorphic to a ring $\mathcal{A}_E$ of sections of an exterior bundle $\wedge E^*$ of the dual $E^*$ of $E$. This isomorphism is not canonical, but usually is fixed from the beginning. Therefore, we restrict our consideration to graded manifolds $(Z,\mathcal{A}_E)$, called the ***simple graded manifolds***, modelled over vector bundles $E \to Z$.

• A common configuration space of even and odd variables is defined as a graded bundle $(Y, \mathcal{A}_F)$ which is a simple graded manifold modelled over a vector bundle $F \to Y$ whose body is $Y$. Its graded $r$-order jet manifold $(J^rY, \mathcal{A}_r = \mathcal{A}_{J^rF})$ is introduced as a simple graded manifold modelled over a vector bundle $J^rF \to J^rY$. ***This definition of graded jet manifolds is compatible with the conventional one of jets of fibre bundles.***

• Lagrangian theory on a fibre bundle $Y$ adequately is phrased in terms of the ***variational bicomplex*** of exterior forms on the infinite order jet manifold $J^\infty Y$ of $Y$. This technique is extended to Lagrangian theory on graded manifolds in terms of the Grassmann-graded variational bicomplex $\mathcal{S}^*_\infty[F;Y]$ of graded exterior forms on a ***graded infinite order jet manifold*** $(J^\infty Y, \mathcal{A}_{J^\infty F})$ which is the inverse limit of graded manifolds $(J^rY, \mathcal{A}_r)$.

• ***Lagrangians*** $L$ and the ***Euler – Lagrange operator*** $\delta L$ are defined as the elements and the coboundary operator of this graded bicomplex. Its cohomology provides the global variational formula

$$dL = \delta L - d_H \Xi_L, \qquad \Xi_L \in \mathcal{S}^{n-1}_\infty[F;Y],, \tag{1}$$

where $\Xi_L$ is a global Lepage equivalent of a graded Lagrangian $L$.

• Given a graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y], L)$, by its ***infinitesimal transformations*** are meant contact graded derivations $\vartheta$ of a graded commutative ring $\mathcal{S}^0_\infty[F;Y]$. Every graded derivation $\vartheta$ yields a Lie derivative $\mathbf{L}_\vartheta$ of a graded algebra $\mathcal{S}^*_\infty[F;Y]$. It is the infinite order jet prolongation $\vartheta = J^\infty \upsilon$ of its restriction $\upsilon$ to a graded commutative ring $S^0[F;Y]$.

# Noether identities

We follow the general analysis of Noether identities (NI) and higher-stage NI of differential operators on fibre bundles when ***trivial and non-trivial NI are represented by boundaries and cycles of a chain complex***.

• One can associate to any graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y],L)$ the chain complex (2) whose one-boundaries vanish on the shell $\delta L=0$. Let us consider the density-dual $\overline{VF}=V^*F\underset{F}{\otimes}\overset{n}{\wedge}T^*X\to F$, of the vertical tangent bundle $VF\to F$, and let us enlarge an original algebra $\mathcal{S}^*_\infty[F;Y]$ with the generating basis $(s^A)$ to $\mathcal{P}^*_\infty[\overline{VF};Y]$ with the generating basis $(s^A,\overline{s}_A)$, $[\overline{s}_A]=[A]+1$. Following the terminology of BRST theory, we call its elements $\overline{s}_A$ the ***antifields*** of antifield number $\mathrm{Ant}[\overline{s}_A]=1$. An algebra $\mathcal{P}^*_\infty[\overline{VF};Y]$ is endowed with the nilpotent right graded derivation $\overline{\delta}=\overleftarrow{\partial}^A\mathcal{E}_A$, where $\mathcal{E}_A=\delta_A L$ are the variational derivatives. Then we have a chain complex

$$0\leftarrow \mathrm{Im}\,\overline{\delta}\overset{\overline{\delta}}{\longleftarrow}\mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1\overset{\overline{\delta}}{\longleftarrow}\mathcal{P}^{0,n}_\infty[\overline{VF};Y]_2 \qquad (2)$$

of graded densities of antifield number $\leq 2$. Its one-boundaries $\overline{\delta}\Phi$, $\Phi\in\mathcal{P}^{0,n}_\infty[\overline{VF};Y]_2$, by very definition, vanish on-shell. Any one-cycle $\Phi$ of the complex (2) is a differential operator on a bundle $\overline{VF}$ such that its kernel contains the graded Euler – Lagrange operator $\delta L$, i.e.,

$$\overline{\delta}\Phi=0,\qquad \sum_{0\leq|\Lambda|}\Phi^{A,\Lambda}d_\Lambda\mathcal{E}_A\omega=0, \qquad (3)$$

where $d_\Lambda$ are total derivatives relative to a multi-index $\Lambda=\lambda_1\cdots\lambda_k$. Refereing to a notion of NI of a differential operator, we say that the one-cycles $\Phi$ define the ***Noether identities*** (3) of an Euler – Lagrange operator $\delta L$.

• One-chains $\Phi$ are necessarily NI if they are boundaries. Therefore, these NI are called ***trivial***. Accordingly, ***non-trivial NI*** modulo trivial ones are associated to elements of the first homology $H_1(\overline{\delta})$ of the complex (2).

***A Lagrangian*** $L$ ***is called degenerate if there are non-trivial NI***.

• Non-trivial NI can obey first-stage NI. To describe them, let us assume that a module $H_1(\overline{\delta})$ is finitely generated. Namely, there exists a graded projective $C^\infty(X)$-module $\mathcal{C}_{(0)} \subset H_1(\overline{\delta})$ of finite rank possessing a local basis $\{\Delta_r\omega\}$ such that any element $\Phi \in H_1(\overline{\delta})$ factorizes as

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r,\Xi} d_\Xi \Delta_r\omega, \qquad \Phi^{r,\Xi} \in \mathcal{S}^0_\infty[F;Y], \tag{4}$$

through elements $\Delta_r\omega$ of $\mathcal{C}_{(0)}$. Thus, all non-trivial NI (3) result from the NI

$$\overline{\delta}\Delta_r = \sum_{0\leq|\Lambda|} \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0, \tag{5}$$

called the ***complete NI***. Then the complex (2) can be extended to the chain complex (6) with a ***boundary operator whose nilpotency is equivalent to the complete NI*** (5). By virtue of Serre – Swan theorem, a graded module $\mathcal{C}_{(0)}$ is isomorphic to that of sections of the density-dual $\overline{E}_0$ of some graded vector bundle $E_0 \to X$. Let us enlarge $\mathcal{P}^*_\infty[\overline{VF};Y]$ to an algebra $\overline{\mathcal{P}}^*_\infty\{0\} = \mathcal{P}^*_\infty[\overline{VF} \underset{X}{\times} \overline{E}_0;Y]$ with a generating basis $(s^A, \overline{s}_A, \overline{c}_r)$ where $\overline{c}_r$ are antifields such that $[\overline{c}_r] = [\Delta_r] + 1$ and $\mathrm{Ant}[\overline{c}_r] = 2$. This algebra admits a derivation $\delta_0 = \overline{\delta} + \overleftarrow{\partial}{}^r \Delta_r$ which is nilpotent iff the complete NI (5) hold. Then $\delta_0$ is a boundary operator of a chain complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \overset{\overline{\delta}}{\leftarrow} \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1 \overset{\delta_0}{\leftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2 \overset{\delta_0}{\leftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_3 \tag{6}$$

of graded densities of antifield number $\leq 3$. One can show that its homology $H_1(\delta_0)$ vanishes, i.e., the complex (6) is one-exact.

• Let us consider the second homology $H_2(\delta_0)$ of the complex (6). Its two-cycles define the ***first-stage NI***

$$\delta_0\Phi = 0, \qquad \sum_{0\leq|\Lambda|} G^{r,\Lambda} d_\Lambda \Delta_r \omega = -\overline{\delta} H. \tag{7}$$

Conversely, let the equality (7) hold. Then it is a cycle condition. The first-stage NI (7) are trivial either if the two-cycle $\Phi$ is a $\delta_0$-boundary or its summand $G$ vanishes on-shell. Therefore, ***non-trivial first-stage NI fails to exhaust the second homology*** $H_2(\delta_0)$ of the complex (6) in general. One can show that non-trivial first-stage NI modulo trivial ones are identified with elements of $H_2(\delta_0)$ iff any $\overline{\delta}$-cycle $\phi \in \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2$ is a $\delta_0$-boundary.

***A degenerate Lagrangian is called reducible if it admits non-trivial first-stage NI***.

• Non-trivial first-stage NI can obey second stage NI, and so on. Iterating the arguments, we say that a degenerate graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y], L)$ is called $N$***-stage reducible*** if it admits finitely generated non-trivial $N$-stage NI, but no non-trivial $(N+1)$-stage ones. It is characterized as follows.

(i) There are graded vector bundles $E_0, \ldots, E_N$ over $X$, and $\mathcal{P}^*_\infty[\overline{VF};Y]$ is enlarged to an algebra

$$\overline{\mathcal{P}}^*_\infty\{N\} = \mathcal{P}^*_\infty[\overline{VF} \underset{X}{\times} \overline{E}_0 \underset{X}{\times} \cdots \underset{X}{\times} \overline{E}_N; Y] \tag{8}$$

with a local generating basis $(s^A, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N})$ where $\overline{c}_{r_k}$ are $k$-stage antifields of antifield number $\mathrm{Ant}[\overline{c}_{r_k}] = k+2$.

(ii) The algebra (8) is provided with a nilpotent right graded derivation

$$\delta_{\rm KT}=\delta_N=\overline{\delta}+\sum_{0\leq|\Lambda|}\overleftarrow{\partial}^r\Delta_r^{A,\Lambda}\overline{s}_{\Lambda A}+\sum_{1\leq k\leq N}\overleftarrow{\partial}^{r_k}\Delta_{r_k}, \tag{9}$$

$$\Delta_{r_k}\omega=\sum_{0\leq|\Lambda|}\Delta_{r_k}^{r_{k-1},\Lambda}\overline{c}_{\Lambda r_{k-1}}\omega+ \tag{10}$$
$$\sum_{0\leq|\Sigma|,|\Xi|}(h_{r_k}^{(r_{k-2},\Sigma)(A,\Xi)}\overline{c}_{\Sigma r_{k-2}}\overline{s}_{\Xi A}+...)\omega\in\overline{\mathcal{P}}^{0,n}_\infty\{k-1\}_{k+1},$$

of antifield number -1. The index $k=-1$ here stands for $\overline{s}_A$. The nilpotent derivation $\delta_{\rm KT}$ (9) is called ***the Koszul – Tate operator***.

(iii) With this derivation, a module $\overline{\mathcal{P}}^{0,n}_\infty\{N\}_{\leq N+3}$ of densities of antifield number $\leq(N+3)$ is split into the exact ***Koszul – Tate chain complex***

$$0\leftarrow{\rm Im}\,\overline{\delta}\overset{\overline{\delta}}{\longleftarrow}\mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1\overset{\delta_0}{\longleftarrow}\overline{\mathcal{P}}^{0,n}_\infty\{0\}_2\overset{\delta_1}{\longleftarrow}\overline{\mathcal{P}}^{0,n}_\infty\{1\}_3\cdots \tag{11}$$
$$\overset{\delta_{N-1}}{\longleftarrow}\overline{\mathcal{P}}^{0,n}_\infty\{N-1\}_{N+1}\overset{\delta_{\rm KT}}{\longleftarrow}\overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+2}\overset{\delta_{\rm KT}}{\longleftarrow}\overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+3}$$

which satisfies the following homology regularity condition.

*Any* $\delta_{k<N}$*-cycle* $\phi\in\overline{\mathcal{P}}^{0,n}_\infty\{k\}_{k+3}\subset\overline{\mathcal{P}}^{0,n}_\infty\{k+1\}_{k+3}$ *is a* $\delta_{k+1}$*-boundary.*

(iv) The nilpotentness of the Koszul – Tate operator (9) is ***equivalent*** to complete non-trivial NI (5) and complete non-trivial $(k\leq N)$-stage NI

$$\sum_{0\leq|\Lambda|}\Delta_{r_k}^{r_{k-1},\Lambda}d_\Lambda(\sum_{0\leq|\Sigma|}\Delta_{r_{k-1}}^{r_{k-2},\Sigma}\overline{c}_{\Sigma r_{k-2}})=-\overline{\delta}(\sum_{0\leq|\Sigma|,|\Xi|}h_{r_k}^{(r_{k-2},\Sigma)(A,\Xi)}\overline{c}_{\Sigma r_{k-2}}\overline{s}_{\Xi A}). \tag{12}$$

• It may happen that a graded Lagrangian system possesses non-trivial NI of any stage. However, we restrict our consideration to $N$-reducible Lagrangians for a finite integer $N$. In this case, the Koszul – Tate operator (9) and the gauge operator (16) below contain finite terms.

# Inverse second Noether theorem

Different variants of the inverse second Noether theorem have been suggested in order to relate reducible NI and gauge symmetries.

[G.Barnich, F.Brandt, M.Henneaux, Local BRST cohomology in gauge theories, *Phys. Rep.* **338** (2000) 439.

R.Fulp, T.Lada, J. Stasheff, Noether variational Theorem II and the BV formalism, *Rend. Circ. Mat. Palermo (2) Suppl.* No. 71 (2003) 115.]

***The inverse second Noether theorem*** (Theorem 1), that we formulate in homology terms, ***associates to the Koszul – Tate complex (11) of non-trivial NI the cochain sequence (15) with the ascent operator*** $\mathbf{u}$ ***(16) whose components are gauge and higher-stage gauge symmetries of a Lagrangian system***.

Given an algebra $\overline{\mathcal{P}}^*_\infty\{N\}$ (8), let us consider the algebras

$$P^*_\infty\{N\} = P^*_\infty[F \underset{X}{\times} E_0 \underset{X}{\times} \cdots \underset{X}{\times} E_N; Y], \tag{13}$$

possessing the generating basis $(s^A, c^r, c^{r_1}, \ldots, c^{r_N})$, $[c^{r_k}] = [\overline{c}_{r_k}] + 1$, and

$$\mathcal{P}^*_\infty\{N\} = \mathcal{P}^*_\infty[\overline{VF} \underset{X}{\times} E_0 \underset{X}{\times} \cdots \underset{X}{\times} E_N \underset{X}{\times} \overline{E}_0 \underset{X}{\times} \cdots \underset{X}{\times} \overline{E}_N; Y] \tag{14}$$

with the generating basis $(s^A, \overline{s}_A, c^r, c^{r_1}, \ldots, c^{r_N}, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N})$. Their elements $c^{r_k}$ are called the ***k-stage ghosts*** of ghost number $\mathrm{gh}[c^{r_k}] = k+1$ and antifield number $\mathrm{Ant}[c^{r_k}] = -(k+1)$. The Koszul – Tate operator $\delta_{\mathrm{KT}}$ (9) naturally is extended to a graded derivation of an algebra $\mathcal{P}^*_\infty\{N\}$.

**Theorem 1.** Given the Koszul – Tate complex (11), a module of graded densities $P_\infty^{0,n}\{N\}$ is decomposed into a cochain sequence

$$0 \to \mathcal{S}_\infty^{0,n}[F;Y] \xrightarrow{\mathbf{u}} P_\infty^{0,n}\{N\}^1 \xrightarrow{\mathbf{u}} P_\infty^{0,n}\{N\}^2 \xrightarrow{\mathbf{u}} \cdots \tag{15}$$

graded in ghost number. Its ascent operator

$$\mathbf{u} = u + u^{(1)} + \cdots + u^{(N)} = u^A \frac{\partial}{\partial s^A} + u^r \frac{\partial}{\partial c^r} + \cdots + u^{r_{N-1}} \frac{\partial}{\partial c^{r_{N-1}}}, \tag{16}$$

is an odd graded derivation of ghost number 1 where

$$u = u^A \frac{\partial}{\partial s^A}, \qquad u^A = \sum_{0\leq|\Lambda|} c_\Lambda^r \eta(\Delta_r^A)^\Lambda, \tag{17}$$

$$\frac{\overleftarrow{\delta}(c^r\Delta_r)}{\delta \overline{s}_A} \mathcal{E}_A \omega = u^A \mathcal{E}_A \omega = d_H \sigma_0, \tag{18}$$

is ***a gauge variational symmetry*** of a graded Lagrangian $L$ and the derivations

$$u^{(k)} = u^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}} = \sum_{0\leq|\Lambda|} c_\Lambda^{r_k} \eta(\Delta_{r_k}^{r_{k-1}})^\Lambda \frac{\partial}{\partial c^{r_{k-1}}}, \quad k = 1, \ldots, N, \tag{19}$$

are ***higher-stage gauge symmetries*** which obey the relations

$$\sum_{0\leq|\Xi|} c^{r_k} \sum_{0\leq|\Sigma|} h_{r_k}^{(r_{k-2},\Sigma)(A,\Xi)} \overline{c}_{\Sigma r_{k-2}} d_\Xi \mathcal{E}_A \omega + u^{r_{k-1}} \sum_{0\leq|\Xi|} \Delta_{r_{k-1}}^{r_{k-2},\Xi} \overline{c}_{\Xi r_{k-2}} \omega = d_H \sigma'_k. \tag{20}$$

□

Since components of the ascent operator $\mathbf{u}$ (16) are gauge and higher-stage gauge symmetries, we agree to call it ***the gauge operator***.

# Direct second Noether theorem

The correspondence of gauge and higher-stage gauge symmetries to NI and higher-stage NI in Theorem 1 is unique due to the following direct second Noether theorem.

**Theorem 2.**

(i) If $u$ (17) is a gauge symmetry, the variational derivative of the $d_H$-exact density $u^A\mathcal{E}_A\omega$ (18) with respect to ghosts $c^r$ leads to the equality

$$\delta_r(u^A\mathcal{E}_A\omega) = \sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}d_\Lambda[u_r^{A\Lambda}\mathcal{E}_A] = \tag{21}$$

$$\sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}d_\Lambda(\eta(\Delta_r^A)^\Lambda\mathcal{E}_A) = \sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}\eta(\eta(\Delta_r^A))^\Lambda d_\Lambda\mathcal{E}_A = 0,$$

which reproduces the complete NI (5).

(ii) Given the $k$-stage gauge symmetry $u^{(k)}$ (19), the variational derivative of the equality (20) with respect to ghosts $c^{r_k}$ leads to the equality, reproducing the $k$-stage NI (12). □

# Gauge symmetries

• Treating gauge symmetries of Lagrangian theory, one traditionally is based on gauge theory of principal connections on principal bundles.

• This notion of gauge symmetries has been generalized to Lagrangian theory on an arbitrary fibre bundle $Y \to X$. Gauge symmetry is defined as ***a differential operator*** on section of some vector bundle $E \to X$ with values in a space of variational symmetries of a Lagrangian $L$. In particular, this is the case of general covariant transformations as gauge symmetries of gravitation theory.

• To define gauge symmetries in the framework of graded Lagrangian formalism, one considers an extension of a simple graded manifold $(Y, \mathcal{A}_F)$ modelled over a composite bundle $F \to Y \to X$ to that $(E^0 \underset{X}{\times} Y, \mathcal{A}_{E \underset{X}{\times} F})$. modelled over a fibre bundle $F \underset{X}{\times} E$, where

$$E = E_1 \underset{X}{\oplus} E_0 \to E_0 \to X$$

is some graded vector bundle over $X$. In this case, gauge symmetries depend on even and odd ghosts as gauge parameters.

• ***A problem is that gauge symmetries need not form an algebra***. Therefore, we replace the notion of the algebra of gauge symmetries with some conditions on the gauge operator. Gauge symmetries are said to be ***algebraically closed*** if the gauge operator admits the nilpotent extension (22), called the BRST operator.

# BRST operator

In contrast with the Koszul – Tate operator (9), ***the gauge operator* u *(15) need not be nilpotent***.

• Let us study its extension to a nilpotent graded derivation

$$\mathbf{b} = \mathbf{u} + \gamma = \mathbf{u} + \sum_{1\leq k\leq N+1} \gamma^{(k)} = \mathbf{u} + \sum_{1\leq k\leq N+1} \gamma^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}} \tag{22}$$

$$= \left( u^A \frac{\partial}{\partial s^A} + \gamma^r \frac{\partial}{\partial c^r} \right) + \sum_{0\leq k\leq N-1} \left( u^{r_k} \frac{\partial}{\partial c^{r_k}} + \gamma^{r_{k+1}} \frac{\partial}{\partial c^{r_{k+1}}} \right)$$

of ghost number 1 by means of antifield-free terms $\gamma^{(k)}$ of higher polynomial degree in ghosts $c^{r_i}$ and their jets $c^{r_i}_\Lambda$, $0 \leq i < k$. We call $\mathbf{b}$ (22) ***the BRST operator***, where $k$-stage gauge symmetries are extended to $k$-stage BRST transformations acting both on $(k-1)$-stage and $k$-stage ghosts. If a BRST operator exists, the sequence (15) is brought into a BRST complex

$$0 \to \mathcal{S}^{0,n}_\infty[F;Y] \xrightarrow{\mathbf{b}} P^{0,n}_\infty\{N\}^1 \xrightarrow{\mathbf{b}} P^{0,n}_\infty\{N\}^2 \xrightarrow{\mathbf{b}} \cdots .$$

• One can show that the gauge operator (15) admits the BRST extension (22) only if the gauge symmetry conditions (20) and the higher-stage NI (12) are satisfied off-shell.

• The Koszul – Tate and BRST complexes provide ***the BRST extension***

$$L_E = L + \mathbf{b} \left( \sum_{0\leq k\leq N} c^{r_{k-1}} \overline{c}_{r_{k-1}} \right) \omega + d_H \sigma,$$

of an original Lagrangian theory by graded ghosts $c^{r_k}$ and antifields $\overline{c}_{r_k}$.

• This extension is a preliminary step towards the BV quantization of reducible degenerate Lagrangian theories.

# Main outcomes

- Gauge theory on principal bundles.
- Gauge gravitation theory on natural bundles.
- Chern – Simons topological theory.
- Topological BF theory
- SUSY gauge theory on graded manifold.s

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